On Sprays with Vanishing \(\chi\)-Curvature

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Abstract

Every Riemannian metric or Finsler metric on a manifold induces a spray via its geodesics. In this paper, we discuss several expressions for the \(\chi\)-curvature of a spray. We show that the sprays obtained by a projective deformation using the S-curvature always have vanishing \(\chi\)-curvature. Then we establish the Beltrami Theorem for sprays with \(\chi = 0\).

Keywords: Sprays, Isotropic curvature, \(\chi\)-curvature and S-curvature.


1 Introduction

A spray \(G\) on a manifold \(M\) is a special vector field on the tangent bundle \(TM\). In a standard local coordinate system \((x^i, y^i)\) in \(TM\), a spray \(G\) can be expressed by

\[
G = y^i \frac{\partial}{\partial x_i} - 2G^i \frac{\partial}{\partial y_i},
\]

where \(G^i = G^i(x, y)\) are local \(C^\infty\) functions on non-zero vectors with the following homogeneity: \(G^i(x, \lambda y) = \lambda^2 G^i(x, y)\), \(\forall \lambda > 0\). Every Finsler metric induces a spray on a manifold. Some geometric quantities of a Finsler metric are actually defined by the induced spray only. These quantities are extremely interesting to us.

For a spray \(G\) on a manifold \(M\), with the Berwald connection, we define two key quantities: the Riemann curvature tensor \(R^i_{j,k,l}\) and the Berwald curvature tensor \(B^i_{j,k,l}\) (see [6]). Certain averaging process gives rise to various notions of Ricci curvature tensor. One of them is the Ricci curvature tensor: \(\text{Ric}_{j,l} := \frac{1}{2} \{ R^m_{j,m,l} + R^m_{l,m,j} \} \) ([2]). The well known Ricci curvature \(\text{Ric} := \text{Ric}_{j,l} y^j y^l = R^m_{j,m,l} y^j y^l\) has been studied for a long time by many people. Besides these quantities, we have another important quantity which is expressed in terms of the vertical derivatives of the Riemann curvature. It is the so-called \(\chi\)-curvature defined by

\[
\chi_k := -\frac{1}{6} \left\{ 2R^m_{k,m} + R^m_{m,k} \right\}.
\]  

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where $R^i_k = y^j R^i_{jk} y^j$. The $\chi$-curvature can be expressed in several forms. For an arbitrary volume form $dV$,

$$
\chi_k = \frac{1}{2} \left\{ S_{k|m} y^m - S_k \right\},
$$

(2)

where $S = S(G,dV)$ is the S-curvature of $(G,dV)$ ([5]). For a spray induced by a Finsler metric, the $\chi$-curvature can be expressed by

$$
\chi_k = \frac{1}{2} \left\{ I_{k|p} y^p y^q + I_m R^m_k \right\},
$$

(3)

where $I_k := g^{ij} C_{ijk}$ denotes the mean Cartan torsion ([4] [1]). These are three typical expressions for the mysterious quantity $\chi$. In this paper, we shall focus on sprays with $\chi = 0$.

For a spray $G$ on a manifold $M$, in the projectively equivalent class of $G$, there is always a spray with $\chi = 0$. More precisely, for any volume form $dV$ on $M$, we may construct a spray $\hat{G}$ by a projective change:

$$
\hat{G}^i := G^i - \frac{S}{n+1} y^i,
$$

where $S$ is the S-curvature of $(G,dV)$. This spray $\hat{G}$ is invariant under a projective change with $dV$ fixed. This projective deformation is first introduced in [6]. We prove the following

**Theorem 1.1** Let $G$ be a spray on a manifold $M$. For any volume form $dV$, the spray $\hat{G}$ associated with $(G,dV)$ has vanishing $\chi$-curvature, $\hat{\chi} = 0$.

Note that $\hat{G}$ is projectively equivalent to $G$. Hence if $G$ is of scalar curvature, then $\hat{G}$ is of scalar curvature too. Hence it is of isotropic curvature since $\hat{\chi} = 0$. Thus $\hat{G}$ must be of isotropic curvature. We obtain the following

**Corollary 1.2** Let $G$ be a spray of scalar curvature on a manifold $M$. For any volume form $dV$, the spray $\hat{G}$ associated with $(G,dV)$ must be of isotropic curvature.

