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Entitled Exact Solutions to the Six-Vertex Model with Domain Wall Boundary Conditions and Uniform Asymptotics of Discrete Orthogonal Polynomials on an Infinite Lattice

For the degree of Doctor of Philosophy

Is approved by the final examining committee:

Pavel Bleher
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EXACT SOLUTIONS TO THE SIX-VERTEX MODEL WITH DOMAIN WALL BOUNDARY CONDITIONS AND UNIFORM ASYMPTOTICS OF DISCRETE ORTHOGONAL POLYNOMIALS ON AN INFINITE LATTICE

A Dissertation

Submitted to the Faculty

of

Purdue University

by

Karl Edmund Liechty

In Partial Fulfillment of the

Requirements for the Degree

of

Doctor of Philosophy

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Indianapolis, Indiana

This one goes out to my mom and my pops,
whose continual support always amazes me.

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ABSTRACT

Liechty, Karl Edmund, Ph.D., Purdue University, August 2010. Exact Solutions to the Six-Vertex Model with Domain Wall Boundary Conditions and Uniform Asymptotics of Discrete Orthogonal Polynomials on an Infinite Lattice. Major Professor: Pavel M. Bleher.

In this dissertation the partition function, Z_n, for the six-vertex model with domain wall boundary conditions is solved in the thermodynamic limit in various regions of the phase diagram. In the ferroelectric phase region, we show that $Z_n = CG^nF^{n^2}(1+O(e^{-n^{1-\varepsilon}}))$ for any $\varepsilon > 0$, and we give explicit formulae for the numbers C, G, and F. On the critical line separating the ferroelectric and disordered phase regions, we show that $Z_n = Cn^{1/4}G^{\sqrt{n}}F^{n^2}(1+O(n^{-1/2}))$, and we give explicit formulae for the numbers G and F. In this phase region, the value of the constant C is unknown. In the antiferroelectric phase region, we show that $Z_n = C\vartheta_4(n\omega)F^{n^2}(1+O(n^{-1}))$, where ϑ_4 is Jacobi's theta function, and explicit formulae are given for the numbers ω and F. The value of the constant C is unknown in this phase region.

In each case, the proof is based on reformulating Z_n as the eigenvalue partition function for a random matrix ensemble (as observed by Paul Zinn-Justin), and evaluation of large n asymptotics for a corresponding system of orthogonal polynomials. To deal with this problem in the antiferroelectric phase region, we consequently develop an asymptotic analysis, based on a Riemann-Hilbert approach, for orthogonal polynomials on an infinite regular lattice with respect to varying exponential weights. The general method and results of this analysis are given in Chapter 5 of this dissertation.

1. INTRODUCTION TO THE SIX-VERTEX MODEL

1.1 Definition of the model

The first four chapters of this dissertation compile the results of the papers [7], [8], and [9], in which the partition function for the six-vertex model with domain wall boundary conditions is solved in the thermodynamic limit in various regions of the phase diagram. Let us therefore begin with a review of the six-vertex model.

The six-vertex model, or the model of two-dimensional ice, is a statistical mechanical model stated on a square $n \times n$ lattice with arrows on edges. The arrows obey the rule that at every vertex there are two arrows pointing in and two arrows pointing out. Such a rule is sometimes called the *ice-rule*. There are only six possible configurations of arrows at each vertex, hence the name of the model, see Figure 1.1, and the states of the system are the possible configurations of arrows which obey this rule at every vertex.

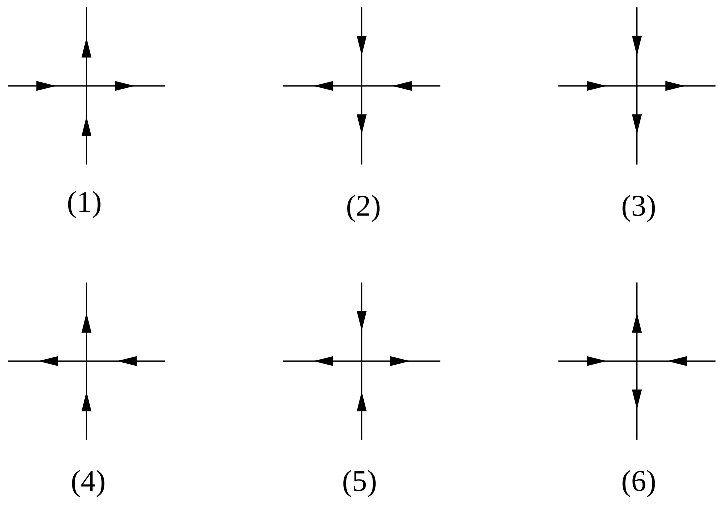

Fig. 1.1. The six arrow configurations allowed at a vertex

The six-vertex model is the prototypical "ice-type" model, meaning that the states of the model can be identified with two-dimensional H_2O crystals. This is achieved by placing an oxygen atom at each vertex of the graph and a hydrogen atom at each edge, and specifying that there is a bond between a hydrogen and an adjacent oxygen if the arrow on that edge points towards the oxygen atom. Other "ice-type" ensembles include the ensemble of alternating sign matrices (ASM's), the three color model, and the eight vertex solid-on-solid model.

The Gibbs measure of this ensemble, which gives the probability of a particular state σ, is defined in the usual way. For each vertex type in Figure 1.1, we assign a weight $w_i > 0$, $i = 1, \ldots, 6$, and define the partition function as a sum over all possible arrow configurations of the product of the vertex weights,

$$Z_n = \sum_{\text{arrow configurations } \sigma} w(\sigma), \qquad w(\sigma) = \prod_{x \in V_n} w_{t(x;\sigma)} = \prod_{i=1}^{6} w_i^{N_i(\sigma)},$$

where V_n is the $n \times n$ set of vertices, $t(x;\sigma) \in \{1, \ldots, 6\}$ is the type of configuration σ at vertex x according to Figure 1.1, and $N_i(\sigma)$ is the number of vertices of type i in the configuration σ. The sum is taken over all possible configurations obeying the given boundary condition. The Gibbs measure is then defined as

$$\mu_n(\sigma) = \frac{w(\sigma)}{Z_n}\,.$$

The six-vertex model was first introduced by J.C. Slater in [43], and solved exactly in the thermodynamic limit by Lieb, [34]–[37], and Sutherland [45], for periodic boundary conditions. As we shall see, the six-vertex model differs from other standard models of statistical mechanics, such as the Ising model, in that it is very sensitive to boundary conditions. The model has been studied with free boundary conditions, anti-periodic boundary conditions, boundary loop conditions, etc. In this dissertation, we consider *domain wall boundary conditions* (DWBC), in which the arrows on the upper and lower boundaries point in the square, and the ones on the left and right boundaries point out. One possible configuration with DWBC on the 4×4 lattice

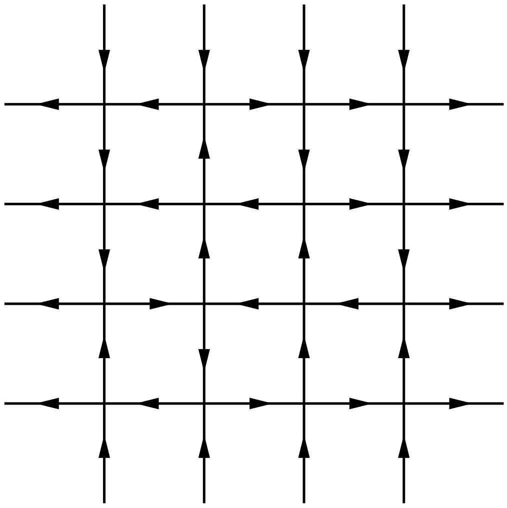

Fig. 1.2. An example of 4×4 configuration with DWBC.

is shown on Figure 1.2. Our main goal is to obtain the large n asymptotics of the partition function Z_n for the six-vertex model with DWBC.

One consequence of domain wall boundary conditions is that they imply some conservation laws in the system which allow us to reduce the number of parameters.

1.2 Height function and reduction of parameters

The six-vertex model has six parameters: the weights w_i. By using some conservation laws it can be reduced to only two parameters. It is convenient to derive the conservation laws from the *height function*. Consider the dual lattice,

$$V' = \{x = (i + \frac{1}{2}, j + \frac{1}{2}), \quad 0 \le i, j \le n\}.$$

Given a configuration σ on E, an integer-valued function $h = h_\sigma$ on V' is called a *height function* of σ if for any two neighboring points $x, y \in V'$, $|x - y| = 1$, we have that

$$h(y) - h(x) = (-1)^s,$$

where $s = 0$ if the arrow σ_e on the edge $e \in E$, crossing the segment $[x, y]$, is oriented in such a way that it points from left to right with respect to the vector $\vec{xy}$, and

$s = 1$ if σ_e is oriented from right to left with respect to $\vec{xy}$. The ice-rule ensures that the height function $h = h_\sigma$ exists for any configuration σ. It is unique up to addition of a constant. An example of a configuration and its corresponding height function is given in Figure 1.3.

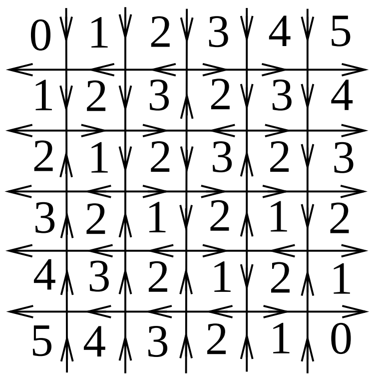

Fig. 1.3. A 5×5 configuration with a height function.

To derive conservation laws, consider the height function $h = h_\sigma$ on a diagonal sequence of points defined by the formula,

$$x_j = x_0 + (j, j), \quad 0 \leq j \leq k,$$

where both x_0 and x_k lie on the boundary B' of the dual lattice V',

$$\begin{aligned} B' = &\left\{ x = \left(i + \frac{1}{2}, \frac{1}{2} \right),\ 0 \leq i \leq n \right\} \cup \left\{ x = \left(m + \frac{1}{2}, j + \frac{1}{2} \right),\ 0 \leq j \leq n \right\} \\ &\cup \left\{ x = \left(i + \frac{1}{2}, n + \frac{1}{2} \right),\ 0 \leq i \leq n \right\} \cup \left\{ x = \left(\frac{1}{2}, j + \frac{1}{2} \right),\ 0 \leq j \leq n \right\}. \end{aligned}$$

Then it follows from the definition of the height function that

$$h(x_j) - h(x_{j-1}) = \begin{cases} 2 \quad \text{if} \quad t(x;\sigma) = 3 \\ -2 \quad \text{if} \quad t(x;\sigma) = 4 \\ 0 \quad \text{if} \quad t(x;\sigma) = 1, 2, 5, 6, \end{cases}$$

where

$$x = \frac{x_j + x_{j-1}}{2}.$$

Hence

$$0 = h(x_k) - h(x_0) = 2N_3(\sigma; L) - 2N_4(\sigma; L),$$

where $N_i(\sigma; L)$ is the number of vertex states of type i in σ on the line

$$L = \{x = x_0 + (t, t),\ t \in \mathbb{R}\}.$$

The line L is parallel to the diagonal $y = x$. By summing up over all possible lines L, we obtain that

$$N_3(\sigma) - N_4(\sigma) = 0,$$

where $N_i(\sigma)$ is the total number of vertex states of the type i in the configuration σ.

Similarly, by considering lines L parallel to the diagonal $y = -x$, we obtain that

$$N_1(\sigma) - N_2(\sigma) = 0.$$

Also,

$$N_5(\sigma) - N_6(\sigma) = n,$$

which follows if we consider lines L parallel to the x-axis.

The conservation laws allow us to reduce the weights $w_1, \dots, w_6$ to 3 parameters. Namely, we have that

$$w_1^{N_1} w_2^{N_2} w_3^{N_3} w_4^{N_4} w_5^{N_5} w_6^{N_6} = C(n) a^{N_1} a^{N_2} b^{N_3} b^{N_4} c^{N_5} c^{N_6},$$

where

$$a = \sqrt{w_1 w_2}, \quad b = \sqrt{w_3 w_4}, \quad c = \sqrt{w_5 w_6},$$

and the constant

$$C(n) = \left(\frac{w_5}{w_6}\right)^{\frac{n}{2}}.$$

This implies the relation between the partition functions,

$$Z_n(w_1, w_2, w_3, w_4, w_5, w_6) = C(n) Z_n(a, a, b, b, c, c),$$

and between the Gibbs measures,

$$\mu_n(\sigma; w_1, w_2, w_3, w_4, w_5, w_6) = \mu_n(\sigma; a, a, b, b, c, c).$$

Therefore, for fixed boundary conditions like DWBC, the general weights are reduced to the case when

$$w_1 = w_2 = a, \quad w_3 = w_4 = b, \quad w_5 = w_6 = c.$$

Furthermore,

$$Z_n(a, a, b, b, c, c) = c^{n^2} Z_n\left(\frac{a}{c}, \frac{a}{c}, \frac{b}{c}, \frac{b}{c}, 1, 1\right) \tag{1.1}$$

and

$$\mu_n(\sigma; a, a, b, b, c, c) = \mu_n\left(\sigma; \frac{a}{c}, \frac{a}{c}, \frac{b}{c}, \frac{b}{c}, 1, 1\right),$$

so that a general weight reduces to the two parameters, $\frac{a}{c}, \frac{b}{c}$.

We would like to remark that the conservation laws are obtained in the paper [23] of Ferrari and Spohn as a corollary of a path representation of the six-vertex model.

1.3 Exact solution of the six-vertex model for a finite n

Introduce the parameter

$$\Delta = \frac{a^2 + b^2 - c^2}{2ab}.$$

There are three physical phases in the six-vertex model: the ferroelectric phase, $\Delta > 1$; the antiferroelectric phase, $\Delta < -1$; and the disordered phase, $-1 < \Delta < 1$. Notice that $|a - b| > c$ in the ferroelectric phase region and $c > a + b$ in the antiferroelectric phase region, while in the disordered phase region a, b, c satisfy the triangle inequalities. In the three phases we parametrize the weights in the standard way: for the ferroelectric phase,

$$a = \sinh(t - \gamma), \quad b = \sinh(t + \gamma), \quad c = \sinh(2|\gamma|), \quad 0 < |\gamma| < t; \tag{1.2}$$

for the antiferroelectric phase,

$$a = \sinh(\gamma - t), \quad b = \sinh(\gamma + t), \quad c = \sinh(2\gamma), \quad |t| < \gamma; \tag{1.3}$$

and for the disordered phase

$$a = \sin(\gamma - t), \quad b = \sin(\gamma + t), \quad c = \sin(2\gamma), \quad |t| < \gamma. \tag{1.4}$$

The phase diagram of the six-vertex model is shown on Figure 1.4.

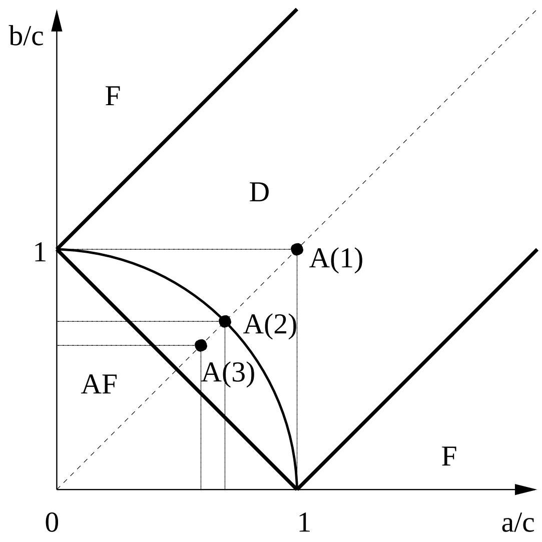

Fig. 1.4. The phase diagram of the model, where **F**, **AF** and **D** mark ferroelectric, antiferroelectric, and disordered phases, respectively. The circular arc corresponds to the so-called "free fermion" line, when $\Delta = 0$, and the three dots correspond to 1-, 2-, and 3-enumeration of alternating sign matrices.

The phase diagram and the Bethe Ansatz solution of the six-vertex model for periodic and anti-periodic boundary conditions are thoroughly discussed in the works of Lieb [34]- [37], Lieb, Wu [38], Sutherland [45], Baxter [2], Batchelor, Baxter, O'Rourke, Yung [3]. See also the work of Wu, Lin [49], in which the Pfaffian solution for the six-vertex model with periodic boundary conditions is obtained on the free fermion line, $\Delta = 0$.

In the paper [28], Korepin derived an important recursion relation for the partition function of the six-vertex model with DWBC. This led to a beautiful determinantal formula of Izergin [24] for the partition function, commonly called the Izergin-Korepin formula. A detailed proof of this formula and its generalizations are given in the paper of Izergin, Coker, and Korepin [25]. When the weights are parameterized according to (1.2)-(1.4), the Izergin-Korepin formula is

$$Z_n = \frac{(ab)^{n^2}}{\left(\prod_{j=0}^{n-1} j!\right)^2}\,\tau_n\,, \tag{1.5}$$

where τ_n is the Hankel determinant,

$$\tau_n = \det \left(\frac{d^{j+k-2}\phi}{dt^{j+k-2}} \right)_{1 \le j,k \le n}, \tag{1.6}$$

and

$$\phi(t) = \frac{c}{ab} = \begin{cases} \dfrac{\sinh(2|\gamma|)}{\sinh(t-\gamma)\sinh(t+\gamma)} & \text{in the ferroelectric phase} \\ \dfrac{\sinh(2\gamma)}{\sinh(\gamma-t)\sinh(\gamma+t)} & \text{in the antiferroelectric phase} \\ \dfrac{\sin(2\gamma)}{\sin(\gamma-t)\sin(\gamma+t)} & \text{in the disordered phase}\,. \end{cases} \tag{1.7}$$

An elegant derivation of the Izergin-Korepin formula from the Yang-Baxter equation is given in the papers of Korepin and Zinn-Justin [31] and of Kuperberg [33] (see also the book of Bressoud [12]).

One of the applications of the Izergin-Korepin formula is that it implies that the function τ_n solves the Toda equation

$$\tau_n \tau_n'' - {\tau_n'}^2 = \tau_{n+1}\tau_{n-1}, \qquad n \ge 1, \qquad (') = \frac{\partial}{\partial t}, \tag{1.8}$$

cf. the work of Sogo [44]. The Toda equation was used by Korepin and Zinn-Justin [31] to derive the free energy of the six-vertex model with DWBC, assuming some Ansatz on the behavior of subdominant terms in the large n asymptotics of the free energy.

