OPPORTUNITIES FOR GENERALIZING WITHIN PRE-SERVICE SECONDARY TEACHERS' SYMBOLIZATION OF COMBINATORIAL TASKS

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This paper reports on an interview study conducted with four pre-service secondary teachers (PSSTs) for the purpose of understanding how symbolizing sets of outcomes supports opportunities for generalizing in the context of solving combinatorial problems. This paper examines opportunities for generalizing based on differences in symbolization as well as differences in PSSTs' conceptualization of seemingly identical symbols.

Keywords: Algebra and Algebraic Thinking, Cognition, Design Experiments

Introduction with Supporting Literature

The act of generalizing is central to mathematical reasoning because it is a primary vehicle for the construction of new mathematical knowledge (Amit & Klass-Tsirulnikov, 2005; Lannin, 2005; Sriraman, 2003). As a result, "developing children's generalizations is one of the principal purposes of school instruction (Davydov, 1972/1990, p. 10)." Current mathematics curricula and standards reflect this principal purpose (e.g., Hirsch, et. al., 2007; Lappan, et. al., 2006), and State and National standards focus on generalization (e.g., Council of Chief State School Officers, 2010; Indiana Academic Standards for Mathematics, 2014).

Researchers studying combinatorics have identified symbolizing sets of outcomes as helpful for students to produce all possible outcomes (English, 1991, 1993; Nunes & Bryant, 1996); avoid, correct, and explain common counting errors (Lockwood, 2014); and establish when two ways of reasoning are isomorphic (Maher, Powell, & Uptegrove, 2010). Researchers investigating combinatorial reasoning have not yet explicitly examined how symbolizing sets of outcomes could support students' opportunities for generalizing. Two reasons such an exploration is of interest in combinatorics are: (a) it is common for students to symbolize sets of outcomes for the same problem in different ways; and (b) it is common for students to symbolize sets of outcomes in ways that look identical, but their symbols have different meanings.

Given these observations, the purpose of this paper is to examine differences in the way pre-service secondary teachers (PSSTs)–who were student-participants in a teaching experiment–symbolized sets of outcomes in their solution of combinatorics problems and to identify how these differences afforded and constrained their opportunities for generalizing. The following research questions guide this paper:

1. What different opportunities for generalizing are available to PSSTs when they symbolize sets of outcomes for the same problem in different ways?
2. What different opportunities for generalizing are available to PSSTs when they symbolize sets of outcomes in the same way but conceive of the symbols differently?

Theoretical Perspectives

We use the term *symbolizing sets of outcomes* to include the creation of graphic items (or the use of other figurative material) in the context of a student implementing her schemes (cf. Von Glasersfeld, 1995). We include in our definition items that are conventional ways of symbolizing a set of outcomes like a written or verbal list, a tree diagram, an array, or an empty slot as well as non-conventional ways of symbolizing sets of outcomes like a drawing, tally marks, curved lines, or a demonstration with concrete materials. We follow Ellis (2007) in differentiating between generalizing actions and reflection generalizations. For the purposes of this paper, we consider symbolizing sets of outcomes to be a generalizing action, which is an action that precedes and may support a formal statement of generalization. Therefore, we focus on differences in the way PSSTs symbolized sets of outcomes and how such differences afforded or constrained opportunities for statements of generalization. We do this even in situations where students may not explicitly make formal statements of generalization.

Methods and Methodology

The data for this study was collected using teaching experiment methodology (Confrey & LaChance, 2000; Steffe & Thompson, 2000). The research team consisted of one university faculty member and six graduate students in mathematics education. Participants in the study were four PSSTs who were concurrently enrolled in their second mathematics methods course at a Midwestern university during the fall of 2018.

Participants were paired for interview purposes, with each pair participating in 13 teaching episodes. Teaching episodes were conducted weekly, lasted 60-90 minutes, and were recorded with three cameras. Two cameras recorded the written work of each PSST, and a third captured the interaction between the teacher-researcher and the PSSTs. The goal of the first 9 teaching episodes was to help the PSSTs see combinatorial structure in common algebraic identities, with an aim at helping them gain a progressively more general understanding of the relationship between common algebraic identities and combinatorial structure (Tillema & Gatza, 2016; Tillema & Gatza, 2017). The remaining 4 episodes focused on how the PSSTs could use their work to teach a lesson sequence during their student teaching.