The well-known Beltrami Theorem in Riemannian geometry says that for two projectively equivalent Riemannian metrics $g_1, g_2$, the metric $g_1$ is of constant curvature if and only if $g_2$ is of constant curvature. In particular, if a Riemannian metric $g$ is locally projectively flat, then it is of constant curvature since $g$ is locally projectively equivalent to the standard Euclidean metric. This theorem can be extended to sprays with $\chi = 0$.

**Theorem 1.3** For two projectively equivalent sprays $G_1, G_2$ with $\chi = 0$, $G_1$ is of isotropic curvature if and only if $G_2$ is of isotropic curvature. In particular, if a spray $G$ is locally projectively flat with $\chi = 0$, then it is of isotropic curvature.

Sprays or Finsler metrics with $\chi = 0$ deserve further study. Spherically symmetric metrics with $\chi = 0$ have been studied in [9].

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2 Preliminaries

A spray $G$ on a manifold $M$ is a vector field on the tangent bundle $TM$ which is locally expressed in the following form

$$G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i},$$

where $G^i = G^i(x, y)$ are local $C^\infty$ function on $TU \equiv U \times \mathbb{R}^n$,

$$G^i(x, \lambda y) = \lambda^2 G^i(x, y), \quad \lambda > 0.$$  

Put

$$N^i_j := \frac{\partial G^i}{\partial y^j}, \quad \Gamma^i_{jk} = \frac{\partial^2 G^i}{\partial y^j \partial y^k}.$$  

Let $\omega^i := dx^i$ and $\omega^{n+i} := dy^i + N^i_j dx^j$ and $\omega^i := \Gamma^i_{jk} dx^k$. We have

$$d\omega^i = \omega^j \wedge \omega^i.$$  

Put

$$\Omega^i_j := d\omega^i_j - \omega^k_j \wedge \omega^i_k.$$  

We obtain two quantities $R$ and $B$:

$$\Omega^i_j = \frac{1}{2} R^i_{jk} \omega^k \wedge \omega^l - B^i_{jl} \omega^k \wedge \omega^{n+l},$$

where $R^i_{jk} + R^i_{kj} = 0$.

$$R^i_{jk} = \frac{\delta \Gamma^i_{jk}}{\delta x^l} - \frac{\delta \Gamma^i_{jl}}{\delta x^k} + \Gamma^i_{ks} \Gamma^s_{jk} - \Gamma^i_{js} \Gamma^s_{lk},$$

$$B^i_{jk} = \frac{\partial \Gamma^i_{kl}}{\partial y^l}.$$  

We have the first set of Bianchi identities

$$R^i_{jk} + R^i_{kj} + R^i_{lj} = 0$$

$$B^i_{jk} = B^i_{kj}.$$  

In fact $B^i_{jk}$ is symmetric in $j, k$, and $y^l B^i_{jk} = 0$. Put

$$R^i_{kl} := y^l R^i_{jk}, \quad R^i_{jk} := R^i_{kl} y^l, \quad R^i_k := y^j R^i_{jk} y^l.$$  

The two-index Riemann curvature tensor $R^i_{kl}$ and the four-index Riemann curvature tensor $R^i_{jkl}$ determine each other by the following identity:

$$R^i_{jkl} = \frac{1}{3} \left\{ R^i_{k\cdot l \cdot j} - R^i_{l\cdot k \cdot j} \right\},$$
We also have
\[ R^i_{jk} = \frac{1}{3} \left\{ 2R^i_{k,j} + R^i_{j,k} \right\}, \quad (8) \]
\[ R^i_{kl} = \frac{1}{3} \left\{ R^i_{k,l} - R^i_{l,k} \right\}, \quad (9) \]
where \( T^*_k \) is the vertical covariant derivative, namely, \( T^*_k = \frac{\partial}{\partial y^k} (T^*_m) \).