Another application of the Izergin-Korepin formula is that τ_n can be expressed in terms of the eigenvalue partition function of a random matrix model and also in terms of related orthogonal polynomials, see the paper [51] of Zinn-Justin. This relation is based on the following lemma, which is well known in the random matrix community.

Lemma 1.3.1 *Let τ_n be the Hankel determinant $\tau_n = \det \left(\frac{d^{j+k-2}\phi}{dt^{j+k-2}} \right)_{1 \le j,k \le n}$, and suppose the function $\phi(t)$ is a Laplace-type transform of some measure μ on the real line, so that, for some constant $c \in \mathbb{R} \setminus \{0\}$,*

$$\phi(t) = \int_{\mathbb{R}} e^{-ctx} d\mu(x).$$

Suppose also that there exists a system of monic orthogonal polynomials $\{P_k(x)\}_{k=0}^{\infty}$ *on* $\mathbb{R}$ *with respect to the measure* $e^{-ctx}\mu(x)$, *so that*

$$\int_{\mathbb{R}} P_j(x)P_k(x)e^{-ctx}d\mu(x) = h_k\delta_{jk}$$

for some normalizing coefficients $\{h_k\}_{k=0}^{\infty}$. *Then*

$$\tau_n = c^{n^2-n}\prod_{k=0}^{n-1} h_k.$$

A proof of this lemma is given in Appendix A. It follows that evaluation of the partition function Z_n in the thermodynamic limit reduces to finding a large n asymptotic formula for the normalizing coefficients of a system of orthogonal polynomials. This can generally be done via a Riemann-Hilbert approach. This is the method employed in the paper [4] of Bleher and Fokin, in which they prove the conjecture of Paul Zinn-Justin [51] that the large n asymptotics of Z_n in the disordered phase has the following form: For some $\varepsilon > 0$,

$$Z_n = Cn^{\kappa}F^{n^2}[1 + O(n^{-\varepsilon})].$$

Furthermore, they find the exact value of the exponent κ,

$$\kappa = \frac{1}{12} - \frac{2\gamma^2}{3\pi(\pi - 2\gamma)}\,.$$

The value of F in the disordered phase is given by the formula,

$$F = \frac{\pi ab}{2\gamma \cos\frac{\pi t}{2\gamma}}\,, \qquad a = \sin(\gamma - t), \qquad b = \sin(\gamma + t)\,,$$

in parametrization (1.4).

In the present study, we obtain the large n asymptotics of Z_n in the remaining two phase regions, as well as on the critical line which separates the ferroelectric and disordered phases.

2. FERROELECTRIC PHASE

2.1 Introduction and formulation of the main results

Here we discuss the ferroelectric phase, and we will use parametrization (1.2) for the weights. Without loss of generality we may assume that

$$\gamma > 0,$$

which corresponds to the region

$$b > a + c.$$

The parameter Δ in the ferroelectric phase reduces to

$$\Delta = \cosh(2\gamma).$$

In this phase, the function $\phi(t)$ in the Hankel determinant τ_n is in fact the Laplace transform of a discrete measure lying on the positive integers. We have that

$$\begin{aligned} \phi(t) = \frac{\sinh(2\gamma)}{\sinh(t+\gamma)\sinh(t-\gamma)} &= 2\left[\frac{e^{-2t+2\gamma} - e^{-2t-2\gamma}}{(1-e^{-2t+2\gamma})(1-e^{-2t-2\gamma})}\right] \\ &= 2\left[\frac{1}{1-e^{-2t+2\gamma}} - \frac{1}{1-e^{-2t-2\gamma}}\right] \\ &= 4\sum_{l=1}^{\infty} e^{-2tl}\sinh(2\gamma l). \end{aligned} \tag{2.1}$$

Introduce now discrete monic polynomials $P_j(x) = x^j + \ldots$ orthogonal on the set $\mathbb{N} = \{l = 1, 2, \ldots\}$ with respect to the weight,

$$w(l) = 2e^{-2tl}\sinh(2\gamma l) = e^{-2tl+2\gamma l} - e^{-2tl-2\gamma l}, \tag{2.2}$$

so that

$$\sum_{l=1}^{\infty} P_j(l)P_k(l)w(l) = h_k\delta_{jk}. \tag{2.3}$$

Then it follows from (2.1), and Lemma 1.3.1 that

$$\tau_n = 2^{n^2} \prod_{k=0}^{n-1} h_k. \tag{2.4}$$

We will prove the following asymptotics of h_k.

Theorem 2.1.1 *For any $\varepsilon > 0$, as $k \to \infty$,*

$$h_k = \frac{(k!)^2 q^{k+1}}{(1-q)^{2k+1}} \left(1 + O(e^{-k^{1-\varepsilon}})\right), \tag{2.5}$$

where

$$q = e^{2\gamma - 2t}.$$

The error term in (2.5) is uniform on any compact subset of the set

$$\{(t, \gamma) : \ 0 < \gamma < t\}. \tag{2.6}$$

Our main result in this chapter is the following theorem.

Theorem 2.1.2 *In the ferroelectric phase with $t > \gamma > 0$, for any $\varepsilon > 0$, as $n \to \infty$,*

$$Z_n = CG^n F^{n^2} \left[1 + O\left(e^{-n^{1-\varepsilon}}\right)\right],$$

where $C = 1 - e^{-4\gamma}$, $G = e^{\gamma - t}$, and $F = \sinh(t + \gamma)$. The error term in (2.5) is uniform on any compact subset of the set (2.6).

Up to the constant factor, this result follows directly from Theorem 2.1.1. To find the constant factor C we will use the Toda equation, combined with the asymptotics of C as $t \to \infty$. The proof of Theorems 2.1.1 and 2.1.2 are given below in Sections 2.2-2.5. To prove Theorem 2.1.1, we compare the orthogonal polynomials (2.3) with the well known Meixner polynomials, which are orthogonal with respect to an exponential weight on the non-negative integers.

2.2 Meixner polynomials

We will use two weights: the weight $w(l)$ defined in (2.2) and the exponential weight on $\mathbb{N}$,

$$w^{\mathrm{Q}}(l) = q^l, \quad l \in \mathbb{N}; \qquad q = e^{2\gamma - 2t} < 1, \tag{2.7}$$

which can be viewed as an approximation to $w(l)$ for large l. The orthogonal polynomials with the weight $w^{\mathrm{Q}}(l)$ are expressed in terms of the Meixner polynomials with $\beta = 1$, which are defined by the formula,

$$M_k(z;q) = \sum_{j=0}^{k} \frac{(1-q^{-1})^j \prod_{i=0}^{j-1}(k-i) \prod_{i=0}^{j-1}(z-i)}{(j!)^2}.$$

They satisfy the orthogonality condition,

$$\sum_{l=0}^{\infty} M_j(l;q) M_k(l;q) q^l = \frac{q^{-k}\delta_{jk}}{1-q},$$

see, e.g. [30]. For the corresponding monic polynomials,

$$P_k^{\mathrm{M}}(z) = \frac{k!}{(1-q^{-1})^k} M_k(z;q)$$

(M in P_k^{M} stands for Meixner), the orthogonality condition reads

$$\sum_{l=0}^{\infty} P_j^{\mathrm{M}}(l) P_k^{\mathrm{M}}(l) q^j = h_k^{\mathrm{M}} \delta_{jk}, \qquad h_k^{\mathrm{M}} = \frac{(k!)^2 q^k}{(1-q)^{2k+1}}. \tag{2.8}$$

They satisfy the three term recurrence relation,

$$zP_k^{\mathrm{M}}(z) = P_{k+1}^{\mathrm{M}}(z) + \frac{kq+k+q}{1-q} P_k^{\mathrm{M}}(z) + \frac{k^2 q}{(1-q)^2} P_{k-1}^{\mathrm{M}}(z),$$

see [30]. According to (2.7), we take $q = e^{2\gamma - 2t}$.

For our purposes it is convenient to introduce a shifted Meixner polynomial,

$$Q_k(z) = P_k^{\mathrm{M}}(z-1) = \frac{(-1)^k k! q^k}{(1-q)^k} M_k(z-1;q), \tag{2.9}$$

which is a monic polynomial as well. Equation (2.8) implies the orthogonality condition,

$$\sum_{l=1}^{\infty} Q_j(l) Q_k(l) q^l = h_k^{\mathrm{Q}} \delta_{jk}, \qquad h_k^{\mathrm{Q}} = \frac{(k!)^2 q^{k+1}}{(1-q)^{2k+1}}. \tag{2.10}$$

By analogy with (2.4), define

$$\tau_n^{\mathrm{Q}} = 2^{n^2} \prod_{k=0}^{n-1} h_k^{\mathrm{Q}}.$$

From (2.8) and (2.10) we obtain that

$$\tau_n^{\mathrm{Q}} = 2^{n^2} \prod_{k=0}^{n-1} \frac{(k!)^2 q^{k+1}}{(1-q)^{2k+1}} = \frac{2^{n^2} q^{(n+1)n/2}}{(1-q)^{n^2}} \prod_{k=0}^{n-1} (k!)^2 \,. \tag{2.11}$$

By analogy with (1.5), define also

$$Z_n^{\mathrm{Q}} = \frac{[\sinh(\gamma+t)\sinh(\gamma-t)]^{n^2}}{\prod_{k=0}^{n-1}(k!)^2}\, \tau_n^{\mathrm{Q}} \,.$$

Then from (2.11) we obtain that

$$Z_n^{\mathrm{Q}} = F^{n^2} G^n,$$

where

$$F = \frac{2\sinh(t-\gamma)\sinh(t+\gamma) q^{1/2}}{1-q} = \frac{2\sinh(t-\gamma)\sinh(t+\gamma) e^{\gamma-t}}{1-e^{2\gamma-2t}} = \sinh(t+\gamma),$$

and

$$G = q^{1/2} = e^{\gamma-t}.$$

Our goal is to compare the normalizing constants for orthogonal polynomials with the weights w and w^{Q}. We begin with the following formula for their difference:

$$h_k - h_k^{\mathrm{Q}} = -\sum_{l=1}^{\infty} P_k(l) Q_k(l)\,[w^{\mathrm{Q}}(l) - w(l)], \tag{2.12}$$

which can be derived as follows. Since P_k and Q_k are monic polynomials, the difference, $P_k - Q_k$, is a polynomial of degree less than k, hence

$$\sum_{l=1}^{\infty} P_k(l)[Q_k(l) - P_k(l)] w(l) = 0.$$

By adding this to equation (2.3) with $j = k$, we obtain that

$$h_k = \sum_{l=1}^{\infty} P_k(l) Q_k(l) w(l).$$

Similarly, from (2.10) we obtain that

$$h_k^{\mathrm{Q}} = \sum_{l=1}^{\infty} P_k(l) Q_k(l) w^{\mathrm{Q}}(l).$$

By subtracting the last two equations, we obtain identity (2.12).

2.3 Evaluation of the ratio h_k/h_k^{Q}

In this section we will prove Theorem 2.1.1. By applying the Cauchy-Schwarz inequality to identity (2.12), we obtain that

$$|h_k - h_k^{\mathrm{Q}}| \leq \left[\sum_{l=1}^{\infty} P_k(l)^2 \left| w(l) - w^{\mathrm{Q}}(l) \right| \right]^{1/2} \left[\sum_{l=1}^{\infty} Q_k(l)^2 \left| w(l) - w^{\mathrm{Q}}(l) \right| \right]^{1/2},$$

so that

$$\left| \frac{h_k}{h_k^{\mathrm{Q}}} - 1 \right| \leq \left[\frac{1}{h_k^{\mathrm{Q}}} \sum_{l=1}^{\infty} P_k(l)^2 \left| w(l) - w^{\mathrm{Q}}(l) \right| \right]^{1/2} \left[\frac{1}{h_k^{\mathrm{Q}}} \sum_{l=1}^{\infty} Q_k(l)^2 \left| w(l) - w^{\mathrm{Q}}(l) \right| \right]^{1/2}, \tag{2.13}$$

From (2.2),

$$|w(l) - w^{\mathrm{Q}}(l)| = e^{-(2t+2\gamma)l} \leq C_0 w(l), \quad l \geq 1; \qquad C_0 = \frac{1}{e^{4\gamma} - 1}, \tag{2.14}$$

hence

$$\frac{1}{h_k^{\mathrm{Q}}} \sum_{l=1}^{\infty} P_k(l)^2 \left| w(l) - w^{\mathrm{Q}}(l) \right| \leq C_0 \frac{1}{h_k^{\mathrm{Q}}} \sum_{l=1}^{\infty} P_k(l)^2 w(l) = \frac{C_0 h_k}{h_k^{\mathrm{Q}}} \leq C_0 (1 + \varepsilon_k),$$

where

$$\varepsilon_k = \left| \frac{h_k}{h_k^{\mathrm{Q}}} - 1 \right|.$$

Thus, by (2.13),

$$\varepsilon_k^2 \leq C_0 (1 + \varepsilon_k) \delta_k, \tag{2.15}$$

where

$$\delta_k = \frac{1}{h_k^{\mathrm{Q}}} \sum_{l=1}^{\infty} Q_k(l)^2 \left| w(l) - w^{\mathrm{Q}}(l) \right|.$$

By (2.14),

$$\delta_k = \frac{1}{h_k^{\mathrm{Q}}} \sum_{l=1}^{\infty} Q_k(l)^2 \, q_0^l, \qquad q_0 = e^{-2(t+\gamma)}. \tag{2.16}$$

Let us evaluate δ_k.

We partition the sum in (2.16) into two parts:

$$\delta'_k = \frac{1}{h_k^{\mathrm{Q}}} \sum_{l=1}^{L} Q_k(l)^2 \, q_0^l, \tag{2.17}$$

and

$$\delta''_k = \frac{1}{h_k^{\mathrm{Q}}} \sum_{l=L+1}^{\infty} Q_k(l)^2 \, q_0^l, \tag{2.18}$$

where

$$L = [k^\lambda], \qquad 0 < \lambda < 1. \tag{2.19}$$

We first estimate δ'_k. We have from (2.9), (2.10) that

$$\frac{Q_k(l)}{(h_k^{\mathrm{Q}})^{1/2}} = \frac{(-1)^k (1-q)^{1/2} q^{k/2}}{q^{1/2}} M_k(l-1;q). \tag{2.20}$$

By (2.2),

$$\begin{aligned} M_k(l-1;q) = 1 + (1-q^{-1}) k(l-1) + (1-q^{-1})^2 \frac{k(k-1)(l-1)(l-2)}{(2!)^2} \\ + (1-q^{-1})^3 \frac{k(k-1)(k-2)(l-1)(l-2)(l-3)}{(3!)^2} + \dots \end{aligned} \tag{2.21}$$

If $l < k$, then the latter sum consists of l nonzero terms. For $l \le L$ it is estimated as

$$M_k(l-1;q) = O(k^L L^{L+1}) = O(e^{L \log k + (L+1) \log L}),$$

hence

$$\frac{Q_k(l)}{(h_k^{\mathrm{Q}})^{1/2}} = O(e^{\frac{k \log q}{2} + L \log k + (L+1) \log L}).$$

Due to our choice of L in (2.19), this implies the estimate,

$$\frac{Q_k(l)}{(h_k^{\mathrm{Q}})^{1/2}} = O(e^{\frac{k \log q}{2} + 2k^\lambda \log k}).$$

Since $0 < q < 1$ and $0 < \lambda < 1$, the expression on the right is exponentially small as $k \to \infty$. From (2.17) we obtain now that

$$\delta'_k = O(e^{k \log q + 4k^\lambda \log k}).$$

Since $\lambda < 1$ and $q < 1$, we obtain that

$$\delta_k' = O(e^{-c_0 k}), \qquad c_0 = -\frac{\log q}{2} > 0. \tag{2.22}$$

Now let us estimate δ_k''. By (2.10),

$$\frac{1}{h_k^{\mathrm{Q}}} \sum_{l=1}^{\infty} Q_k(l)^2 q^l = 1,$$

hence

$$\delta_k'' = \frac{1}{h_k^{\mathrm{Q}}} \sum_{l=L+1}^{\infty} Q_k(l)^2\, q_0^l < \left(\frac{q_0}{q}\right)^L \frac{1}{h_k^{\mathrm{Q}}} \sum_{l=L+1}^{\infty} Q_k(l)^2\, q^l < \left(\frac{q_0}{q}\right)^L = e^{-4\gamma L}. \tag{2.23}$$

Thus,

$$\delta_k'' < e^{-4\gamma(k^\lambda - 1)}. \tag{2.24}$$

Since $0 < \lambda < 1$ is an arbitrary number, we obtain from (2.22) and (2.24) that for any $\eta > 0$,

$$\delta_k = O\left(e^{-k^{1-\eta}}\right). \tag{2.25}$$

Let us return back to inequality (2.15). Consider two cases: (1) $\varepsilon_k > 1$ and (2) $\varepsilon_k \le 1$. In the first case (2.15) implies that

$$\varepsilon_k \le 2C_0\delta_k,$$

which is impossible, because of (2.25). Hence, for large k, $\varepsilon_k \le 1$, in which case (2.15) gives that

$$\varepsilon_k^2 \le 2C_0\delta_k.$$

Estimate (2.25) implies now that for any $\eta > 0$,

$$\varepsilon_k = O\left(e^{-k^{1-\eta}}\right),$$

so that as $k \to \infty$,

$$h_k = h_k^{\mathrm{Q}}(1 + \tilde{\varepsilon}_k), \qquad |\tilde{\varepsilon}_k| = \varepsilon_k = O\left(e^{-k^{1-\eta}}\right). \tag{2.26}$$

This proves Theorem 2.1.1.

From (2.26) we obtain that for any $\eta > 0$,

$$Z_n = Z_n^{\mathrm{Q}} \prod_{k=0}^{n} (1+\tilde{\varepsilon}_k) = C Z_n^{\mathrm{Q}} \left[1 + O\left(e^{-n^{1-\eta}}\right)\right],$$

where

$$C = \prod_{k=0}^{\infty} (1+\tilde{\varepsilon}_k) > 0. \tag{2.27}$$

Thus, we have proved the following result.