Data analysis included the first and second author independently watching video segments from each of the four participants to analyze how symbolizing sets of outcomes afforded or constrained opportunities for generalizing. The first and second author then discussed their interpretations of these video segments until they reached a common interpretation. These interpretations were then shared with the third author and again discussed until there was a common interpretation. The example in this paper focuses on opportunities for generalizing based on PSSTs' symbolizing sets of outcomes in a combinatorial situation where they were making meaning for the term $10x^3y^2$ in the expansion of $(x + y)^5$. We choose this example because the ten arrangements of three x's and 2 y's was not obvious to the PSSTs who had not yet developed a systematic way to count arrangements for middle terms in the identity (i.e., $10x^3y^2$, $10x^2y^3$).

Results and Discussion

In this paper, we provide one example of how symbolizing a set of outcomes affords different opportunities for generalizing. This case highlights differences in the ways Olive and Aaron symbolized sets of outcomes. We will provide additional examples across the four PSSTs during the presentation.

During the ninth teaching episode, Olive and Aaron generated a list related to the algebraic identity $(A + B)^5 = 1A^5 + 5A^4B + 10A^3B^2 + 10A^2B^3 + 5AB^4 + 1B^5$. At this point in the study, Olive and Aaron had established: (a) that A and B were variables representing the number of possible options in a binary situation; (b) that each term on the right-hand side of the equivalence represented 5 selections of either variable A or variable B; and (c) that the coefficients represented the number of ways they could position the As or Bs in 5 slots (e.g., the number of different ways to position 3 *A*s and 2 *B*s for the term A^3B^2). They, however, did not have a systematic way to count the arrangements of *A*s and *B*s for the middle terms of the identity (i.e., A^3B^2 and A^2B^3). The teacher-researcher pressed both PSSTs to show the different ways to arrange 3 *A*s and 2 *B*s.

For the A^3B^2-term, Olive focused on the variable appearing fewer times, i.e. *B* (Figure 1).

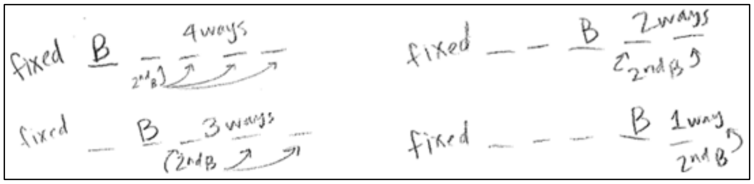

Figure 1: Olive's representation of the 10 variations of A^3B^2.

Olive's way of symbolizing the set of outcomes communicated a systematic organization, and so the teacher-researcher asked Aaron to make a conjecture regarding Olive's thinking.

> She fixed her *B*s and then rotated where the other [second] *B* could be. Because you only have two *B*s. She fixed [a *B*] in the first position and then moved [the fixed *B*] to the second position. But you know that you can't have another B back in the first position because you already counted that for the first one. And then you keep working your way down, rotating over the fixed *B* and counting the other ways, knowing the others behind the fixed *B* were already counted.

Olive agreed that Aaron had adequately explained her method. Without prompting, Olive symbolized her set of outcomes as 4 + 3 + 2 + 1 = 10. After accurately explaining Olive's list, the teacher-researcher asked Aaron to explain his own (Table 1, left).