Further covariant derivatives yield the second set of Bianchi identities:
\[ R^i_{jk l} |_m + R^i_{jl m} |_k + R^i_{jm k} |_l = 0 \quad (10) \]
\[ R^i_{kl m} = B_j^i m l | k - B_j^i m k | l \quad (11) \]
\[ B_j^i k l = B_j^i k m l. \quad (12) \]

Contracting (10) with \( y^l \) yields
\[ R^i_{kl m} + R^i_{lm k} + R^i_{mk l} = 0. \quad (13) \]
Contracting (13) with \( y^l \) yields
\[ R^i_{k|m} - R^i_{m|k} + R^i_{mk|l} y^l = 0. \quad (14) \]

3 The \( \chi \)-curvature

The \( \chi \)-curvature defined by the Riemann curvature tensor in (1) can be expressed in several ways.

**Lemma 3.1**
\[ \chi_k = -\frac{1}{2} R^m_{m k} = -\frac{1}{2} R^m_{m k} y^l. \quad (15) \]

**Proof:** It follows from (8). Q.E.D.

Lemma 3.1 tells us that if \( R^m_{m k} = 0 \), then \( \chi = 0 \).

Put
\[ T^i_k := R^i_k - \left\{ R^i_k - \frac{1}{2} R y^l \right\}, \quad (16) \]
where \( R := \frac{1}{n-1} R^m_m \). By definition, \( G \) is of isotropic curvature if \( T^i_k = 0 \). Note that
\[ \text{trace}(T) := T^m_{m} = 0. \]

By a direct computation, we can obtain another expression for \( \chi_k \).

**Lemma 3.2**
\[ \chi_k = -\frac{1}{3} T^m_{k.m}. \quad (17) \]
Lemma 3.2 tells us that if $G$ is of isotropic curvature, then $\chi = 0$.

Recall the definition of the Weyl curvature

$$W^i_k := A^i_k - \frac{1}{n+1} A^m_{k,m} y^i,$$  \hspace{1cm} (18)

where $A^i_k := R^i_k - R \delta^i_k$. Clearly,

$$W^m_{k,m} = 0.$$

We obtain a nice formula for the Weyl curvature.

**Lemma 3.3** The Weyl curvature is given by

$$W^i_k = R^i_k - \left\{ R \delta^i_k - \frac{1}{2} R_{k,i} y^i \right\} + \frac{3}{n+1} \chi y^i. \hspace{1cm} (19)$$

**Proof:** One can easily rewrite $W^i_k$ as

$$W^i_k = R^i_k - \left\{ R \delta^i_k - \frac{1}{2} R_{k,i} y^i \right\} - \frac{1}{2(n+1)} \left\{ 2R^m_{k,m} + (n-1)R_y \right\} y^i.$$

By (1), we prove the lemma. Q.E.D.

Given a volume $dV = \sigma(x) dx^1 \cdots dx^n$, the S-curvature of $(G, dV)$ is defined by

$$S := \Pi - y^m \frac{\partial}{\partial x^m} \left( \ln \sigma \right).$$

We have the following expression for $\chi$.

**Lemma 3.4** ([2])

$$\chi^k = \frac{1}{2} \left\{ S_{k|m} y^m - S_{k} \right\}. \hspace{1cm} (20)$$

In local coordinates, by (20), one can easily get

$$\chi^k = \frac{1}{2} \left\{ \Pi_{x^m y^k y^m} - \Pi_{x^k} - 2\Pi_{x^k y^m} G^m \right\}, \hspace{1cm} (21)$$

where $\Pi := \frac{\partial G^m}{\partial y^m}$. Clearly, $\chi$ is independent of $dV$.