Proposition 2.3.1 *For any $\varepsilon > 0$, as $n \to \infty$,*

$$Z_n = C F^{n^2} G^n \left[1 + O\left(e^{-n^{1-\varepsilon}}\right)\right], \tag{2.28}$$

where $C > 0$, $F = \sinh(t+\gamma)$, and $G = e^{\gamma - t}$.

To finish the proof of Theorem 2.1.2, it remains to find the constant C.

2.4 Evaluation of the constant factor

In the next two sections we will find the exact value of the constant C in formula (2.28). This will be done in two steps: first, with the help of the Toda equation, we will find the form of the dependence of C on t, and second, we will find the large t asymptotics of C. By combining these two steps, we will obtain the exact value of C. In this section we carry out the first step of our program.

By dividing the Toda equation, (1.8), by τ_n^2, we obtain that

$$\frac{\tau_n \tau_n'' - \tau_n'^2}{\tau_n^2} = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2}, \qquad (') = \frac{\partial}{\partial t}. \tag{2.29}$$

The left hand side can be written as

$$\frac{\tau_n \tau_n'' - \tau_n'^2}{\tau_n^2} = \left(\frac{\tau_n'}{\tau_n}\right)' = (\log \tau_n)''.$$

From (2.4) we obtain that

$$\frac{\tau_{n+1}}{\tau_n} = 2^{2n+1} h_n,$$

hence equation (2.29) implies that

$$(\log \tau_n)'' = \frac{4h_n}{h_{n-1}}\,. \tag{2.30}$$

From (2.5) we obtain that

$$\frac{4h_n}{h_{n-1}} = \frac{4n^2 q}{(1-q)^2} + O\left(e^{-n^{1-\varepsilon}}\right). \tag{2.31}$$

We have that

$$\frac{4q}{(1-q)^2} = \frac{4e^{2\gamma-2t}}{(1-e^{2\gamma-2t})^2} = \left[\frac{(-2)}{1-e^{2\gamma-2t}}\right]' = \left[-\log(1-e^{2\gamma-2t})\right]'',$$

hence from (2.30), (2.31) we obtain that

$$(\log \tau_n)'' = n^2\left[-\log(1-e^{2\gamma-2t})\right]'' + O\left(e^{-n^{1-\varepsilon}}\right).$$

By (1.5) this implies that

$$(\log Z_n)'' = n^2\left[\log \frac{\sinh(t-\gamma)\sinh(t+\gamma)}{1-e^{2\gamma-2t}}\right]'' + O\left(e^{-n^{1-\varepsilon}}\right). \tag{2.32}$$

Since

$$\log\left[\frac{\sinh(t-\gamma)\sinh(t+\gamma)}{1-e^{2\gamma-2t}}\right] = \log[\sinh(t+\gamma)] + (t-\gamma) - \log 2,$$

we can simplify (2.32) to

$$(\log Z_n)'' = n^2\left[\log \sinh(t+\gamma)\right]'' + O\left(e^{-n^{1-\varepsilon}}\right).$$

Observe that the error term in the last formula is uniform when t belongs to a compact set on (γ,∞), hence by integrating it we obtain that

$$\log Z_n = n^2 \log\sinh(t+\gamma) + c_1 t + c_0 + O\left(e^{-n^{1-\varepsilon}}\right),$$

where c_0, c_1 do not depend on t. In general, c_0, c_1 depend on γ and n. By substituting formula (2.28) into the preceding equation, we obtain that

$$\log C + n(\gamma - t) = c_1 t + c_0 + O\left(e^{-n^{1-\varepsilon}}\right). \tag{2.33}$$

Denote

$$d_0 = c_0 - n\gamma, \qquad d_1 = c_1 + n.$$

Then equation (2.33) reads

$$\log C = d_1 t + d_0 + O\left(e^{-n^{1-\varepsilon}}\right). \tag{2.34}$$

Observe that $C = C(\gamma, t)$ does not depend on n, while $d_{0,1} = d_{0,1}(\gamma, n)$ does not depend on t. Take any $0 < \gamma < t_1 < t_2$. Then

$$\log C(\gamma, t_2) - \log C(\gamma, t_1) = d_1(t_2 - t_1) + O\left(e^{-n^{1-\varepsilon}}\right).$$

From this formula we obtain that the limit,

$$\lim_{n\to\infty} d_1(\gamma, n) = d_1(\gamma),$$

exists. This in turn implies that the limit,

$$\lim_{n\to\infty} d_0(\gamma, n) = d_0(\gamma),$$

exists. By taking the limit $n \to \infty$ in (2.34), we obtain that

$$\log C = d_1(\gamma)t + d_0(\gamma).$$

Thus we have proved the following result.

Proposition 2.4.1 *The constant C in asymptotic formula (2.28) has the form*

$$C = e^{d_1(\gamma)t + d_0(\gamma)}.$$

2.5 Explicit formula for C

In this section we will find the exact value of C, and by doing this we will finish the proof of Theorem 2.1.2. Let us consider the following regime:

$$\gamma \text{ is fixed}, \quad t \to \infty,$$

and let us evaluate the asymptotics of C in this regime. By (2.2) we have that

$$h_0 = \sum_{l=1}^{\infty} w(l) = \sum_{l=1}^{\infty} \left(e^{-2tl+2\gamma l} - e^{-2tl-2\gamma l}\right) = \frac{e^{-2t+2\gamma}}{1 - e^{-2t+2\gamma}} - \frac{e^{-2t-2\gamma}}{1 - e^{-2t-2\gamma}}.$$

Similarly, by (2.10),

$$h_0^{\mathrm{Q}} = \frac{e^{-2t+2\gamma}}{1-e^{-2t+2\gamma}}\,,$$

hence

$$\frac{h_0}{h_0^{\mathrm{Q}}} = 1 - e^{-4\gamma} + O(e^{-2t}), \qquad t \to \infty. \tag{2.35}$$

Let us evaluate $\varepsilon_k = \left|\frac{h_k}{h_k^{\mathrm{Q}}} - 1\right|$ for $k \ge 1$.

By (2.15),

$$\varepsilon_k^2 \le C_0(1+\varepsilon_k)\delta_k, \qquad C_0 = \frac{1}{e^{4\gamma}-1}\,. \tag{2.36}$$

In the partition of δ_k as $\delta_k' + \delta_k''$ in (2.17), (2.18), let us choose

$$L = [k^{2/3} + t^{2/3}]. \tag{2.37}$$

From (2.20), (2.21) we obtain that for $l \le L$,

$$\frac{|Q_k(l)|}{(h_k^{\mathrm{Q}})^{1/2}} \le q^{(k-1)/2} k^L L^{L+1}, \qquad q = e^{2\gamma-2t},$$

hence

$$\delta_k' \le \frac{q_0 q^{k-1} k^L L^{L+1}}{1-q_0} \le \frac{q^k k^L L^{L+1}}{1-q_0}\,, \qquad q_0 = e^{-2\gamma-2t}.$$

In addition, by (2.23),

$$\delta_k'' \le e^{-4\gamma L}.$$

Our choice of L in (2.37) ensures that there exists $t_0 > 0$ such that for any $t \ge t_0$ and any $k \ge 1$,

$$\delta_k = \delta_k' + \delta_k'' \le e^{-k^{1/2}-t^{1/2}}.$$

From (2.36) we obtain now that for $k \ge 1$ and large t,

$$\varepsilon_k \le C_1 e^{-\frac{k^{1/2}}{2}-\frac{t^{1/2}}{2}}, \qquad C_1 = (2C_0)^{1/2}. \tag{2.38}$$

By (2.27),

$$\log C = \sum_{k=0}^{\infty} \log(1+\tilde\varepsilon_k), \qquad |\tilde\varepsilon_k| = \varepsilon_k.$$

From equations (2.35) and (2.38) we obtain now that

$$\log C = \log(1-e^{-4\gamma}) + O(e^{-\frac{t^{1/2}}{2}}), \qquad t \to \infty.$$

On the other hand, by (2.34)

$$\log C = d_1(\gamma)t + d_0(\gamma)$$

This implies that

$$d_1(\gamma) = 0, \qquad d_0(\gamma) = \log(1 - e^{-4\gamma}),$$

so that

$$C = 1 - e^{-4\gamma}. \tag{2.39}$$

By substituting expression (2.39) into formula (2.28), we prove Theorem 2.1.2.

As a final note in this chapter, we compare the asymptotics of the free energy in the ferroelectric phase with the energy of the ground state.

2.6 Ground state configuration

The ground state in the ferroelectric phase region is the configuration

$$\sigma^{\mathrm{gs}}(x) = \begin{cases} \sigma_5 & \text{if } x \text{ is on the diagonal} \\ \sigma_3 & \text{if } x \text{ is above the diagonal} \\ \sigma_4 & \text{if } x \text{ is below the diagonal,} \end{cases}$$

see Figure 2.1. The weight of the ground state configuration is

$$w(\sigma^{\mathrm{gs}}) = b^{n^2}\left(\frac{c}{b}\right)^n = F^{n^2}G_0^n,$$

where

$$F = \sinh(t+\gamma), \qquad G_0 = \frac{\sinh(2\gamma)}{\sinh(t+\gamma)}.$$

The ratio $Z_n/w(\sigma^{\mathrm{gs}})$ is evaluated as

$$\frac{Z_n}{w(\sigma^{\mathrm{gs}})} = G_1^n,$$

where

$$G_1 = \frac{G}{G_0} = \frac{e^{\gamma-t}\sinh(t+\gamma)}{\sinh 2\gamma} = \frac{e^{2\gamma} - e^{-2t}}{e^{2\gamma} - e^{-2\gamma}} > 1.$$

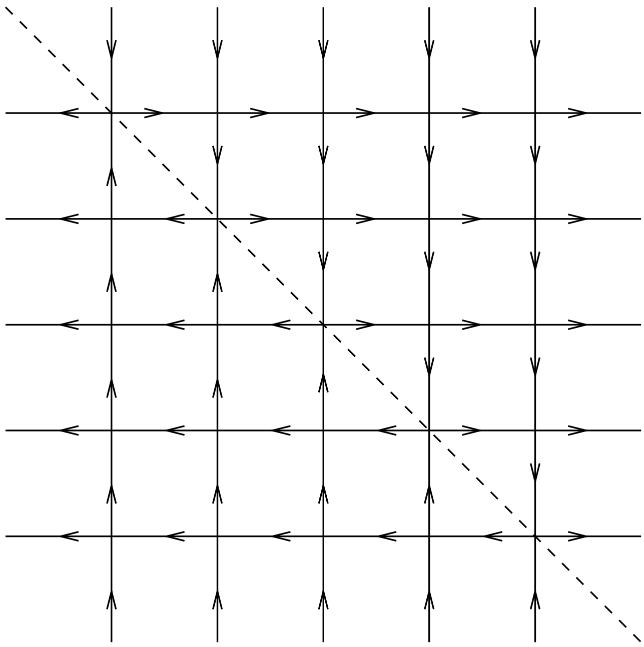

Fig. 2.1. A ground state configuration in the ferroelectric phase region.

Observe that

$$\lim_{n\to\infty} \frac{\log Z_n}{n^2} = \lim_{n\to\infty} \frac{\log w(\sigma^{\mathrm{gs}})}{n^2} = \log F,$$

so that the free energy is determined by the free energy of the ground state configuration. This can be explained by the fact that low energy excited states are local perturbations of the ground state around the diagonal. Namely, it is impossible to create a new configuration by perturbing the ground state locally away of the diagonal: the conservation law $N_3(\sigma) = N_4(\sigma)$ forbids such a configuration. Therefore, typical configurations of the six-vertex model in the ferroelectric phase are frozen outside of a relatively small neighborhood of the diagonal.

This behavior of typical configurations in the ferroelectric phase is in a big contrast with the situation in the disordered and antiferroelectric phases. Extensive rigorous, theoretical and numerical studies, see, e.g., the works of Cohn, Elkies, Propp [13], Eloranta [21], Syljuasen, Zvonarev [46], Allison, Reshetikhin [1], Kenyon, Okounkov [27], Kenyon, Okounkov, Sheffield [29], Sheffield [42], Ferrari, Spohn [23], Colomo, Pronko [14], Zinn-Justin [52], and references therein, show that in the disordered and antiferroelectric phases the "arctic circle" phenomenon persists, so that there are macroscopically big frozen and random domains in typical configurations, separated in the limit $n \to \infty$ by an "arctic curve".

3. CRITICAL LINE BETWEEN FERROELECTRIC AND DISORDERED PHASES

3.1 Introduction and formulation of the main result

In this chapter, we consider the partition function Z_n on the critical line

$$\frac{b}{c} - \frac{a}{c} = 1.$$

We fix a point,

$$\frac{a}{c} = \frac{\alpha - 1}{2}, \quad \frac{b}{c} = \frac{\alpha + 1}{2}; \qquad \alpha > 1,$$

on this line, and we are interested in the large n asymptotics of the partition function

$$Z_n = Z_n\left(\frac{\alpha-1}{2}, \frac{\alpha-1}{2}, \frac{\alpha+1}{2}, \frac{\alpha+1}{2}, 1, 1\right).$$

Let us first derive a formula for Z_n on the critical line. To that end, consider the limit of the Izergin-Korepin formula in the ferroelectric phase, (1.5)−(1.7), as

$$t, \gamma \to +0, \qquad \frac{t}{\gamma} \to \alpha. \tag{3.1}$$

Observe that in this limit,

$$\frac{a}{c} = \frac{\sinh(t-\gamma)}{\sinh(2\gamma)} \to \frac{\alpha-1}{2}, \qquad \frac{b}{c} = \frac{\sinh(t+\gamma)}{\sinh(2\gamma)} \to \frac{\alpha+1}{2}.$$

By (1.1),

$$Z_n\left(\frac{a}{c}, \frac{a}{c}, \frac{b}{c}, \frac{b}{c}, 1, 1\right) = \frac{Z_n(a,a,b,b,c,c)}{c^{n^2}},$$

hence by (1.5), and (2.4),

$$Z_n\left(\frac{a}{c}, \frac{a}{c}, \frac{b}{c}, \frac{b}{c}, 1, 1\right) = \left[\frac{2\sinh(t-\gamma)\sinh(t+\gamma)}{\sinh(2\gamma)}\right]^{n^2} \prod_{k=0}^{n-1} \frac{h_k}{(k!)^2}, \tag{3.2}$$

where h_k are the normalizing coefficients associated with the orthogonal polynomials $\{P_k(l)\}_{k=0}^{\infty}$ defined in (2.3), so that

$$\sum_{l=1}^{\infty} P_j(l)P_k(l)w(l) = h_k\delta_{jk}\,; \quad w(l) = e^{-2tl+2\gamma l} - e^{-2tl-2\gamma l}. \tag{3.3}$$

To deal with limit (3.1) we need to rescale the orthogonal polynomials $P_k(l)$. Introduce the rescaled variable,

$$x = 2tl - 2\gamma l,$$

and the rescaled limiting weight,

$$w_\alpha(x) = \lim_{t,\gamma\to+0,\ \frac{t}{\gamma}\to\alpha} (e^{-2tl+2\gamma l} - e^{-2tl-2\gamma l}) = e^{-x} - e^{-rx}, \qquad r = \frac{\alpha+1}{\alpha-1} > 1\,.$$

Consider monic orthogonal polynomials $P_j(x;\alpha)$ satisfying the orthogonality condition,

$$\int_0^\infty P_j(x;\alpha)P_k(x;\alpha)w_\alpha(x)dx = h_{k,\alpha}\delta_{jk}\,. \tag{3.4}$$

To find a relation between $P_k(l)$ and $P_k(x;\alpha)$, introduce the monic polynomials

$$\tilde{P}_k(x) = \Delta^k P_k(x/\Delta), \tag{3.5}$$

where

$$\Delta = 2t - 2\gamma,$$

and rewrite orthogonality condition (3.3) in the form

$$\sum_{l=1}^{\infty} \tilde{P}_j(l\Delta)\tilde{P}_k(l\Delta)w_\alpha(l\Delta)\Delta = \Delta^{2k+1}h_k\delta_{jk},$$

which is a Riemann sum for the integral in orthogonality condition (3.4). Therefore,

$$\lim_{t,\gamma\to+0,\ \frac{t}{\gamma}\to\alpha} \tilde{P}_k(x) = P_k(x;\alpha),$$

and

$$\lim_{t,\gamma\to+0,\ \frac{t}{\gamma}\to\alpha} \Delta^{2k+1}h_k = h_{k,\alpha}.$$

Let us rewrite formula (3.2) as

$$Z_n\left(\frac{a}{c},\frac{a}{c},\frac{b}{c},\frac{b}{c},1,1\right)=\left[\frac{2\sinh(t-\gamma)\sinh(t+\gamma)}{\sinh(2\gamma)\Delta}\right]^{n^2}\prod_{k=0}^{n-1}\frac{\Delta^{2k+1}h_k}{(k!)^2},$$

and take limit (3.1). In the limit we obtain that

$$Z_n=Z_n\left(\frac{\alpha-1}{2},\frac{\alpha-1}{2},\frac{\alpha+1}{2},\frac{\alpha+1}{2},1,1\right)=\left(\frac{\alpha+1}{2}\right)^{n^2}\prod_{k=0}^{n-1}\frac{h_{k,\alpha}}{(k!)^2}. \tag{3.6}$$

Our main technical result in this chapter will be the proof of the following asymptotics of $h_{k,\alpha}$. Let, as usual,

$$\zeta(s)=1+\frac{1}{2^s}+\frac{1}{3^s}+\dots,\qquad \operatorname{Re}s>1.$$

Theorem 3.1.1 *As $k\to\infty$,*

$$\log\left[\frac{h_{k,\alpha}}{(k!)^2}\right]=-\frac{\zeta(\frac{3}{2})}{2\sqrt{\pi(r-1)}k^{1/2}}+\frac{1}{4k}+O(k^{-3/2}),\qquad r=\frac{\alpha+1}{\alpha-1}. \tag{3.7}$$

The main result of this chapter is the following asymptotics of Z_n on the critical line between these two phases.