Table 1: Aaron symbolizing outcomes

Aaron's original list of outcomes:			Aaron's list emphasizing his thinking:		
xxxyy xxyxy xxyyx	xyxxy xyxyx xyyxx	yxxxy yxxyx yxyxx yyxxx	xxxyy xxyxy xxyyx	xyxxy xyxyx xyyxx	yxxxy yxxyx yxyxx yyxxx

Aaron explained that he focused on placing three *x*'s for the outcomes in the first column. He fixed two *x*'s in the first two positions and then rotated the third *x* through the remaining three positions (i.e., either position 3 or 4 or 5) (Table 1, right). In the second column, he repeated this process by fixing *x* in the first and third positions while rotating the third *x* through the remaining two positions. Aaron then fixed *x*'s in the first and fourth positions, placing the third *x* in the last position. Realizing he had exhausted possible arrangements with *x* in the first position, Aaron considered arrangements with *y* in the first position. At this point, Aaron's thinking changed *from* placing *x* in three out of five positions *to* placing *y* in two out of five positions. Aaron stated "the *y*'s are easier to explain. I fixed the first *y* and started the other at the outside and just worked its way in."

Although Aaron interpreted Olive's method before explaining his own, he did not seem to alter his explanation based on new insight provided by Olive's method. We make this inference based on Aaron's reaction to (a) Olive's re-voicing of his thinking and (b) Olive subsequently explaining how she could see her own structure in Aaron's list. During Olive's explanation, Aaron gestured between the two lists (i.e., Olive's and his own) as if attempting to confirm the connection Olive suggested.

Based on Aaron's thinking, the total number of outcomes in his list could be symbolized as (3 + 2 + 1) + 4, where (3 + 2 + 1) represents arrangements when the x positions were foregrounded and 4 represents the foregrounding of the y positions. However, Aaron did not numerically symbolize his set of outcomes in either way; he simply wrote that there were 10 ways. Writing 10, taken together with Aaron's statement that "the y's are easier to explain", indicates that this mixed method of listing constrained his ability to easily symbolize a structure that he could subsequently generalize with algebraic notation.

Response to Research Question 1

Olive and Aaron worked on the same task; however, they symbolized the set of outcomes differently. Olive's method *explicitly* demonstrated a process, but left the individual outcomes *implied*. The explicit nature of Olive's listing method allowed Aaron to accurately infer the process she used and enabled Olive to independently translate her method into the sum 4 + 3 + 2 + 1 ways to produce the A^3B^2-term. Aaron, on the other hand, listed individual outcomes *explicitly*, switching processes part way through. When Aaron used numerical symbols for his solution he did not use them to show the structure of his reasoning; this is in part because he switched the process for generating outcomes as he created his list. We see these distinctions as affording differences in opportunities to generalize. Olive's explanation suggested she could project how she would reason in new cases whereas Aaron's explanation suggested he would have difficulty projecting his reasoning onto new cases. We took the PSSTs' ability (or inability) to provide explanations that could be applied to new cases as a key indicator of opportunity to generalize.

Response to Research Question 2

Aaron generated his final four outcomes using a new strategy. Aaron could have continued to focus on the position of the x's after exhausting outcomes with x in the first position. With this method, Aaron's list (Figure 2) could be symbolized as (3 + 2 + 1) + (2 + 1) + (1)–i.e., as a sum of sums–with the partial sum (2 + 1) symbolizing the number of outcomes with the first x fixed in the second position, and the final partial sum (1) as symbolizing the number of outcomes with the first x fixed in the third position.

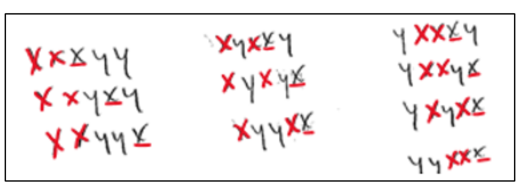

Figure 2: Aaron's list

We see other students, who produce the same list as Aaron but conceive of their list as a sum of sums by persisting with focus on x's, as afforded greater opportunity to generalize.

Our data illustrates both that *different ways* of symbolizing sets of outcomes offer different opportunities to generalize and that different meanings for the *same way* of symbolizing a set of outcomes offers different opportunities to generalize. These findings support the conclusion that careful attention both to the way students symbolize sets of outcomes as well as to the meaning students have for the outcomes they have symbolized is crucial for teachers interested in supporting generalization in the domain of combinatorics.

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