### 4 Sprays with $\chi = 0$

A spray is said to be $S$-closed if in local coordinates, $\Pi = \frac{\partial G^m}{\partial y^m}$ is a closed local 1-form. The spray induced by a Riemannian metric $g = g_{ij}(x)y^i y^j$ is $S$-closed. In fact

$$\Pi = y^k \frac{\partial}{\partial x^k} \left[ \ln \sqrt{\det(g_{ij}(x))} \right]. \hspace{1cm} (22)$$
By (22), for any volume form \(dV = \sigma(x)dx^1 \wedge \cdots \wedge dx^n\), the S-curvature of \((G, dV)\) is a closed 1-form,

\[ S = y^k \frac{\partial}{\partial x^k} [\ln \varphi(x)], \]

where \(\varphi(x) = \sqrt{\det(g_{ij}(x))}/\sigma(x)\).

We have the following

**Proposition 4.1** If a spray is S-closed, then \(\chi = 0\). In particular, if for some volume form \(dV = \sigma dx^1 \cdots dx^n\), the S-curvature of \((G, dV)\) is a closed 1-form, then \(\chi = 0\).

**Proof:** By assumption,

\[ S = \Pi - y^m \frac{\partial}{\partial x^m} (\ln \sigma) = \eta_k y^k, \]

with \((\eta_k)_x^l = (\eta_l)_x^k\). Then by (21), \(\chi_k = 0\).

Q.E.D.

Let \(\tilde{F}\) be a Finsler metric and \(G\) be a spray on a manifold \(M\). The spray coefficients \(\tilde{G}^i\) of \(\tilde{F}\) can be expressed as follows

\[ \tilde{G}^i = G^i + Py^i + \frac{1}{2} \tilde{F} g^{ik} \{ \tilde{F}_{m} y^m - \tilde{F}_{|k} \}. \]

(23)

where \(P = \tilde{F}_{m} y^m / (2 \tilde{F})\). Thus \(\tilde{F}\) is projectively equivalent to \(G\) if and only if

\[ \tilde{F}_{m} y^m - \tilde{F}_{|k} = 0. \]

(24)

This is the generalized version of the famous Rapcsák Theorem. By (20), we obtain the following

**Theorem 4.2** Let \(G\) be a spray with \(\chi = 0\) and \(dV\) be a volume form. If for the S-curvature \(S\) of \((G, dV)\), \(\tilde{F} = |S|\) is a Finsler metric, then it is projectively equivalent to \(G\).

### 5 Sprays of isotropic curvature

A spray \(G\) is said to be of scalar curvature if

\[ R^i_k = R \delta^i_k - \tau_k y^i, \]

(25)

where \(\tau_k\) is a positively homogeneous function of degree one with \(\tau_k y^k = R\). This is equivalent to that \(W^i_k = 0\). By (19), we see that (25) is equivalent to the following

\[ R^i_k = R \delta^i_k - \frac{1}{2} R_k y^k - \frac{3}{n+1} \chi_k y^i. \]

(26)

The \(\chi\)-curvature characterizes sprays of isotropic curvature among sprays of scalar curvature. By (26), we obtain the following
Theorem 5.1 ([3]) Let $G$ be a spray of scalar curvature. $G$ is of isotropic curvature if and only if $\chi = 0$.

Proof of Theorem 1.3: If $G_1$ is of isotropic curvature, then $G_2$ is of scalar curvature by the projective equivalence. Since $\chi = 0$, we see that $G_2$ is of isotropic curvature by Proposition 5.1. Q.E.D.

If $G$ is of isotropic curvature, then
\[
R^i_{jkl} = \frac{1}{2} \left\{ R_{i|j|m} \delta^i_k - R_{-m|j|l} \delta^i_l \right\}.
\]

Assume that $G$ is of isotropic curvature. By (10), we obtain
\[
(R_{i|j|m} - R_{m|j|i}) \delta^i_k + (R_{m|j|k} - R_{-k|j|m}) \delta^i_l + (R_{k|j|i} - R_{-l|j|m}) \delta^i_m = 0. \tag{27}
\]

This yields
\[
(R_{i|m} - R_{m|i}) \delta^i_k + (R_{m|k} - R_{-k|m}) \delta^i_l + (R_{k|l} - R_{-l|k}) \delta^i_m = 0. \tag{28}
\]