Theorem 3.1.2 *As $n\to\infty$,*

$$Z_n\left(\frac{\alpha-1}{2},\frac{\alpha-1}{2},\frac{\alpha+1}{2},\frac{\alpha+1}{2},1,1\right)=Cn^{\kappa}G^{\sqrt{n}}F^{n^2}[1+O(n^{-1/2})],$$

where $C>0$,

$$\kappa=\frac{1}{4},\qquad G=\exp\left[-\zeta\left(\frac{3}{2}\right)\sqrt{\frac{\alpha-1}{2\pi}}\right],$$

and

$$F=\frac{\alpha+1}{2}.$$

The proof of Theorem 3.1.2 follows easily from Theorem 3.1.1. Namely, from formula (3.6) and asymptotics (3.7) we obtain that

$$\log\left[\frac{Z_n\left(\frac{\alpha-1}{2},\frac{\alpha-1}{2},\frac{\alpha+1}{2},\frac{\alpha+1}{2},1,1\right)}{\left(\frac{\alpha+1}{2}\right)^{n^2}}\right]=\sum_{k=0}^{n-1}\log\left[\frac{h_{k,\alpha}}{(k!)^2}\right]$$
$$=\sum_{k=0}^{n-1}\left[-\frac{\zeta(\frac{3}{2})}{2\sqrt{\pi(r-1)}\,k^{1/2}}+\frac{1}{4k}+O(k^{-3/2})\right]$$
$$=-\zeta\left(\frac{3}{2}\right)\sqrt{\frac{(\alpha-1)}{2\pi}}\,n^{1/2}+\frac{\log n}{4}+C_0+O(n^{-1/2}),$$

which implies Theorem 3.1.2.

3.2 Large k asymptotics of $h_{k,\alpha}$

We will use asymptotic formulae for orthogonal polynomials on $[0,\infty)$, obtained in the paper [48] of Vanlessen. To formulate and to apply the Vanlessen's asymptotic formula we will need to introduce some notations and to evaluate some parameters. Let us write

$$w_\alpha(z) = e^{-z} - e^{-rz} = ze^{-Q(z)},$$

so that

$$Q(z) = z + \log \frac{z}{1 - e^{-(r-1)z}}\,, \tag{3.8}$$

where for the logarithm we take the principal branch with a cut on $(-\infty, 0]$. Observe that the function $Q(z)$ is analytic in a strip $|\mathrm{Im}\, z| \le c_0$, $c_0 > 0$. Define the Mashkar-Rakhmanov-Saff (MRS) numbers $\beta_k = \beta_k(\alpha)$ as a solution to the equation

$$\frac{1}{2\pi}\int_0^{\beta_k} Q'(x)\sqrt{\frac{x}{\beta_k - x}}\,dx = k. \tag{3.9}$$

As shown in [48], for large k there is a unique solution to this equation. The purpose of the MRS numbers is to rescale the equilibrium measure, introduced in Section 3.4, so that it is supported by the interval $[0,1]$.

3.3 Evaluation of β_k

By the change of variable, $x = \beta_k u$, equation (3.9) reduces to

$$\frac{\beta_k}{2\pi}\int_0^1 Q'(\beta_k u)\sqrt{\frac{u}{1-u}}\,du = k. \tag{3.10}$$

Set

$$b_k = \frac{\beta_k}{4k}\,. \tag{3.11}$$

Then equation (3.10) reduces to

$$\frac{2b_k}{\pi}\int_0^1 Q'(4b_k k u)\sqrt{\frac{u}{1-u}}\,du = 1. \tag{3.12}$$

From (3.8),

$$Q'(z) = 1 + \frac{1}{z} - \frac{r-1}{e^{(r-1)z} - 1}\,.$$

Observe that the function $Q'(z)$ has poles at the points

$$z = \frac{2m\pi i}{r-1}, \quad m = \pm 1, \pm 2, \dots$$

After evaluating integrals of the first two terms of $Q'(4b_k ku)$, equation (3.12) reads

$$b_k + \frac{1}{2k} - \frac{2}{\pi}\int_0^1 \frac{(r-1)b_k}{e^{4(r-1)b_k ku}-1}\sqrt{\frac{u}{1-u}}\,du = 1.$$

By the change of variable $x = ku$, it reduces to

$$b_k + \frac{1}{2k} - \frac{2}{\pi k^{3/2}}\int_0^k \frac{(r-1)b_k}{e^{4(r-1)b_k x}-1}\sqrt{\frac{x}{1-(x/k)}}\,dx = 1.$$

Set

$$\varepsilon = \frac{1}{k^{1/2}},$$

and consider the function,

$$f(b,\varepsilon) = b + \frac{\varepsilon^2}{2} - \frac{2\varepsilon^3}{\pi}\int_0^{1/\varepsilon^2} \frac{(r-1)b}{e^{4(r-1)bx}-1}\sqrt{\frac{x}{1-\varepsilon^2 x}}\,dx - 1, \qquad \frac{1}{2} \le b \le 2.$$

Observe that as $\varepsilon \to 0$,

$$\begin{aligned}\int_0^{1/\varepsilon^2} \frac{(r-1)b}{e^{4(r-1)bx}-1}\sqrt{\frac{x}{1-\varepsilon^2 x}}\,dx &= \int_0^\infty \frac{(r-1)b\sqrt{x}\,dx}{e^{4(r-1)bx}-1} + O(\varepsilon^2) \\ &= \frac{\sqrt{\pi}\,\zeta(\frac{3}{2})}{16\sqrt{b(r-1)}} + O(\varepsilon^2),\end{aligned} \tag{3.13}$$

hence

$$f(b,\varepsilon) = (b-1) + \frac{\varepsilon^2}{2} - \frac{\varepsilon^3\zeta(\frac{3}{2})}{8\sqrt{\pi b(r-1)}} + O(\varepsilon^5). \tag{3.14}$$

It is easy to see that equation (3.13) can be differentiated in b infinitely many times, and hence the function $f(b,\varepsilon)$ is C^∞ in a neighborhood of the point $b = 1,\ \varepsilon = 0$. In addition,

$$f(1,0) = 0, \qquad \frac{\partial f(1,0)}{\partial b} = 1.$$

By the implicit function theorem, there is a C^∞-solution $b(\varepsilon)$ of the equation $f(b,\varepsilon) = 0$. From (3.14) we obtain that

$$b(\varepsilon) = 1 - \frac{\varepsilon^2}{2} + \frac{\varepsilon^3\zeta(\frac{3}{2})}{8\sqrt{\pi(r-1)}} + O(\varepsilon^5), \qquad \varepsilon \to 0.$$

Since $b_k = b(k^{-1/2})$, this gives

$$b_k = 1 - \frac{1}{2k} + \frac{\zeta(\frac{3}{2})}{8\sqrt{\pi(r-1)}k^{3/2}} + O(k^{-5/2}), \qquad k \to \infty. \tag{3.15}$$

By (3.11),

$$\beta_k = 4kb_k. \tag{3.16}$$

3.4 Evaluation of the equilibrium measure

Set

$$V_k(x) = \frac{1}{k}\, Q(\beta_k x), \tag{3.17}$$

and consider the following minimization problem:

$$E = \inf_{\mu} I(\mu),$$

where

$$I(\mu) = -\iint \log|x-y|\, d\mu(x)d\mu(y) + \int V_k(x)d\mu(x).$$

and $\inf_{\mu}$ is taken over all probability measures on $[0,\infty)$. There exists a unique minimizer, $\mu = \mu_k$, called the equilibrium measure. Let us review the analytical properties of the equilibrium measure discussed by Vanlessen in [48].

The equilibrium measure has the following properties:

1. The support of μ_k is the interval $[0,1]$.

2. The measure μ_k is absolutely continuous with respect to the Lebesgue measure.

3. The density function of μ_k has the form,

$$\frac{d\mu_k(x)}{dx} \equiv \psi_k(x) = \frac{1}{2\pi}\sqrt{\frac{1-x}{x}}\, q_k(x), \tag{3.18}$$

   where $q_k(x)$ is analytic and positive on $[0,1]$.

The equilibrium measure μ_k is characterized by the Euler-Lagrange variational conditions: there exists $l_k \in \mathbb{R}$ such that

$$\begin{aligned} 2\int_0^1 \log|x-y|d\mu_k(y) - V_k(x) - l_k &= 0 \quad \text{for } x \in [0,1], \\ 2\int_0^1 \log|x-y|d\mu_k(y) - V_k(x) - l_k &\le 0 \quad \text{for } x \notin [0,1]. \end{aligned} \tag{3.19}$$

The function $q_k(z)$ in (3.18) is given by the formula,

$$q_k(z) = \frac{1}{2\pi i}\oint_\Gamma \sqrt{\frac{y}{y-1}}\,\frac{V_k'(y)\,dy}{y-z}\,, \qquad z \in \operatorname{Int}\Gamma,$$

where $\sqrt{\frac{y}{y-1}}$ is taken on the principal branch, with cut on $[0,1]$, and Γ is a positively oriented contour containing $[0,1]\cup\{z\}$ in its interior, with the additional condition that the function $V_k'(y)$ is analytic inside Γ. By (3.17) and (3.8),

$$V_k(z) = 4b_k z + \frac{1}{k}\log\frac{4b_k kz}{1-e^{-4(r-1)b_k kz}}, \tag{3.20}$$

hence

$$V_k'(z) = 4b_k + \frac{1}{kz} - \frac{\gamma_k}{e^{\gamma_k kz}-1}\,, \tag{3.21}$$

where

$$\gamma_k = 4(r-1)b_k,$$

hence

$$q_k(z) = \frac{1}{2\pi i}\oint_\Gamma \sqrt{\frac{y}{y-1}}\left[4b_k + \frac{1}{ky} - \frac{\gamma_k}{e^{\gamma_k ky}-1}\right]\frac{dy}{y-z}\,.$$

By taking the residue at infinity, we obtain that

$$q_k(z) = 4b_k + s_k(z), \qquad s_k(z) = -\frac{\gamma_k}{2\pi i}\oint_\Gamma \sqrt{\frac{y}{y-1}}\,\frac{dy}{(e^{\gamma_k ky}-1)(y-z)}\,. \tag{3.22}$$

Observe that the function $V_k'(z)$ has poles at the points

$$z_n = \frac{2\pi n i}{\gamma_k k}\,, \quad n = \pm 1, \pm 2, \dots, \tag{3.23}$$

hence the contour Γ has to pass close to 0. We choose Γ such that

$$\frac{c_1}{k} \ge \operatorname{dist}(0,\Gamma) \ge \frac{c_2}{k}, \quad c_1 \ge \operatorname{dist}(1,\Gamma) \ge c_2 > 0, \tag{3.24}$$

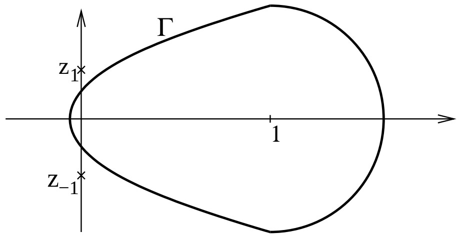

Fig. 3.1. The contour Γ.

see Figure 3.1. More precisely, for a given $z \in \mathbb{C}$, let $m(z) \in [0,1]$ be the closest point from z on $[0,1]$, so that

$$\inf\{|z-u|,\ u \in [0,1]\} = |z - m(z)|.$$

Then we define, for a given $\delta > 0$,

$$\Gamma = \Gamma(\delta, k) = \{z \in \mathbb{C} :\ |z - m(z)| = \delta \left[\frac{1}{k} + m(z)\right],$$

and we choose δ to be sufficiently small so that the points z_n in (3.23) lie outside of Γ, see Figure 3.1. Observe that $\Gamma(0,k) = [0,1]$.

With the help of the change of variables, $u = ky$, we obtain that

$$s_k(z) = -\frac{\gamma_k}{2\pi i k^{1/2}} \oint_{k\Gamma} \sqrt{\frac{u}{(u/k)-1}}\, \frac{du}{(e^{\gamma_k u}-1)(u-kz)}\,, \tag{3.25}$$

which implies that

$$\sup_{0\le z\le 1} |s_k(z)| = O(k^{-1/2}), \tag{3.26}$$

or even that

$$\sup_{0\le d\le \frac{\delta}{2}}\ \sup_{z\in\Gamma(d,k)} |s_k(z)| = O(k^{-1/2}). \tag{3.27}$$

For $z > 1$, the function $s_k(z)$ can be reduced to

$$s_k(z) = -\gamma_k \sqrt{\frac{z}{z-1}}\, \frac{1}{(e^{\gamma_k k z}-1)} - \frac{\gamma_k}{\pi} \int_0^1 \sqrt{\frac{y}{1-y}}\, \frac{dy}{(e^{\gamma_k k y}-1)(z-y)}\,.$$

It implies that

$$s_k(z) = \frac{a_k}{z} + r_k(z)\,, \tag{3.28}$$

where

$$a_k = -\frac{\gamma_k}{\pi}\int_0^1 \sqrt{\frac{y}{1-y}}\,\frac{dy}{(e^{\gamma_k k y}-1)} \tag{3.29}$$

and $r_k(z)$ satisfies the estimate,

$$|r_k(z)| \le \frac{C}{z\sqrt{z-1}\,k^{5/2}}\,, \quad z>1, \tag{3.30}$$

with some $C>0$. Indeed,

$$r_k(z) = -\gamma_k\sqrt{\frac{z}{z-1}}\,\frac{1}{e^{\gamma_k k z}-1} - \frac{\gamma_k}{\pi}\int_0^1 \sqrt{\frac{y}{1-y}}\,\frac{y\,dy}{(e^{\gamma_k k y}-1)(z-y)z}\,.$$

The first term on the right is exponentially small in z and k, and it obviously satisfies estimate (3.30). In the second term on the right, let us split the integral in two integrals, from 0 to $\frac{1}{2}$ and from $\frac{1}{2}$ to 1. The first part is estimated as follows:

$$\begin{aligned} 0 &\le \int_0^{\frac{1}{2}} \sqrt{\frac{y}{1-y}}\,\frac{y\,dy}{(e^{\gamma_k k y}-1)(z-y)z} \le \frac{4}{z^2}\int_0^{\frac{1}{2}} \frac{y^{3/2}dy}{(e^{\gamma_k k y}-1)} \\ &\le \frac{4}{z^2 k^{5/2}}\int_0^{\infty} \frac{u^{3/2}du}{(e^{\gamma_k u}-1)} \le \frac{C_0}{z^2 k^{5/2}}\,, \quad u = ky, \end{aligned}$$

hence it satisfies estimate (3.30). For the second part we have that

$$\begin{aligned} 0 &\le \int_{\frac{1}{2}}^1 \sqrt{\frac{y}{1-y}}\,\frac{y\,dy}{(e^{\gamma_k k y}-1)(z-y)z} \le \frac{1}{z(e^{\gamma_k k/2}-1)}\int_{\frac{1}{2}}^1 \frac{dy}{(z-y)\sqrt{1-y}} \\ &= \frac{1}{z(e^{\gamma_k k/2}-1)}\int_0^{\frac{1}{2}} \frac{du}{(z-1+u)\sqrt{u}} \le \frac{C_1}{z\sqrt{z-1}\,(e^{\gamma_k k/2}-1)}\,, \quad u = 1-y, \end{aligned}$$

which satisfies estimate (3.30). Thus, (3.30) is proved.

From (3.22) and (3.28) we obtain that

$$q_k(z) = 4b_k + \frac{a_k}{z} + r_k(z)\,, \tag{3.31}$$

where a_k is given by formula (3.29) and $r_k(z)$ satisfies estimate (3.30).

3.5 Evaluation of the Lagrange multiplier

Introduce the function

$$g_k(z) = \int_0^1 \log(z-x)\,d\mu_k(x), \quad z \in \mathbb{C}\setminus[0,1], \tag{3.32}$$

where the branch of log is taken on the principal sheet, with a cut on $(-\infty, 0]$. Also, let

$$\omega_k(z) \equiv g_k'(z) = \int_0^1 \frac{d\mu_k(x)}{z-x}, \quad z \in \mathbb{C} \setminus [0,1], \tag{3.33}$$

be the resolvent of the equilibrium measure μ_k. From equation (3.32) it follows that, as $z \to \infty$,

$$g_k(z) = \log z + O(z^{-1}),$$

and from (3.33), that

$$\omega_k(z) = \frac{1}{z} + O(z^{-2}).$$

From equation (3.19) it follows that

$$\omega_k(z) = \frac{V_k'(z)}{2} - \sqrt{\frac{z-1}{z}}\,\frac{q_k(z)}{2}\,, \tag{3.34}$$

see, e.g., equations (3.27), (3.29) in [48], and

$$l_k = 2g_k(1) - V_k(1).$$

Since

$$\begin{aligned} -g_k(1) = \lim_{u\to\infty}[g_k(u) - \log u - g_k(1)] &= \lim_{u\to\infty} \int_1^u \left[\omega_k(z) - \frac{1}{z}\right] dz \\ &= \int_1^\infty \left[\frac{V_k'(z)}{2} - \sqrt{\frac{z-1}{z}}\,\frac{q_k(z)}{2} - \frac{1}{z}\right] dz, \end{aligned}$$

we obtain that

$$g_k(1) = -\int_1^\infty \left[\frac{V_k'(z)}{2} - \sqrt{\frac{z-1}{z}}\,\frac{q_k(z)}{2} - \frac{1}{z}\right] dz,$$

hence

$$l_k = -\int_1^\infty \left[V_k'(z) - \sqrt{\frac{z-1}{z}}\,q_k(z) - \frac{2}{z}\right] dz - V_k(1). \tag{3.35}$$

We split the last integral as

$$\begin{aligned} \int_1^\infty \left[V_k'(z) - \sqrt{\frac{z-1}{z}}\,q_k(z) - \frac{2}{z}\right] dz &= \int_1^\infty \left[V_k'(z) - 4b_k - \frac{1}{kz}\right] dz \\ - \int_1^\infty \left[\sqrt{\frac{z-1}{z}}\,q_k(z) + \frac{2}{z} - 4b_k - \frac{1}{kz}\right] dz &= I_1 - I_2. \end{aligned} \tag{3.36}$$

From (3.21) we have that

$$I_1 = \int_1^\infty \left[V_k'(z) - 4b_k - \frac{1}{kz} \right] dz = -\int_1^\infty \frac{\gamma_k dz}{e^{\gamma_k kz} - 1} = O(e^{-c_0 k}) \,, \quad c_0 > 0.$$

Let us evaluate I_2. By (3.31),

$$I_2 = \int_1^\infty \left[\sqrt{\frac{z-1}{z}}\, 4b_k + \sqrt{\frac{z-1}{z}}\, \frac{a_k}{z} + \sqrt{\frac{z-1}{z}}\, r_k(z) + \frac{2}{z} - 4b_k - \frac{1}{kz} \right] dz.$$