Contracting (28) with $y^m$ yields
\[
(R_{i|m} y^m - 2R_{i|m}) \delta^i_k + (2R_{m|k} - R_{-k|m} y^m) \delta^i_l + (R_{k|l} - R_{-l|k}) y^i = 0. \tag{29}
\]

Taking trace $i = k$ in (29), we obtain
\[
(n - 2)(R_{i|m} y^m - 2R_{i|l}) = 0. \tag{30}
\]

Theorem 5.2 If $G$ is an $n$-dimensional spray of isotropic curvature $R$ ($n \geq 3$), then $R$ satisfies
\[
\frac{1}{2} R_{i|m} y^m - R_{i|l} = 0. \tag{31}
\]

Proof: By assumption $n \geq 3$, we obtain from (30),
\[
R_{i|l} - \frac{1}{2} R_{i|m} y^m = 0.
\]

Q.E.D.

For a spray $G$, we introduce a new quantity $\eta = \eta_k dx^k$,
\[
\eta_k := \frac{1}{2} R_{k|m} y^m - R_{k|l}, \tag{32}
\]

where $R := \frac{1}{n-1} \text{Ric}$.

For a spray of isotropic curvature $R$ on $n$-dimensional manifold $M$ ($n \geq 3$), By Theorem 5.2, $\eta = 0$. 

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Let $L := \tilde{F}^2$ be a Finsler metric and $G$ be a spray on a manifold. By (23), the spray coefficients of $\tilde{F}$ can be expressed as

$$\tilde{G}^i = G^i + \frac{1}{4}L^{-1}L_{[m}y^m y^i} - \frac{1}{8}L^{-1}\tilde{g}^{ik}L_{[m}y^m L_{k]} + \frac{1}{4}\tilde{g}^{ik}\left\{L_{[k]}m y^m - L_{[k]}\right\}.$$

We obtain

$$\tilde{G}^i = G^i + \frac{1}{4}\tilde{g}^{ik}L_{[k]} + \frac{1}{2}\tilde{g}^{ik}\left\{\frac{1}{2}L_{[k]}m y^m - L_{[k]}\right\}. \quad (33)$$

In [7], we introduced a notion of dually flat Finsler metrics. This concept can be generalized as follows. $\tilde{F}$ is said to be dually equivalent to $G$ if $L := F^2$ satisfies

$$\tilde{G}^i = G^i + \frac{1}{4}\tilde{g}^{ik}L_{[k]}.$$

By (33), this is equivalent to

$$\frac{1}{2}L_{[k]}m y^m - L_{[k]} = 0. \quad (34)$$

For a spray $G$ on an $n$-dimensional manifold $M$ with isotropic scalar curvature $R$. Assume that $R$ is a Finsler metric, by Theorem 5.2, $R$ satisfies (31). Thus one can see that $R$ is dually equivalent to $G$.

### 6 Projective change by the S-curvature

Let $G$ be a spray and $dV$ be a volume form on an $n$-dimensional manifold $M$. We deform $G$ to another spray $\hat{G}$ by

$$\hat{G} := G - \frac{S}{n+1}y^i,$$

where $S$ denotes the S-curvature of $(G, dV)$. From the definition, we see that $\hat{G}$ is projectively equivalent to $G$.

**Lemma 6.1** Let $G$ be a spray and $dV$ a volume form on a manifold $M$. Let $\hat{G}$ be the spray associated with $(G, dV)$. Then the S-curvature of $(G, dV)$ vanishes. Hence, $\hat{\chi} = 0$.

**Proof:** Recall

$$\hat{\chi}_k = \frac{1}{2}\left\{\hat{S}_{[m,k}y^m - \hat{S}_{[k]}\right\}.$$

On the other hand, $\hat{G}^i = G^i + Py^i$ with $P = -\frac{S}{n+1}$. Thus

$$\hat{S} = S + (n + 1)P = 0.$$

This yields that $\hat{\chi} = 0$. Q.E.D.
Lemma 6.2 If \( G_1 \) and \( G_2 \) are two projectively equivalent sprays on a manifold \( M \), then for any volume form \( dV \), the spray \( \hat{G}_1 \) associated with \( (G_1, dV) \) and \( \hat{G}_2 \) associated with \( (G_2, dV) \) are equal, i.e., \( G_1 = G_2 \).