Since

$$\int_1^\infty \left(\sqrt{\frac{z-1}{z}} - 1 + \frac{1}{2z} \right) dz = \frac{1}{2} - \log 2,$$

we obtain that

$$I_2 = b_k(2 - 4\log 2) + \int_1^\infty \left[\sqrt{\frac{z-1}{z}}\, \frac{a_k}{z} + \sqrt{\frac{z-1}{z}}\, r_k(z) - \frac{2b_k - 2}{z} - \frac{1}{kz} \right] dz.$$

From estimate (3.30) we obtain that

$$\int_1^\infty \sqrt{\frac{z-1}{z}}\, r_k(z) dz = O(k^{-5/2}),$$

hence

$$I_2 = b_k(2 - 4\log 2) + \int_1^\infty \left[\sqrt{\frac{z-1}{z}}\, \frac{a_k}{z} - \frac{2b_k - 2 + \frac{1}{k}}{z} \right] dz + O(k^{-5/2}). \tag{3.37}$$

From equation (3.34) we have that

$$\omega_k(z) = \frac{V_k'(z)}{2} - \frac{1}{2}\sqrt{\frac{z-1}{z}} \left[4b_k + \frac{a_k}{z} + r_k(z) \right].$$

By equating terms of the order $\frac{1}{z}$ for large z on both sides, we obtain that

$$1 = \frac{1}{2k} - \frac{a_k}{2} + b_k,$$

hence

$$a_k = 2b_k - 2 + \frac{1}{k}\,.$$

By substituting this expression into (3.37) we obtain that

$$I_2 = b_k(2 - 4\log 2) + \left(2b_k - 2 + \frac{1}{k} \right) \int_1^\infty \left(\sqrt{\frac{z-1}{z}} - 1 \right) \frac{dz}{z} + O(k^{-5/2}).$$

Since

$$\int_1^\infty \left(\sqrt{\frac{z-1}{z}}-1\right)\frac{dz}{z} = 2\log 2 - 2,$$

we obtain that

$$\begin{aligned} I_2 &= b_k(2-4\log 2) + \left(2b_k - 2 + \frac{1}{k}\right)(2\log 2 - 2) + O(k^{-5/2}) \\ &= -2b_k - 4\log 2 + 4 + \frac{2\log 2 - 2}{k} + O(k^{-5/2}) \\ &= 2 - 4\log 2 + \frac{2\log 2 - 1}{k} - \frac{\zeta(\frac{3}{2})}{4\sqrt{\pi(r-1)}k^{3/2}} + O(k^{-5/2}). \end{aligned}$$

By (3.35), (3.36),

$$l_k = I_2 - I_1 - V_k(1),$$

and by (3.20) and (3.15),

$$\begin{aligned} V_k(1) &= 4b_k + \frac{\log(4b_k k)}{k} + O(k^{-5/2}) \\ &= 4 + \frac{\log k}{k} + \frac{2\log 2 - 2}{k} + \frac{\zeta(\frac{3}{2})}{2\sqrt{\pi(r-1)}k^{3/2}} - \frac{1}{2k^2} + O(k^{-5/2}), \end{aligned}$$

hence

$$l_k = -2 - 4\log 2 - \frac{\log k}{k} + \frac{1}{k} - \frac{3\zeta(\frac{3}{2})}{4\sqrt{\pi(r-1)}k^{3/2}} + \frac{1}{2k^2} + O(k^{-5/2}), \tag{3.38}$$

3.6 Evaluation of $h_{k,\alpha}$

According to Vanlessen's asymptotic formula, see [48],

$$h_{k,\alpha} = \frac{\pi}{8}\beta_k^{2k+2}e^{kl_k}\left[1 + \left(\frac{3}{4q_k(0)} + \frac{47}{12q_k(1)} - \frac{q_k'(1)}{4q_k(1)^2}\right)\frac{1}{k} + O(k^{-2})\right]. \tag{3.39}$$

Observe that in [48] this formula is proved under the assumption that the weight for the orthogonal polynomials has the form

$$w(x) = x^\alpha e^{-Q(x)}, \qquad \alpha > -1,$$

where $Q(x)$ is a polynomial. In Appendix B at the end of the paper we show what changes in the paper of Vanlessen [48] should be made to prove (3.39) for $Q(x)$ given by (3.8). By (3.26),

$$q_k(0) = 4 + O(k^{-1/2}), \quad q_k(1) = 4 + O(k^{-1/2}). \tag{3.40}$$

By (3.25),

$$q_k'(1) = -\frac{\gamma_k}{2\pi i k^{1/2}} \oint_{k\Gamma} \sqrt{\frac{u}{(u/k)-1}} \frac{k du}{(e^{\gamma_k u}-1)(u-k)^2},$$

hence

$$q_k'(1) = O(k^{-1/2}), \tag{3.41}$$

because by condition (3.24), the function $\frac{k}{u-k}$ is bounded by $1/c$ for $u \in k\Gamma$. From (3.40) and (3.41) we obtain that

$$1 + \left(\frac{3}{4q_k(0)} + \frac{47}{12q_k(1)} - \frac{q_k'(1)}{4q_k(1)^2} \right) \frac{1}{k} = 1 + \frac{7}{6k} + O(k^{-3/2}). \tag{3.42}$$

By substituting formulae (3.16), (3.15), (3.38), and (3.42) into (3.39) and by using the Stirling formula for $k!$, we obtain that

$$\log \frac{h_{k,\alpha}}{(k!)^2} = -\frac{\zeta(\frac{3}{2})}{2\sqrt{\pi(r-1)}k^{1/2}} + \frac{1}{4k} + O(k^{-3/2})$$

(we use MAPLE for this calculation). Theorem 3.1.1 is proved.

4. ANTIFERROELECTRIC PHASE

4.1 Introduction

In this chapter we discuss the antiferroelectric phase region, and we will use parameterization (1.3) with two parameters t, γ such that $|t| < \gamma$. The parameter Δ in the antiferroelectric phase region reduces to

$$\Delta = -\cosh(2\gamma).$$

In this phase region, the function $\phi(t)$ appearing in the Hankel determinant τ_n is the Laplace transform of a discrete measure lying on $\mathbb{Z}$, so that

$$\begin{aligned}\phi(t) &= \frac{\sinh(2\gamma)}{\sinh(\gamma+t)\sinh(\gamma-t)} = \frac{2(1-e^{4\gamma})}{(1-e^{-2(\gamma-t)})(1-e^{-2(\gamma+t)})} \\ &= \frac{2}{1-e^{-2(\gamma-t)}} + \frac{2}{1-e^{-2(\gamma+t)}} - 2 = 2\sum_{l=0}^{\infty} e^{-2l(\gamma-t)} + 2\sum_{l=0}^{\infty} e^{-2l(\gamma+t)} - 2 \\ &= 2\sum_{l=0}^{\infty} e^{-2l(\gamma-t)} + 2\sum_{l=1}^{\infty} e^{-2l(\gamma+t)} = 2\sum_{l=-\infty}^{\infty} e^{2tl}e^{-2\gamma|l|}.\end{aligned} \tag{4.1}$$

Introduce the discrete monic polynomials $P_j(x) = x^j + \dots$ orthogonal on the set $\mathbb{Z}$ with respect to the weight

$$w(l) = e^{2tl}e^{-2\gamma|l|},$$

so that

$$\sum_{l=-\infty}^{\infty} P_j(l)P_k(l)w(l) = h_k\delta_{jk}. \tag{4.2}$$

Then it follows from (4.1), and Lemma 1.3.1 that

$$\tau_n = 2^{n^2}\prod_{k=0}^{n-1} h_k. \tag{4.3}$$

We will evaluate the asymptotics of the orthogonal polynomials (4.2) by reformulating them as solutions to a Riemann-Hilbert problem which can be evaluated in the large n limit by the Deift-Zhou steepest descent method. To employ this method, it is convenient to rescale the polynomials. Set

$$\Delta_n = \frac{2\gamma}{n}\,, \qquad x = l\Delta_n, \qquad w_n(x) = e^{-n(|x|-\zeta x)}, \qquad \zeta = \frac{t}{\gamma}\,, \tag{4.4}$$

and

$$P_{nk}(x) = \Delta_n^k P_k\left(\frac{x}{\Delta_n}\right).$$

Consider also the lattice

$$L_n = \left\{x = \frac{2\gamma k}{n}\,,\ k \in \mathbb{Z}\right\}. \tag{4.5}$$

Then from (4.2) we obtain that the monic polynomials $P_{nk}(x)$ satisfy the orthogonality condition,

$$\sum_{x\in L_n} P_{nj}(x)P_{nk}(x)w_n(x)\Delta_n = h_{nk}\delta_{jk}, \qquad h_{nk} = h_k\Delta_n^{2k+1}. \tag{4.6}$$

We can then combine equations (1.5), (4.3), and (4.6) to obtain

$$Z_n = \left(\frac{nab}{\gamma}\right)^{n^2}\prod_{k=0}^{n-1}\frac{h_{nk}}{(k!)^2}\,, \qquad a = \sinh(\gamma - t), \qquad b = \sinh(\gamma + t). \tag{4.7}$$

For what follows we will need to extend the weight $w_n(x)$ to the complex plane. We do so by defining $w_n(z)$ on the complex plane as

$$w_n(z) = e^{-nV(z)}$$

where

$$V(z) = \begin{cases} z - \zeta z & \text{when} \quad \operatorname{Re} z \geq 0 \\ -z - \zeta z & \text{when} \quad \operatorname{Re} z \leq 0, \end{cases}$$

so that $V(z)$, and thus $w_n(z)$, is two-valued on the imaginary axis.

4.2 Main result: Asymptotics of the partition function

The main result of this chapter is a large n asymptotic formula for Z_n in the antiferroelectric phase region. The formulation of this result and the proofs involve the Jacobi theta functions. Let us review their definition and basic properties.

There are four Jacobi theta functions:

$$
\begin{aligned}
\vartheta_1(z) &= 2\sum_{n=0}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2}\sin\big((2n+1)z\big)\,,\\
\vartheta_2(z) &= 2\sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2}\cos\big((2n+1)z\big)\,,\\
\vartheta_3(z) &= 1+2\sum_{n=1}^{\infty} q^{n^2}\cos(2nz)\,,\\
\vartheta_4(z) &= 1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos(2nz)\,,
\end{aligned}
\tag{4.8}
$$

where q is the *elliptic nome*. We will assume that $1 > q > 0$. Figure 4.1 shows the graphs of ϑ_1, ϑ_2 (left figure) and ϑ_3, ϑ_4 (right figure) on the interval $[0,\pi]$ for $q = 0.5$. Observe that ϑ_1, ϑ_4 are increasing on $[0,\frac{\pi}{2}]$ while ϑ_2, ϑ_3 are decreasing on this interval.

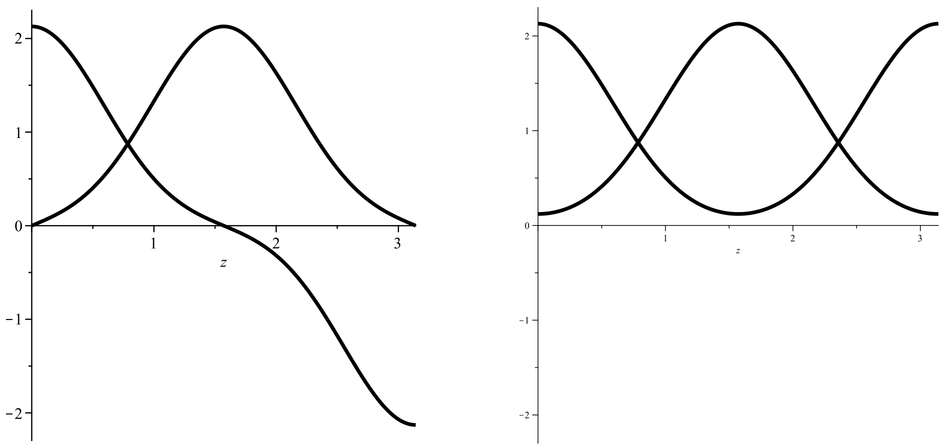

Fig. 4.1. The graphs of ϑ_1, ϑ_2 (left figure) and ϑ_3, ϑ_4 (right figure) on the interval $[0,\pi]$ for $q = 0.5$.

The Jacobi theta functions satisfy the following periodicity conditions:

$$
\begin{aligned}
\vartheta_1(z+\pi) &= -\vartheta_1(z), & \vartheta_1(z+\pi\tau) &= -e^{-2iz}q^{-1}\vartheta_1(z),\\
\vartheta_2(z+\pi) &= -\vartheta_2(z), & \vartheta_2(z+\pi\tau) &= e^{-2iz}q^{-1}\vartheta_2(z),\\
\vartheta_3(z+\pi) &= \vartheta_3(z), & \vartheta_3(z+\pi\tau) &= e^{-2iz}q^{-1}\vartheta_3(z),\\
\vartheta_4(z+\pi) &= \vartheta_4(z), & \vartheta_4(z+\pi\tau) &= -e^{-2iz}q^{-1}\vartheta_4(z),
\end{aligned}
\tag{4.9}
$$

where τ is a pure imaginary number related to q by the equation,

$$q = e^{i\pi\tau}.$$

The theta functions also satisfy the symmetry conditions

$$\vartheta_1(-z) = -\vartheta_1(z), \quad \vartheta_2(-z) = \vartheta_2(z), \quad \vartheta_3(-z) = \vartheta_3(z), \quad \vartheta_4(-z) = \vartheta_4(z),$$

and the equations,

$$\vartheta_1(z) = \vartheta_2\left(z - \frac{\pi}{2}\right), \quad \vartheta_3(z) = \vartheta_4\left(z + \frac{\pi}{2}\right), \quad \vartheta_1(z) = -ie^{iz+\frac{i\pi\tau}{4}}\vartheta_4\left(z + \frac{\pi\tau}{2}\right). \tag{4.10}$$

The only zeroes of the theta functions are

$$\vartheta_1(0) = 0, \quad \vartheta_2\left(\frac{\pi}{2}\right) = 0, \quad \vartheta_3\left(\frac{\pi}{2} + \frac{\pi\tau}{2}\right) = 0, \quad \vartheta_4\left(\frac{\pi\tau}{2}\right) = 0,$$

and their shifts by $m\pi + n\pi\tau$; $m, n \in \mathbb{Z}$. There are many non-trivial identities satisfied by the theta functions. A list of those identities used in this chapter is given in Appendix E.

In what follows we will use the following parameters, which are simply related to the parameters t and γ:

$$\zeta = \frac{t}{\gamma} \in (-1, 1), \qquad \omega = \frac{\pi(1+\zeta)}{2} \in (0, \pi).$$

The elliptic nome for all Jacobi theta functions in this chapter will be equal to

$$q = e^{-\frac{\pi^2}{2\gamma}}.$$

Our main result in this chapter is the following asymptotic formula for Z_n in the antiferroelectric phase:

Theorem 4.2.1 *As $n \to \infty$,*

$$Z_n = C\vartheta_4(n\omega)\, F^{n^2}(1 + O(n^{-1})), \tag{4.11}$$

where $C > 0$ is a constant, and

$$F = \frac{\pi \sinh(\gamma - t)\sinh(\gamma + t)\vartheta_1'(0)}{2\gamma\vartheta_1(\omega)}.$$

The asymptotic formula (4.11) proves the conjecture of Zinn-Justin in [51]. The proof of Theorem 4.2.1 will be based on the Riemann-Hilbert approach to discrete orthogonal polynomials. An important first step in this approach is a construction of the equilibrium measure.

4.3 Equilibrium measure

The heuristic idea behind the equilibrium measure is the following. We can write τ_n in a form similar to the eigenvalue partition function of a random matrix ensemble, so that

$$\tau_n = \frac{2^{n^2}}{n!} \sum_{l_1,\ldots,l_n=-\infty}^{\infty} \Delta(l)^2 \prod_{i=1}^{n} e^{2tl_i - 2\gamma|l_i|}, \tag{4.12}$$

(see Appendix A). If we scale the variables in (4.12) as $\mu_i = \frac{2\gamma l_i}{n}$, then we can rewrite formula (4.12) as

$$\tau_n = \frac{2^{n^2}}{n!} \sum_{\mu \in \frac{2\gamma}{n}\mathbb{Z}^n} e^{-n^2 H(\nu_\mu)}, \tag{4.13}$$

where

$$d\nu_\mu(x) = \frac{1}{n}\sum_{j=1}^{n} \delta(x-\mu_j),$$

and

$$H(\nu) = \iint_{x\neq y} \log\frac{1}{|x-y|} d\nu(x)d\nu(y) + \int (|x| - \zeta x)d\nu(x),$$

where all integrals are over $\mathbb{R}$.

Due to the factor $(-n^2)$ in the exponent of (4.13), we expect the sum, in the large n limit, to be focused in a neighborhood of a global minimum of the functional H. Clearly, we have that ν_μ is a probability measure and

$$\nu_\mu(a,b) < \frac{b-a}{2\gamma} + \frac{1}{n} \quad \text{for any} \quad -\infty < a < b < \infty, \tag{4.14}$$

because in (4.3), $\mu_j \in \frac{2\gamma}{n}\mathbb{Z}$ and $\mu_j \neq \mu_k$ if $j \neq k$. With these constraints in mind, we define

$$E_0 = \inf_\nu H(\nu)$$

where the infimum is taken over all probability measures satisfying

$$\nu(a,b) \le \frac{b-a}{2\gamma} \quad \text{for any} \quad -\infty < a < b < \infty, \tag{4.15}$$

which is (4.14) in the large n limit. It is possible to prove that there exists a unique minimizer ν_0, so that

$$E_0 = H(\nu_0),$$

see, e.g., the works of Saff and Totik [41], Dragnev and Saff [20] and Kuijlaars [32]. Furthermore, ν_0 has support on a finite number of intervals, and is absolutely continuous with respect to the Lebesgue measure. The minimizer ν_0 is called the *equilibrium measure*.

Denote the density function of the equilibrium measure as $\rho(x)$, and its resolvent as ω, so we have

$$\frac{d\nu_0}{dx} = \rho(x), \quad \omega(z) = \int \frac{\rho(x)dx}{z-x},$$

and

$$\rho(x) = \frac{1}{2\pi i}\big(\omega(x-i0) - \omega(x+i0)\big). \tag{4.16}$$

The structure of the equilibrium measure ν_0 is studied in the paper of Zinn-Justin [51], who shows that ν_0 has support on an interval $[\alpha, \beta]$, with a saturated region $[\alpha', \beta']$ in which

$$\rho(x) = \frac{1}{2\gamma}, \qquad x \in [\alpha', \beta'],$$

and two unsaturated regions, $[\alpha, \alpha']$ and $[\beta', \beta]$, in which

$$0 < \rho(x) < \frac{1}{2\gamma}, \qquad x \in (\alpha, \alpha') \cup (\beta', \beta), \tag{4.17}$$

see Figure 4.2. We also have that

$$\alpha < \alpha' < 0 < \beta' < \beta,$$

so that the origin, which is a singular point of the potential $V(x) = |x| - \zeta x$, lies inside the saturated region $[\alpha', \beta']$.

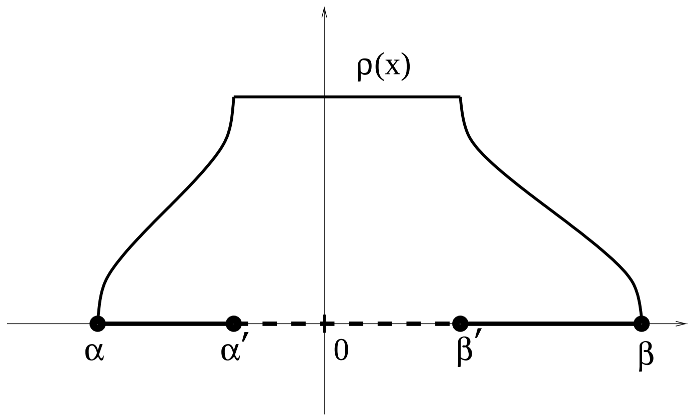

Fig. 4.2. The equilibrium density function $\rho(x)$.

The measure ν_0 is uniquely determined by the Euler-Lagrange variational conditions

$$2\int \log|x-y|d\nu_0(y) - (|x| - \zeta x) \begin{cases} = l & \text{for} \quad x \in [\alpha, \alpha'] \cup [\beta', \beta] \\ \geq l & \text{for} \quad x \in [\alpha', \beta'] \\ \leq l & \text{for} \quad x \notin [\alpha, \beta], \end{cases} \tag{4.18}$$

where l is the Lagrange multiplier. The Euler-Lagrange variational conditions imply

$$\omega(x - i0) + \omega(x + i0) = -\zeta + \operatorname{sgn}(x) \quad \text{for} \quad x \in [\alpha, \alpha'] \cup [\beta', \beta], \tag{4.19}$$

whereas in the saturated region, we have

$$\rho(x) = \frac{1}{2\pi i}\big(\omega(x - i0) - \omega(x + i0)\big) = \frac{1}{2\gamma} \quad \text{for} \quad x \in [\alpha', \beta']. \tag{4.20}$$

Now we will give a detailed description of the equilibrium measure. We begin with explicit formulae for the end-points of the support of the equilibrium measure.

4.4 Explicit formulae for the end-points

The locations of the end-points of the support of the equilibrium measure are given in the following proposition.

Proposition 4.4.1 *The end-points of the support of the equilibrium measure ν_0 are equal to*

$$\alpha = -\pi \frac{\vartheta_1'(\frac{\omega}{2})}{\vartheta_1(\frac{\omega}{2})}, \qquad \alpha' = -\pi \frac{\vartheta_4'(\frac{\omega}{2})}{\vartheta_4(\frac{\omega}{2})},$$
$$\beta' = -\pi \frac{\vartheta_3'(\frac{\omega}{2})}{\vartheta_3(\frac{\omega}{2})}, \qquad \beta = -\pi \frac{\vartheta_2'(\frac{\omega}{2})}{\vartheta_2(\frac{\omega}{2})}.$$

The differences between the end-points are equal to

$$\begin{aligned} &\alpha' - \alpha = \pi\vartheta_4^2(0)\frac{\vartheta_2(\frac{\omega}{2})\vartheta_3(\frac{\omega}{2})}{\vartheta_1(\frac{\omega}{2})\vartheta_4(\frac{\omega}{2})}, \qquad \beta' - \alpha' = \pi\vartheta_2^2(0)\frac{\vartheta_1(\frac{\omega}{2})\vartheta_2(\frac{\omega}{2})}{\vartheta_3(\frac{\omega}{2})\vartheta_4(\frac{\omega}{2})}, \\ &\beta - \beta' = \pi\vartheta_4(0)^2\frac{\vartheta_1(\frac{\omega}{2})\vartheta_4(\frac{\omega}{2})}{\vartheta_2(\frac{\omega}{2})\vartheta_3(\frac{\omega}{2})}. \end{aligned} \tag{4.21}$$

and

$$\begin{aligned} &\beta - \alpha = \pi\vartheta_2^2(0)\frac{\vartheta_3(\frac{\omega}{2})\vartheta_4(\frac{\omega}{2})}{\vartheta_1(\frac{\omega}{2})\vartheta_2(\frac{\omega}{2})}, \qquad \beta - \alpha' = \pi\vartheta_3(0)^2\frac{\vartheta_1(\frac{\omega}{2})\vartheta_3(\frac{\omega}{2})}{\vartheta_2(\frac{\omega}{2})\vartheta_4(\frac{\omega}{2})}, \\ &\beta' - \alpha = \pi\vartheta_3^2(0)\frac{\vartheta_2(\frac{\omega}{2})\vartheta_4(\frac{\omega}{2})}{\vartheta_1(\frac{\omega}{2})\vartheta_3(\frac{\omega}{2})}. \end{aligned} \tag{4.22}$$

Finally, we have the Zinn-Justin formula for the centroid of the end-points,

$$\frac{\alpha + \alpha' + \beta' + \beta}{4} = -\frac{\pi}{2}\frac{\vartheta_2'(\frac{\pi\zeta}{2})}{\vartheta_2(\frac{\pi\zeta}{2})}.$$

For a proof of Proposition 4.4.1 see Section 4.8.

4.5 Equilibrium density function

The equilibrium density function is described in the following proposition.

Proposition 4.5.1 *The equilibrium density function $\rho(x)$ is given by the formula,*

$$\rho(x) = \begin{cases} \dfrac{1}{\pi}\displaystyle\int_\alpha^x \frac{dx'}{\sqrt{(x'-\alpha)(\alpha'-x')(\beta'-x')(\beta-x')}}, & \alpha \le x \le \alpha' \\ \dfrac{1}{2\gamma}, & \alpha' \le x \le \beta' \\ \dfrac{1}{\pi}\displaystyle\int_x^\beta \frac{dx'}{\sqrt{(x'-\alpha)(x'-\alpha')(x'-\beta')(\beta-x')}}, & \beta' \le x \le \beta. \end{cases} \tag{4.23}$$

Also,

$$\int_0^\beta \rho(x)\,dx = \frac{1+\zeta}{2}. \tag{4.24}$$

The resolvent $\omega(z)$ of the equilibrium measure is given as

$$\omega(z) = \int_z^\infty \frac{dz'}{\sqrt{(z'-\alpha)(z'-\alpha')(z'-\beta')(z'-\beta)}}, \tag{4.25}$$

where integration takes place on the sheet of

$$\sqrt{R(z')} \equiv \sqrt{(z'-\alpha)(z'-\alpha')(z'-\beta')(z'-\beta)}$$

for which $\sqrt{R(z')} > 0$ for $z' > \beta$, with cuts on $[\alpha, \alpha']$ and $[\beta', \beta]$.

For a proof of this proposition see Section 4.8.

4.6 The g-function

Define the g-function on $\mathbb{C} \setminus [-\infty, \beta]$ as

$$g(z) = \int_\alpha^\beta \log(z-x) d\nu_0(x) \tag{4.26}$$

where we take the principal branch for logarithm.

Properties of $g(z)$:

1. $g(z)$ is analytic in $\mathbb{C} \setminus (-\infty, \beta]$.

2. For large z,
$$g(z) = \log z - \sum_{j=1}^\infty \frac{g_j}{z^j}, \qquad g_j = \int_\alpha^\beta \frac{x^j}{j} d\nu_0(x). \tag{4.27}$$

3. $g'(z) = \omega(z)$.

4. From the first relation in (4.18) we have that
$$g_+(x) + g_-(x) = |x| - \zeta x + l \quad \text{for} \quad x \in [\alpha, \alpha'] \cup [\beta', \beta], \tag{4.28}$$
where g_+ and g_- refer to the limiting values of g from the upper and lower half-planes, respectively. By differentiating this equation we obtain that
$$\omega_+(x) + \omega_-(x) = g'_+(x) + g'_-(x) = \operatorname{sgn} x - \zeta \quad \text{for} \quad x \in [\alpha, \alpha'] \cup [\beta', \beta]. \tag{4.29}$$

Consider the function

$$f(x) = g_+(x) + g_-(x) - (|x| - \zeta x + l).$$

We have from (4.28), (4.29) that

$$f(x) = f'(x) = 0 \quad \text{for} \quad x = \alpha, \alpha', \beta', \beta,$$

and from (4.25) that

$$f''(x) = -\frac{1}{\sqrt{(x-\alpha)(x-\alpha')(x-\beta')(x-\beta)}} \text{ for } x \in (-\infty, \alpha) \cup (\alpha', \beta') \cup (\beta, \infty).$$

Since

$$f''(x) < 0 \quad \text{for} \quad x \in (-\infty, \alpha) \cup (\beta, \infty).$$

and

$$f''(x) > 0 \quad \text{for} \quad x \in (\alpha', \beta'),\ x \neq 0,$$

we obtain that

$$g_+(x) + g_-(x) \begin{cases} = |x| - \zeta x + l & \text{for} \quad x \in [\alpha, \alpha'] \cup [\beta', \beta], \\ > |x| - \zeta x + l & \text{for} \quad x \in (\alpha', \beta'), \\ < |x| - \zeta x + l & \text{for} \quad x \in \mathbb{R} \setminus [\alpha, \beta]. \end{cases} \tag{4.30}$$

5. Equation (4.26) implies that the function

$$G(x) \equiv g_+(x) - g_-(x)$$

is pure imaginary for all real x, and

$$G(x) = \begin{cases} 2\pi i & \text{for} \quad -\infty < x \leq \alpha \\ 2\pi i - 2\pi i \displaystyle\int_\alpha^x \rho(s)\, ds & \text{for} \quad \alpha \leq x \leq \alpha' \\ 2\pi i \left(\dfrac{1+\zeta}{2} - \dfrac{x}{2\gamma} \right) & \text{for} \quad \alpha' \leq x \leq \beta' \\ 2\pi i \displaystyle\int_x^\beta \rho(s)\, ds & \text{for} \quad \beta' \leq x \leq \beta \\ 0 & \text{for} \quad \beta \leq x < \infty. \end{cases} \tag{4.31}$$

From (4.30) and (4.31) we obtain that

$$2g_\pm(x) = \begin{cases} |x| - \zeta x + l \pm \left[2\pi i - 2\pi i \int_\alpha^x \rho(s)\, ds\right] & \text{for} \quad \alpha \le x \le \alpha' \\ |x| - \zeta x + l \pm 2\pi i \int_x^\beta \rho(s)\, ds & \text{for} \quad \beta' \le x \le \beta. \end{cases} \tag{4.32}$$

6. Also, from (4.31) we see that the function $G(x)$ is real analytic on each of the intervals $(-\infty, \alpha)$, (α, α'), (α', β'), (β', β), and (β, ∞), and can therefore be extended to the complex plane in a neighborhood of any of these intervals. The Cauchy-Riemann equations imply that

$$\left.\frac{dG(x+iy)}{dy}\right|_{y=0} = 2\pi\rho(x) > 0, \quad x \in (\alpha, \beta).$$

Observe that from (4.28) we have that

$$G(x) = 2g_+(x) - V(x) - l = -[2g_-(x) - V(x) - l], \quad x \in [\alpha, \alpha'] \cup [\beta', \beta],$$

where $V(x) \equiv |x| - \zeta x$.

4.7 Evaluation of the Lagrange multiplier l

We have the following proposition.

Proposition 4.7.1 *The Lagrange multiplier l solves the equation,*

$$e^{\frac{l}{2}} = \frac{\pi \vartheta_1'(0)}{2e\vartheta_1(\omega)}. \tag{4.33}$$

For a proof of this proposition see the next section.

4.8 Proof of Propositions 4.4.1, 4.5.1, and 4.7.1

Proof of Proposition 4.4.1. Following Zinn-Justin [51], we make the following elliptic change of variables:

$$u(z) = \frac{1}{2}\sqrt{(\beta' - \alpha)(\beta - \alpha')} \int_\beta^z \frac{dz'}{\sqrt{(z'-\alpha)(z'-\alpha')(z'-\beta')(z'-\beta)}}, \tag{4.34}$$

where integration takes place on the sheet of $\sqrt{R(z')}$ specified in Proposition 4.5.1. To understand this integral in terms of the Jacobi elliptic functions, we first make the change of variables

$$v(z') = \frac{(\beta - z')(\beta' - \alpha)}{(\beta' - z')(\beta - \alpha)}, \tag{4.35}$$

so that

$$z' = \frac{\beta'(\beta - \alpha)v - \beta(\beta' - \alpha)}{(\beta - \alpha)v - (\beta' - \alpha)}.$$

Note that $v(\beta) = 0$, $v(\beta') = \infty$, and $v(\alpha) = 1$. When we substitute v into equation (4.34), we have

$$u(z) = \frac{1}{2k}\int_0^{v(z)} \frac{dv}{\sqrt{v(v-1)(v-\frac{1}{k^2})}},$$

where

$$k = \sqrt{\frac{(\beta - \alpha)(\beta' - \alpha')}{(\beta' - \alpha)(\beta - \alpha')}}.$$

We next take $v' = \sqrt{v}$, obtaining

$$u(z) = \int_0^{\sqrt{v(z)}} \frac{dv'}{\sqrt{(1 - v'^2)(1 - k^2 v'^2)}},$$

which corresponds to $\sqrt{v(z)} = \mathrm{sn}(u, k)$, so that

$$\frac{(\beta - z)(\beta' - \alpha)}{(\beta' - z)(\beta - \alpha)} = \mathrm{sn}^2(u), \qquad \mathrm{sn}(u) = \mathrm{sn}(u, k). \tag{4.36}$$

Notice that u maps the upper z-plane conformally and bijectively onto the rectangle $[0, K] \times [0, iK']$, and the lower z-plane conformally and bijectively onto the rectangle $[0, K] \times [-iK', 0]$, where

$$K = u(\alpha) = \int_0^1 \frac{dv'}{\sqrt{(1 - v'^2)(1 - k^2 v'^2)}} \qquad \text{and}$$

$$K' = -iu(\beta') = \int_1^{\frac{1}{k}} \frac{dv'}{\sqrt{(v'^2 - 1)(1 - k^2 v'^2)}}$$

are the usual complete integrals of the first kind. More specifically (see Figure 4.3),

1. The upper (resp. lower) cusp of the interval $[\beta', \beta]$ is mapped onto the interval $[0, iK']$ (resp. $[0, -iK']$).

2. The upper (resp. lower) cusp of the interval $[\alpha, \alpha']$ is mapped onto the interval $[K, K+iK']$ (resp. $[K, K-iK']$).

3. The interval $[\alpha', \beta']$ is mapped onto the inteval $[iK', K+iK']$ or the inteval $[-iK', K-iK']$, depending on the path of integration.

4. The remaining part of the real axis, $[-\infty, \alpha] \cup [\beta, \infty]$, is mapped onto the interval $[0, K]$, with $u(\infty) = u_* = u_\infty$.

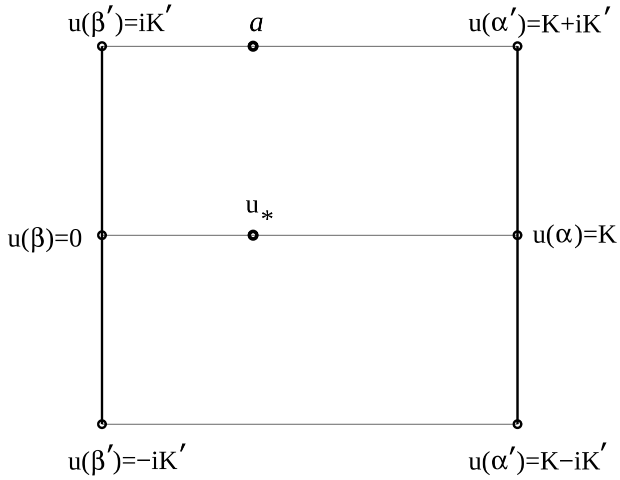

Fig. 4.3. The u-plane. Here $u_* = u_\infty \equiv u(\infty)$ and $a = u_* + iK'$.