Proof: It is easy to see that if \( \hat{G}_1 = G_1^i + Py^i \), then
\[
S_1 = S_2 + (n+1)P.
\]

Then
\[
\hat{G}_1^i = G_1^i - \frac{S_1}{n+1}y^j = [G_2^i + Py^i] - \frac{S_2 + (n+1)P}{n+1}y^i = G_2^i - \frac{S_2}{n+1}y^i = \hat{G}_2.
\]

Q.E.D.

Proof of Corollary 1.2: First by definition, \( \hat{G} \) is projectively equivalent to \( G \). Thus \( \hat{G} \) is of scalar curvature. Since \( \hat{\chi} = 0 \), by Lemma 5.1, we see that \( \hat{G} \) is of isotropic curvature.

Q.E.D.

By the above lemma, any geometric quantity of \( \hat{G} \) is a projective invariant of \( G \) with respect to a fixed volume form \( dV \). Further, if the geometric quantity of \( \hat{G} \) is independent of the volume form \( dV \), then the quantity is a projective quantity of \( G \).

Lemma 6.3 Let \( G \) be a spray and \( dV \) a volume form on a manifold \( M \). For the spray \( \hat{G} \) associated with \( (G, dV) \), the Riemann curvature of \( \hat{G} \) is given by
\[
\hat{R}^i_k = R^i_k + \tau \delta^i_k - \frac{1}{2} \tau_{,ik}y^j + \frac{3\chi_k}{n+1}y^i,
\]
where
\[
\tau := \left( \frac{S}{n+1} \right)^2 + \frac{1}{n+1}S_{,m}y^m.
\]

Proof: By a direct argument. Q.E.D.

By (35), we get the projective Ricci curvature tensor \( \tilde{\text{Ric}}_{ij} := \frac{1}{2} \{ \hat{R}_{jml} + \hat{R}_{i,mj} \} \) and the projective Ricci curvature \( \tilde{\text{Ric}} := \tilde{\text{Ric}}_{ij}y^iy^j \).

\[
\tilde{\text{Ric}}_{ij} = \text{Ric}_{ij} + \frac{n-1}{2} \tau_{,ij} + H_{ij},
\]
\[
\tilde{\text{Ric}} = \text{Ric} + (n-1)\tau,
\]

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where $\widehat{\text{Ric}} = \widehat{\text{Ric}}_{jil} y^i y^j$ is the Ricci curvature of $\widehat{G}$ and

$$H_{ij} := \frac{1}{2} \left\{ \chi_{i,j} + \chi_{j,i} \right\}.$$ 

It is natural to consider other quantities of $\widehat{G}$, such as the Berwald curvature defined in (4) and the T-curvature defined in (16)

$$\hat{B}^{i}_{jkl} = \frac{\partial^3 \hat{G}^i}{\partial y^j \partial y^k \partial y^l},$$

$$\hat{T}^i_k = \hat{R}^i_k - \left\{ \hat{R}^j_k - \frac{1}{2} \hat{R} k y^j \right\}.$$ 

Clearly, $\hat{B}$ and $\hat{T}$ are projective invariants with a fixed volume form $dV$. We have the following

**Proposition 6.4** Let $G$ be a spray on a manifold and $\hat{G}$ a spray associated with $(G, dV)$ for some volume form $dV$. Then the Berwald curvature $\hat{B}$ and $\hat{T}$ are independent of $dV$, hence they are projective invariants of $G$. In fact $\hat{B} = D$ is the Douglas curvature and $\hat{T} = W$ is the Weyl curvature of $G$.

Here we provide another description of the Douglas curvature and the Weyl curvature of a spray.

Let $G$ be a spray and $\hat{G}$ be the spray associated with $(G, dV)$ for some volume form $dV$. Let $\hat{\eta}$ be the quantity of $\hat{G}$ defined in (32). Then $\hat{\eta}$ is a projective invariant of $G$ for a fixed volume form $dV$. In fact, $\hat{\eta} = W^o$ the so-called Berwald-Weyl curvature ([6]). If $G$ is of scalar curvature, then $\hat{G}$ is of isotropic curvature. Thus $\hat{\eta} = 0$ when $n = \dim M \geq 3$ by Theorem 5.2.