We will denote the rectangle $[0, K] \times [-iK', iK']$ as R, the *fundamental domain* of the function $z(u)$. We can now define

$$\tilde{\omega}(u) = \omega(z(u)) \quad \text{for} \quad u \in R.$$

The Euler-Lagrange equation (4.19) and the equation (4.20) then become

$$\begin{aligned} \tilde{\omega}(u) + \tilde{\omega}(-u) &= 1 - \zeta && \text{for } u \in [-iK', iK'] \\ \tilde{\omega}(u) + \tilde{\omega}(-u + 2K) &= -1 - \zeta && \text{for } u \in [K - iK', K + iK'] \\ \tilde{\omega}(u + 2iK') - \tilde{\omega}(u) &= -\frac{i\pi}{\gamma} && \text{for } u \in [-iK', K - iK']. \end{aligned} \tag{4.37}$$

The function $\omega(z)$ is analytic in $\mathbb{C} \setminus [\alpha, \beta]$, but can be analytically continued from either above or below through any of the cuts $[\alpha, \alpha']$, $[\alpha', \beta']$, and $[\beta', \beta]$. These analytic continuations in the z-plane give an analytic continuation of $\tilde{\omega}$ in the u-plane into a neighborhood of R, which can then be continued by equations (4.37) to the entire u-plane. We therefore have that $\tilde{\omega}$ is analytic and satisfies equations (4.37) throughout the u-plane. The first two equations of (4.37) can be combined as

$$\tilde{\omega}(u + 2K) = \tilde{\omega}(u) - 2. \tag{4.38}$$

It therefore follows that $\tilde{\omega}$ is a linear function of u, as its derivative is a doubly periodic entire function. We also know from the fact that $\omega(z) \sim \frac{1}{z}$ at infinity that

$$\tilde{\omega}(u) = -\frac{2}{\sqrt{(\beta' - \alpha)(\beta - \alpha')}}(u - u_\infty) + O(u - u_\infty)^2 \tag{4.39}$$

in some neighborhood of u_∞, where u_∞ is the image of infinity under the map $u(z)$. It thus follows from (4.37), (4.38), and (4.39) that

$$\tilde{\omega}(u) = -\frac{1}{K}(u - u_\infty), \tag{4.40}$$

and that

$$\frac{K'}{K} = \frac{\pi}{2\gamma}, \tag{4.41}$$

$$\sqrt{(\beta' - \alpha)(\beta - \alpha')} = 2K, \tag{4.42}$$

$$\frac{u_\infty}{K} = \frac{1 - \zeta}{2}. \tag{4.43}$$

From (4.36) we obtain that

$$\frac{\beta' - \alpha}{\beta - \alpha} = \mathrm{sn}^2(u_\infty). \tag{4.44}$$

This implies that

$$\begin{aligned} \mathrm{cn}^2(u_\infty) &= 1 - \mathrm{sn}^2(u_\infty) = 1 - \frac{\beta' - \alpha}{\beta - \alpha} = \frac{\beta - \beta'}{\beta - \alpha}, \\ \mathrm{dn}^2(u_\infty) &= 1 - k^2\mathrm{sn}^2(u_\infty) = 1 - \frac{(\beta - \alpha)(\beta' - \alpha')}{(\beta' - \alpha)(\beta - \alpha')}\frac{(\beta' - \alpha)}{(\beta - \alpha)} = \frac{\beta - \beta'}{\beta - \alpha'}. \end{aligned} \tag{4.45}$$

From equations (4.42), (4.44), (4.45) we obtain the distances between the turning points in terms of u_∞:

$$\begin{aligned} \beta - \alpha &= 2K\frac{\mathrm{dn}(u_\infty)}{\mathrm{sn}(u_\infty)\mathrm{cn}(u_\infty)}, \quad \beta - \alpha' = 2K\frac{\mathrm{cn}(u_\infty)}{\mathrm{sn}(u_\infty)\mathrm{dn}(u_\infty)}, \\ \beta - \beta' &= 2K\frac{\mathrm{cn}(u_\infty)\mathrm{dn}(u_\infty)}{\mathrm{sn}(u_\infty)}. \end{aligned} \tag{4.46}$$

The functions sn, cn, and dn are expressed in terms of Jacobi theta functions as follows (see e.g. [50]),

$$\mathrm{sn}(u) = \frac{\vartheta_3(0)}{\vartheta_2(0)}\frac{\vartheta_1(\frac{\pi u}{2K})}{\vartheta_4(\frac{\pi u}{2K})}, \quad \mathrm{cn}(u) = \frac{\vartheta_4(0)}{\vartheta_2(0)}\frac{\vartheta_2(\frac{\pi u}{2K})}{\vartheta_4(\frac{\pi u}{2K})}, \quad \mathrm{dn}(u) = \frac{\vartheta_4(0)}{\vartheta_3(0)}\frac{\vartheta_3(\frac{\pi u}{2K})}{\vartheta_4(\frac{\pi u}{2K})}.$$

By (4.41), the half-period ratio τ and the elliptic nome q of the theta functions are

$$\tau = \frac{iK'}{K} = \frac{i\pi}{2\gamma} \quad \text{and} \quad q = e^{\frac{-\pi K'}{K}} = e^{\frac{-\pi^2}{2\gamma}}. \tag{4.47}$$

If we take into account the fact that

$$\vartheta_3(0)^2 = \frac{2K}{\pi}$$

and equation (4.43), we can write equations for the distances between the turning points that involve the original parameters only:

$$\begin{aligned} \beta - \alpha &= \pi\vartheta_2^2(0)\frac{\vartheta_3(\frac{\omega}{2})\vartheta_4(\frac{\omega}{2})}{\vartheta_1(\frac{\omega}{2})\vartheta_2(\frac{\omega}{2})}, \qquad \beta - \alpha' = \pi\vartheta_3(0)^2\frac{\vartheta_1(\frac{\omega}{2})\vartheta_3(\frac{\omega}{2})}{\vartheta_2(\frac{\omega}{2})\vartheta_4(\frac{\omega}{2})}, \\ \beta' - \alpha &= \pi\vartheta_3^2(0)\frac{\vartheta_2(\frac{\omega}{2})\vartheta_4(\frac{\omega}{2})}{\vartheta_1(\frac{\omega}{2})\vartheta_3(\frac{\omega}{2})}, \end{aligned} \tag{4.48}$$

which is (4.22). These equations determine the end-points $\alpha, \alpha', \beta', \beta$ up to a shift. To fix the shift we use the equation (4.18) at the points α' and β' to obtain

$$\int_{\alpha'}^{\beta'} \big(\omega(z+i0) + \omega(z-i0)\big)dz = (1-\zeta)\beta' + (1+\zeta)\alpha'.$$

Writing this integral in terms of u gives

$$\int_{iK'}^{K+iK'} \frac{1}{K}(u-u_\infty)r'(u)du + \int_{-iK'}^{K-iK'} \frac{1}{K}(u-u_\infty)r'(u)du = (1-\zeta)\beta' + (1+\zeta)\alpha',$$

where

$$r(u) = \frac{\beta'(\beta-\alpha)\mathrm{sn}^2(u) - \beta(\beta'-\alpha)}{(\beta-\alpha)\mathrm{sn}^2(u) - (\beta'-\alpha)} = \frac{\beta - \frac{\beta'\mathrm{sn}^2(u)}{\mathrm{sn}^2(u_\infty)}}{1 - \frac{\mathrm{sn}^2(u)}{\mathrm{sn}^2(u_\infty)}} \quad \text{and} \quad r'(u) = \frac{d}{du}r(u). \tag{4.49}$$

Note that $r(\pm iK') = \beta'$ and $r(K \pm iK') = \alpha'$. Integrating by parts gives

$$\frac{2}{K}((K - u_\infty)\alpha' + \beta' u_\infty) - \int_{iK'}^{K+iK'} \frac{r(u)}{K} du - \int_{-iK'}^{K-iK'} \frac{r(u)}{K} du = (1-\zeta)\beta' + (1+\zeta)\alpha',$$

or equivalently

$$\int_{iK'}^{K+iK'} r(u)du + \int_{-iK'}^{K-iK'} r(u)du = 0. \tag{4.50}$$

We can evaluate these integrals by first writing $r(u)$ in the form

$$r(u) = \beta + \left(\frac{\beta - \beta'}{\mathrm{sn}^2(u_\infty)}\right) \frac{\mathrm{sn}^2(u)}{1 - \frac{\mathrm{sn}^2(u)}{\mathrm{sn}^2(u_\infty)}}$$

and using the functions

$$\Theta(u) = \vartheta_4\left(\frac{\pi u}{2K}\right), \quad Z(u) = \frac{\Theta'(u)}{\Theta(u)}.$$

The addition formulae for the sn and Z functions are (see [50])

$$\mathrm{sn}(u \pm a) = \frac{\mathrm{sn}(u)\mathrm{cn}(a)\mathrm{dn}(a) \pm \mathrm{sn}(a)\mathrm{cn}(u)\mathrm{dn}(u)}{1 - k^2\mathrm{sn}^2(a)\mathrm{sn}^2(u)},$$

$$Z(u \pm a) = Z(u) \pm Z(a) \mp k^2\mathrm{sn}(u)\mathrm{sn}(a)\mathrm{sn}(u \pm a).$$

Thus we have

$$\begin{aligned} Z(u-a) - Z(u+a) + 2Z(a) &= k^2\mathrm{sn}(u)\mathrm{sn}(a)\big(\mathrm{sn}(u+a) + \mathrm{sn}(u-a)\big) \\ &= \frac{k^2\mathrm{sn}(u)\mathrm{sn}(a)\,[2\mathrm{sn}(u)\mathrm{cn}(a)\mathrm{dn}(a)]}{1 - k^2\mathrm{sn}^2(a)\mathrm{sn}^2(u)} \\ &= \frac{2k^2\mathrm{sn}(a)\mathrm{cn}(a)\mathrm{dn}(a)\mathrm{sn}^2(u)}{1 - k^2\mathrm{sn}^2(a)\mathrm{sn}^2(u)}. \end{aligned} \tag{4.51}$$

We also have the half- and quarter-period identities

$$\mathrm{sn}(u + iK') = \frac{1}{k\mathrm{sn}(u)}, \quad \mathrm{cn}(u+iK') = \frac{-i}{k}\frac{\mathrm{dn}(u)}{\mathrm{sn}(u)}, \quad \mathrm{dn}(u+iK') = -i\frac{\mathrm{cn}(u)}{\mathrm{sn}(u)}. \tag{4.52}$$

In particular, notice that $\frac{1}{\mathrm{sn}(u_\infty)} = k\mathrm{sn}(u_\infty + iK')$. Using the addition formula (4.51) we can write $r(u)$ as

$$r(u) = \beta + \left(\frac{\beta - \beta'}{2k^2\mathrm{sn}(a)\mathrm{cn}(a)\mathrm{dn}(a)\mathrm{sn}^2(u_\infty)}\right)\big(Z(u-a) - Z(u+a) + 2Z(a)\big), \tag{4.53}$$

where $a = u_\infty + iK'$, (see Figure 4.3). From (4.52) and (4.46), it follows that

$$\frac{\beta - \beta'}{2k^2 \mathrm{sn}(u_\infty + iK')\mathrm{cn}(u_\infty + iK')\mathrm{dn}(u_\infty + iK')\mathrm{sn}^2(u_\infty)} = -K.$$

Thus we can write (4.53) as

$$r(u) = \beta - K\left[Z(u - u_\infty - iK') - Z(u + u_\infty + iK') + 2Z(u_\infty + iK')\right]. \tag{4.54}$$

If we write $u = x + iK'$ in the first integral of (4.50), and $u = x - iK'$ in the second, we obtain

$$\begin{aligned}\int_0^K \big[2\beta - 4KZ(u_\infty + iK') - K[Z(x - u_\infty) - Z(x + u_\infty + 2iK')\\ + Z(x - u_\infty - 2iK') - Z(x + u_\infty)]\big]\, dx = 0\end{aligned} \tag{4.55}$$

From the periodic properties of ϑ_4, it follows that

$$Z(u \pm 2iK') = Z(u) \mp \frac{\pi i}{K},$$

so we can write (4.55) as

$$\int_0^K \Big(2\beta - 4KZ(u_\infty + iK') - 2\pi i + 2K\big[Z(x + u_\infty) - Z(x - u_\infty)\big]\Big)\, dx = 0$$

This equation is readily integrated, as Z is the logarithmic derivative of the Θ function. Integrating gives

$$\begin{aligned}0 &= \left[(2\beta - 4KZ(u_\infty + iK') - 2\pi i)x + 2K\log\frac{\Theta(x + u_\infty)}{\Theta(x - u_\infty)}\right]_{x=0}^{K}\\ &= 2K\beta - 4K^2 Z(u_\infty + iK') - 2K\pi i + 2K\log\left(\frac{\Theta(K + u_\infty)}{\Theta(K - u_\infty)}\frac{\Theta(-u_\infty)}{\Theta(u_\infty)}\right).\end{aligned}$$

The logarithmic term in this equation is zero due to the evenness and periodicity (period $2K$) of the Θ function and the fact that the relevant term in the integration is real on the entire contour of integration. Thus we have that

$$\beta = 2KZ(u_\infty + iK') + \pi i. \tag{4.56}$$

From (4.10), we can deduce that

$$Z(u_\infty + iK') = -\frac{\pi}{2K}\left(\frac{\vartheta_2'(\frac{\omega}{2})}{\vartheta_2(\frac{\omega}{2})} + i\right)$$

and write (4.56) as

$$\beta = -\pi \, \frac{\vartheta_2'(\frac{\omega}{2})}{\vartheta_2(\frac{\omega}{2})} \, . \tag{4.57}$$

This equation, together with equations (4.48) and (4.47), determine the end-points $\alpha, \alpha', \beta', \beta$. In fact, similar to (4.57) we have the following explicit formulae for the other end-points:

$$\alpha = -\pi \, \frac{\vartheta_1'(\frac{\omega}{2})}{\vartheta_1(\frac{\omega}{2})} \, , \qquad \alpha' = -\pi \, \frac{\vartheta_4'(\frac{\omega}{2})}{\vartheta_4(\frac{\omega}{2})} \, , \qquad \beta' \; = -\pi \, \frac{\vartheta_3'(\frac{\omega}{2})}{\vartheta_3(\frac{\omega}{2})} \, . \tag{4.58}$$

This follows from (4.48), (4.57), and the identities (E.2). Similarly, in addition to the formulae (4.48) for distances between turning points, we get (4.21).

Proof of the formula of Zinn-Justin for $\frac{\alpha+\alpha'+\beta'+\beta}{4}$. We immediately have from (4.58) that

$$\frac{\alpha+\alpha'+\beta'+\beta}{4} = -\frac{\pi}{4}\left[\frac{\vartheta_1'(\frac{\omega}{2})}{\vartheta_1(\frac{\omega}{2})} + \frac{\vartheta_2'(\frac{\omega}{2})}{\vartheta_2(\frac{\omega}{2})} + \frac{\vartheta_3'(\frac{\omega}{2})}{\vartheta_3(\frac{\omega}{2})} + \frac{\vartheta_4'(\frac{\omega}{2})}{\vartheta_4(\frac{\omega}{2})}\right].$$

From the identity (E.6), we can deduce that

$$\frac{\vartheta_1'(2z)}{\vartheta_1(2z)} = \frac{1}{2}\left[\frac{\vartheta_1'(z)}{\vartheta_1(z)} + \frac{\vartheta_2'(z)}{\vartheta_2(z)} + \frac{\vartheta_3'(z)}{\vartheta_3(z)} + \frac{\vartheta_4'(z)}{\vartheta_4(z)}\right],$$

thus we have

$$\frac{\alpha+\alpha'+\beta'+\beta}{4} = -\frac{\pi}{2}\frac{\vartheta_1'(\omega)}{\vartheta_1(\omega)} = -\frac{\pi}{2}\frac{\vartheta_1'(\frac{\pi}{2}+\frac{\pi\zeta}{2})}{\vartheta_1(\frac{\pi}{2}+\frac{\pi\zeta}{2})} = -\frac{\pi}{2}\frac{\vartheta_2'(\frac{\pi\zeta}{2})}{\vartheta_2(\frac{\pi\zeta}{2})}.$$

Proof of Proposition 4.5.1. From equations (4.40), (4.34), and (4.42) we obtain formula (4.25), cf. [51]. From formula (4.25) and equations (4.16), (4.20) we obtain that the equilibrium density function $\rho(x)$ is given by formulae (4.23). We are left to prove formula (4.24).

By (4.34), (4.42), and (4.23), on the interval $[\beta', \beta]$,

$$\rho(x) = \frac{1}{iK\pi} u_+(x) \quad \text{for} \quad x \in [\beta', \beta].$$

It follows that

$$\int_{\beta'}^{\beta} \rho(x)dx = \frac{1}{iK\pi}\int_{\beta'}^{\beta} u_+(x)dx = \frac{1}{iK\pi}\int_{iK'}^{0} ur'(u)du, \tag{4.59}$$

where $r(u)$ is defined in (4.49). If we use equation (4.54), together with formula (4.56), we can write $r(u)$ as

$$r(u) = i\pi - K\big[Z(u - u_\infty - iK') - Z(u + u_\infty + iK')\big]. \tag{4.60}$$

Integrating (4.59) by parts, we get

$$\int_{iK'}^{0} ur'(u)du = -\beta' iK' - \pi K' - K\left[\log\left(\frac{\Theta(-u_\infty - iK')\Theta(u_\infty + 2iK')}{\Theta(-u_\infty)\Theta(u_\infty + iK')}\right)\right]. \tag{4.61}$$

Using the fact that Θ is an even function and the identity

$$\Theta(u + 2iK') = e^{i\pi} e^{-i\pi\tau} e^{\frac{-i\pi u}{K}} \Theta(u),$$

we can write (4.61) as

$$\begin{aligned}\int_{iK'}^{0} ur'(u)du &= -\beta' iK' - \pi K' + K\left(i\pi + \pi\frac{K'}{K} - \frac{i\pi u_\infty}{K}\right) \\ &= i(K\pi - \beta' K' - \pi u_\infty).\end{aligned} \tag{4.62}$$

Remark: There is perhaps some question here as to which branch of the logarithm to take, but it is clear that we have chosen the correct branch, as it is the only one that gives $0 < \int_{\beta'}^{\beta} \rho(x)dx < 1$.

Thus, from (4.59) and (4.62), we have

$$\int_{\beta'}^{\beta} \rho(x)dx = \frac{1}{iK\pi} i(K\pi - \beta' K' - \pi u_\infty) = 1 - \frac{\beta' K'}{\pi K} - \frac{u_\infty}{K} = 1 - \frac{\beta'}{2\gamma} - \frac{1-\zeta}{2},$$

hence by (4.23),

$$\int_0^{\beta} \rho(x)dx = \int_0^{\beta'} \rho(x)dx + \int_{\beta'}^{\beta} \rho(x)dx = \frac{\beta'}{2\gamma} + 1 - \frac{\beta'}{2\gamma} - \frac{1-\zeta}{2} = \frac{1+\zeta}{2},$$

which proves formula (4.24).