**Proposition 6.5** Let $G$ be a spray on a manifold and $\hat{G}$ a spray associated with $(G, dV)$ for some volume form $dV$. Assume that $G$ is of scalar curvature. Then the projective invariant $\hat{\eta} = 0$ in dimension $n \geq 3$.

### 7 Examples

In this section, we shall give some sprays of isotropic curvature.

**Example 7.1** Let $F = \alpha + \beta$ be a Randers metric on an $n$-dimensional manifold $M$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x) y^i$ is a 1-form on $M$. Let $\nabla \beta = b_{ij} y^i dx^j$ denote the covariant derivative of $\beta$ with respect to $\alpha$. Let

$$r_{ij} := \frac{1}{2} \left( b_{ij} + b_{ji} \right), \quad s_{ij} := \frac{1}{2} \left( b_{ij} - b_{ji} \right), \quad s_j := b^i s_{ij},$$

$$q_{ij} := r_{im} s^m_j, \quad t_{ij} := s_{im} s^m_j, \quad t_j := b^i t_{ij}.$$
Let
\[ \hat{G}^i := G^i_\alpha + \alpha s^i_\alpha. \] (39)
In fact \( \hat{G} \) is the spray associated with \((G, dV_\alpha)\). It is proved that \( \hat{G} \) is of scalar curvature if and only if the Riemann curvature \( \bar{R}^i_k \) of \( \alpha \) and the covariant derivatives of \( \beta \) satisfy the following equations ([8])

\[ \bar{R}^i_k = \kappa \left\{ \alpha^2 \delta^i_k - y_k y^i \right\} + \alpha^2 t^i_k + t_{00} \delta^i_k - t_{0i} y_k - 3 s^i_0 s^0, \] (40)

\[ s_{ij|k} = \frac{1}{n-1} \left\{ a_{ik}s^m_j|m - a_{jk}s^m_i|m \right\}. \] (41)

where \( \kappa = \kappa(x) \) is a scalar function on \( M \). In this case, \( \hat{G} \) is actually of isotropic curvature. \( \bar{R}^i_k = \bar{R}\delta^i_k - \frac{1}{2} \bar{R} y^i \). By a simple computation, we obtain a formula for \( \bar{R} := \frac{1}{n-1} \text{Ric} \):

\[ \bar{R} = \kappa \alpha^2 + t_{00} + \frac{2}{n-1} \alpha s^m_0|m. \]

**Example 7.2** Consider a spray on an open subset \( U \subset R^2 \),

\[ G = y^1 \frac{\partial}{\partial x^1} + y^2 \frac{\partial}{\partial x^2}, \]

where

\[ G^1 = B(y^1)^2 + 2C y^1 y^2 + D(y^2)^2 + \frac{1}{3} (f_{x^1}(y^1)^2 + f_{x^2}(y^1)^2) \]

\[ G^2 = -A(y^1)^2 + 2B y^1 y^2 - C(y^2)^2 + \frac{1}{3} (f_{x^1} y^1 y^2 + f_{x^2}(y^1)^2) \]

where

\[ A = A(x^1, x^2), \ B = B(x^1, x^2), \ C = C(x^1, x^2), \ D = D(x^1, x^2), \ f = f(x^1, x^2) \]

are \( C^\infty \) functions on \( U \). The geodesics are the graphs of \( x^2 = \phi(x^1) \)

\[ \phi'' = 2A(x^1, \phi) + 6B(x^1, \phi) \phi' + 6C(x^1, \phi)(\phi')^2 + 2D(x^1, \phi)(\phi')^3. \]

We have

\[ \Pi = \frac{\partial G^m}{\partial y^m} = f_{x^1} y^1 + f_{x^2} y^2. \]

Thus \( \chi_k = 0 \). Further computation shows that \( G \) is of isotropic curvature.
References


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