Proof of Proposition 4.7.1. By taking $x = \beta$ we obtain from (4.30) that

$$l = 2g(\beta) - V(\beta) = 2g(\beta) - (1 - \zeta)\beta.$$

We also have that

$$\lim_{A\to\infty} [g(A) - \log A] = 0,$$

hence

$$l = -2 \lim_{A\to\infty} [g(A) - g(\beta) - \log A] - (1-\zeta)\beta$$
$$= -2 \lim_{A\to\infty} \left[\int_\beta^A \omega(z)\, dz - \log A \right] - (1-\zeta)\beta.$$

Writing this integral in terms of u (so $z = r(u)$) gives

$$l = 2 \lim_{A\to\infty} \left[\int_0^B \frac{1}{K}(u-u_\infty) r'(u)\, du + \log A \right] - (1-\zeta)\beta$$
$$= 2 \lim_{B\to u_\infty} \left[\int_0^B \frac{1}{K}(u-u_\infty) r'(u)\, du + \log r(B) \right] - (1-\zeta)\beta,$$

where $A = r(B)$. Integrating by parts gives

$$l = 2 \lim_{B\to u_\infty} \left[\frac{1}{K}(u-u_\infty) r(u) \Big|_{u=0}^{B} - \frac{1}{K}\int_0^B r(u)\, du + \log r(B) \right] - (1-\zeta)\beta.$$

From (4.49) we obtain that $r(0) = \beta$ and

$$\lim_{B\to u_\infty} (B-u_\infty) r(B) = -\frac{(\beta-\beta')\mathrm{sn}(u_\infty)}{2\mathrm{sn}'(u_\infty)} = -\frac{(\beta-\beta')\mathrm{sn}(u_\infty)}{2\mathrm{cn}(u_\infty)\mathrm{dn}(u_\infty)} = -K\,,$$

hence

$$\begin{aligned} l &= 2\left[-1 + \frac{\beta(1-\zeta)}{2}\right] - 2 \lim_{B\to u_\infty} \left[\frac{1}{K}\int_0^B r(u)\, du - \log r(B) \right] - (1-\zeta)\beta \\ &= -2 - 2 \lim_{B\to u_\infty} \left[\frac{1}{K}\int_0^B r(u)\, du - \log r(B) \right]. \end{aligned} \tag{4.63}$$

Using equation (4.60) for $r(u)$, we immediately get that

$$\frac{1}{K}\int_0^B r(u) du = \frac{B\pi i}{K} + \log\left[\frac{\Theta(B+u_\infty+iK')}{\Theta(B-u_\infty-iK')}\right]$$

Now using equation (4.49) for $r(u)$, we have

$$\begin{aligned} &\lim_{B\to u_\infty} \left[\frac{1}{K}\int_0^B r(u)\, du - \log r(B) \right] \\ &= \frac{u_\infty \pi i}{K} + \lim_{B\to u_\infty} \log\left[\frac{\Theta(B+u_\infty+iK')\big(\mathrm{sn}^2(u_\infty) - \mathrm{sn}^2(B)\big)}{\Theta(B-u_\infty-iK')\big(\beta\mathrm{sn}^2(u_\infty) - \beta'\mathrm{sn}^2(B)\big)} \right] \\ &= \frac{u_\infty \pi i}{K} + \log\left[\frac{\Theta(2u_\infty+iK') 2\mathrm{sn}(u_\infty)\mathrm{sn}'(u_\infty)}{\Theta'(iK')(\beta-\beta')\mathrm{sn}^2(u_\infty)} \right] \\ &= \frac{u_\infty \pi i}{K} + \log\left[\frac{2e^{\frac{-i\pi u_\infty}{K}} \vartheta_1(\frac{\pi u_\infty}{K})}{\pi\vartheta_1'(0)} \right] \\ &= \log\left[\frac{2\vartheta_1(\omega)}{\pi\vartheta_1'(0)} \right]. \end{aligned}$$

Plugging this into (4.63) gives

$$l = -2 + 2\log\left(\frac{\pi\vartheta_1'(0)}{2\vartheta_1(\omega)}\right),$$

and thus we obtain that

$$e^{\frac{l}{2}} = \frac{\pi\vartheta_1'(0)}{2e\vartheta_1(\omega)}.$$

4.9 Riemann-Hilbert approach: Interpolation problem

The Riemann-Hilbert approach to discrete orthogonal polynomials is based on the following Interpolation Problem (IP), which was introduced in the paper [11] of Borodin and Boyarchenko under the name of the discrete Riemann-Hilbert problem. See also the monograph [5] of Baik, Kriecherbauer, McLaughlin, and Miller, in which it is called the Interpolation Problem.

We will consider the lattice L_n defined in (4.5) and the weight $w_n(x)$ defined in (4.4).

Interpolation Problem. For a given $n = 0, 1, \ldots$, find a 2×2 matrix-valued function $\mathbf{P}_n(z) = (\mathbf{P}_{nij}(z))_{1\le i,j\le 2}$ with the following properties:

1. *Analyticity*: $\mathbf{P}_n(z)$ is an analytic function of z for $z \in \mathbb{C} \setminus L_n$.

2. *Residues at poles*: At each node $x \in L_n$, the elements $\mathbf{P}_{n11}(z)$ and $\mathbf{P}_{n21}(z)$ of the matrix $\mathbf{P}_n(z)$ are analytic functions of z, and the elements $\mathbf{P}_{n12}(z)$ and $\mathbf{P}_{n22}(z)$ have a simple pole with the residues,

$$\operatorname*{Res}_{z=x} \mathbf{P}_{nj2}(z) = w_n(x)\mathbf{P}_{nj1}(x), \quad j = 1, 2. \tag{4.64}$$

3. *Asymptotics at infinity*: There exists a function $r(x) > 0$ on L_n such that

$$\lim_{x\to\infty} r(x) = 0,$$

and such that as $z \to \infty$, $\mathbf{P}_n(z)$ admits the asymptotic expansion,

$$\mathbf{P}_n(z) \sim \left(I + \frac{\mathbf{P}_1}{z} + \frac{\mathbf{P}_2}{z^2} + \ldots\right)\begin{pmatrix} z^n & 0 \\ 0 & z^{-n}\end{pmatrix}, \qquad z \in \mathbb{C} \setminus \left[\bigcup_{x\in L_n}^{\infty} D\big(x, r(x)\big)\right], \tag{4.65}$$

where $D(x, r(x))$ denotes a disk of radius $r(x) > 0$ centered at x and I is the identity matrix.

It is not difficult to see (see [5] and [11]) that the IP has a unique solution, which is

$$\mathbf{P}_n(z) = \begin{pmatrix} P_{nn}(z) & C(w_n P_{nn})(z) \\ (h_{n,n-1})^{-1} P_{n,n-1}(z) & (h_{n,n-1})^{-1} C(w_n P_{n,n-1})(z) \end{pmatrix}, \qquad (4.66)$$

where the Cauchy transformation C is defined by the formula

$$C(f)(z) = \sum_{x \in L_n} \frac{f(x)}{z - x},$$

and $P_{nk}(z) = z^k + \ldots$ are monic polynomials orthogonal with respect to the weight $w_n(x)$. Because of the orthogonality condition, as $z \to \infty$,

$$C(w_n P_{nk})(z) = \sum_{x \in L_n}^{\infty} \frac{w_n(x) P_{nk}(x)}{z - x} \sim \sum_{x \in L_n}^{\infty} w_n(x) P_{nk}(x) \sum_{j=0}^{\infty} \frac{x^j}{z^{j+1}} = \frac{h_{nk}}{z^{k+1}} + \sum_{j=k+2}^{\infty} \frac{a_j}{z^j},$$

which justifies asymptotic expansion (4.65). We have that

$$h_{nn} = [\mathbf{P}_1]_{12}, \qquad h_{n,n-1}^{-1} = [\mathbf{P}_1]_{21}. \qquad (4.67)$$

4.10 Reduction of IP to RHP

We would like to reduce the Interpolation Problem to a Riemann-Hilbert Problem (RHP). That is, we would like to replace the condition (4.64) on poles of the matrix $\mathbf{P}$ with a jump condition on some contour on the complex plane. In order to do so, we begin with some preliminaries. Introduce the function,

$$\Pi(z) = \frac{2\gamma}{n\pi} \sin\left(\frac{n\pi z}{2\gamma}\right).$$

Observe that

$$\Pi(x_k) = 0, \quad \Pi'(x_k) = (-1)^k, \quad \exp\left(\frac{in\pi x_k}{2\gamma}\right) = (-1)^k \quad \text{for} \quad x_k = \frac{2\gamma k}{n} \in L_n.$$

Introduce the upper triangular matrices,

$$\mathbf{D}_{\pm}^{u}(z) = \begin{pmatrix} 1 & -\frac{w_n(z)}{\Pi(z)} e^{\pm\frac{in\pi z}{2\gamma}} \\ 0 & 1 \end{pmatrix},$$

and the lower triangular matrices,

$$\mathbf{D}^l_\pm = \begin{pmatrix} \Pi(z)^{-1} & 0 \\ -\frac{1}{w_n(z)} e^{\pm \frac{in\pi z}{2\gamma}} & \Pi(z) \end{pmatrix} = \begin{pmatrix} \Pi(z)^{-1} & 0 \\ 0 & \Pi(z) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{\Pi(z) w_n(z)} e^{\pm \frac{in\pi z}{2\gamma}} & 1 \end{pmatrix}.$$

Define the matrix-valued functions,

$$\mathbf{R}^u_n = \mathbf{P}_n(z) \times \begin{cases} \mathbf{D}^u_+(z) & \text{when} \quad \operatorname{Im} z \geq 0 \\ \mathbf{D}^u_-(z) & \text{when} \quad \operatorname{Im} z \leq 0 \end{cases} \tag{4.68}$$

and

$$\mathbf{R}^l_n = \mathbf{P}_n(z) \times \begin{cases} \mathbf{D}^l_+(z), & \text{when} \quad \operatorname{Im} z \geq 0 \\ \mathbf{D}^l_-(z), & \text{when} \quad \operatorname{Im} z \leq 0. \end{cases}$$

From (4.66) we have that

$$\mathbf{R}^u_n(z) = \begin{pmatrix} P_{nn}(z) & -\frac{w_n(z) P_{nn}(z)}{\Pi(z)} e^{\pm \frac{in\pi z}{2\gamma}} + C(w_n P_{nn})(z) \\ h^{-1}_{n,n-1} P_{n,n-1}(z) & -\frac{w_n(z) h^{-1}_{n,n-1} P_{n,n-1}(z)}{\Pi(z)} e^{\pm \frac{in\pi z}{2\gamma}} + h^{-1}_{n,n-1} C(w_n P_{n,n-1})(z) \end{pmatrix}$$

$$\text{when} \quad \pm \operatorname{Im} z \geq 0,$$

and

$$\mathbf{R}^l_n(z) = \begin{pmatrix} \frac{P_{nn}(z)}{\Pi(z)} - \frac{C(w_n P_{nn})(z)}{w_n(z)} e^{\pm \frac{in\pi z}{2\gamma}} & \Pi(z) C(w_n P_{nn})(z) \\ \frac{h^{-1}_{n-1} P_{n,n-1}(z)}{\Pi(z)} - \frac{h^{-1}_{n,n-1} C(w_n P_{n,n-1})(z)}{w_n(z)} e^{\pm \frac{in\pi z}{2\gamma}} & \Pi(z) h^{-1}_{n,n-1} C(w_n P_{n,n-1})(z) \end{pmatrix}$$

$$\text{when} \quad \pm \operatorname{Im} z \geq 0.$$

Observe that the functions $\mathbf{R}^u_n(z)$ and $\mathbf{R}^l_n(z)$ are meromorphic on the closed quadrants of the complex plane, and that they are two-valued on the real and imaginary axes. Their possible poles are located on the lattice L_n. An important result is that, due to some cancellations, they do not have any poles at all. We have the following proposition.

Proposition 4.10.1 *The matrix-valued functions $\mathbf{R}^u_n(z)$ and $\mathbf{R}^l_n(z)$ have no poles and on the real line they satisfy the following jump conditions at $x \in \mathbb{R}$:*

$$\mathbf{R}^u_{n+}(x) = \mathbf{R}^u_{n-}(x) j^u_R(x), \qquad j^u_R(x) = \begin{pmatrix} 1 & -\frac{n\pi i w_n(x)}{\gamma} \\ 0 & 1 \end{pmatrix}, \tag{4.69}$$

and

$$\mathbf{R}^l_{n+}(x) = \mathbf{R}^l_{n-}(x) j^l_R(x), \qquad j^l_R(x) = \begin{pmatrix} 1 & 0 \\ -\frac{n\pi i}{\gamma w_n(x)} & 1 \end{pmatrix}. \tag{4.70}$$

Proof It follows from the definition of $\mathbf{R}^u_n(z)$ that all possible poles of $\mathbf{R}^u_n(z)$ are located on the lattice L_n. Let us show that the residue of all these poles is equal to zero. Consider any $x_k \in L_n$. The residue of the matrix element $\mathbf{R}^u_{n,12}(z)$ at x_k is equal to

$$\operatorname*{Res}_{z=x_k} \mathbf{R}^u_{n,12}(z) = -\frac{w_n(x_k)P_{nn}(x_k)}{(-1)^k}(-1)^k + w_n(x_k)P_{nn}(x_k) = 0.$$

Similarly we get that

$$\operatorname*{Res}_{z=x_k} \mathbf{R}_{n,22}(z) = 0,$$

hence $\mathbf{R}^u_n(z)$ has no pole at x_k.

Similarly, the residue of the matrix element $\mathbf{R}^l_{n,11}(z)$ at x_k is equal to

$$\operatorname*{Res}_{z=x_k} \mathbf{R}^l_{n,11}(z) = \frac{P_{nn}(x_k)}{(-1)^k} - \frac{w_n(x_k)P_{nn}(x_k)(-1)^k}{w_n(x_k)} = 0.$$

In the same way we obtain that

$$\operatorname*{Res}_{z=x_k} \mathbf{R}_{n,21}(z) = 0.$$

In the entry $\mathbf{R}^l_{n,21}(z)$, the pole of the function $C(w_nP_n)(z)$ at $z = x_k$ is cancelled by the zero of the function $\Pi(z)$, hence $\mathbf{R}^l_{n,21}(z)$ has no pole at x_k. Similarly, $\mathbf{R}^l_{n,22}(z)$ has no pole at x_k as well, hence $\mathbf{R}^l_n(z)$ has no pole at x_k.

Let us evaluate the jump matrices at $x \in \mathbb{R}$. From (4.68) we have that

$$j^u_R(x) = \mathbf{D}^u_-(x)^{-1}\mathbf{D}^u_+(x) = \begin{pmatrix} 1 & -\frac{w_n(x)}{\Pi(x)} 2i \sin\frac{n\pi x}{2\gamma} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{n\pi i w_n(x)}{\gamma} \\ 0 & 1 \end{pmatrix},$$

which proves (4.69). Similarly,

$$j^l_R(x) = \mathbf{D}^l_-(x)^{-1}\mathbf{D}^l_+(x) = \begin{pmatrix} 1 & 0 \\ -\frac{1}{\Pi(x)w_n(x)} 2i \sin\frac{n\pi x}{2\gamma} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{n\pi i}{\gamma w_n(x)} & 1 \end{pmatrix},$$

which proves (4.70). ■

We are now in a position to reduce the Interpolation Problem to a Riemann-Hilbert Problem. We follow the work [5] with some modifications. Denote

$$\Delta = L_n \cap [\alpha', \beta'], \qquad \nabla = L_n \setminus \Delta.$$

Consider the oriented contour Σ on the complex plane depicted on Figure 4.4, in which the horizontal lines are $\operatorname{Im} z = \varepsilon, 0, -\varepsilon$, where $\varepsilon > 0$ is a small positive constant which will be determined later, and the vertical segments pass through the points $z = \alpha'$ and $z = \beta'$. Consider the regions $\Omega_\pm^\Delta$ and $\Omega_\pm^\nabla$ bounded by the contour Σ, see Figure 4.4. Observe that the regions $\Omega_\pm^\nabla$ consist of two connected components, to the left and to the right of $\Omega_\pm^\Delta$.

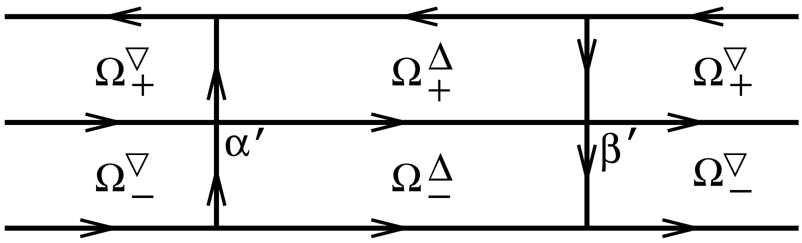

Fig. 4.4. The contour Σ.

Define

$$\mathbf{R}_n(z) = \begin{cases} \mathbf{K}_n \mathbf{R}_n^u(z) \mathbf{K}_n^{-1} & \text{for} \quad z \in \Omega_\pm^\nabla \\ \mathbf{K}_n \mathbf{R}_n^l(z) \mathbf{K}_n^{-1} & \text{for} \quad z \in \Omega_\pm^\Delta \\ \mathbf{K}_n \mathbf{P}_n(z) \mathbf{K}_n^{-1} & \text{otherwise.} \end{cases} \tag{4.71}$$

where $\mathbf{K}_n = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{n\pi i}{\gamma} \end{pmatrix}$.

Define a contour Σ_R by adding to the contour Σ a vertical segment $[i\varepsilon, -i\varepsilon]$, see Figure 4.5. If $A \subset \mathbb{C}$ is a set on the complex plane and $b \in \mathbb{C}$ then, as usual, we denote

$$A + b = \{z = a + b,\ a \in A\}.$$

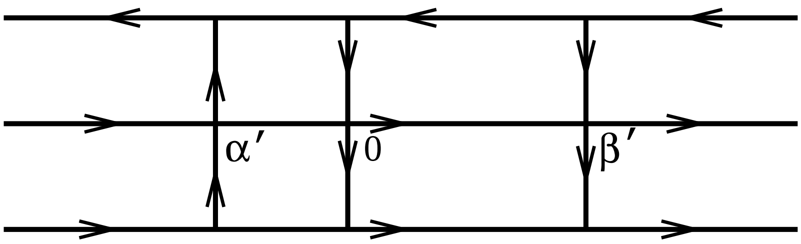

Fig. 4.5. The contour Σ_R.

Proposition 4.10.2 *The matrix-valued function* $\mathbf{R}_n(z)$ *has the following jumps on the contour* Σ_R*:*

$$\mathbf{R}_{n+}(z) = \mathbf{R}_{n-}(z) j_R(z),$$

where

$$j_R(z) = \begin{cases} \begin{pmatrix} 1 & w(z) \\ 0 & 1 \end{pmatrix} \quad \text{when} \quad z \in (-\infty, \alpha') \cup (\beta', \infty) \\ \begin{pmatrix} 1 & 0 \\ -(\frac{n\pi}{\gamma})^2 w(z)^{-1} & 1 \end{pmatrix} \quad \text{when} \quad z \in [\alpha', \beta'] \\ \mathbf{K}_n \mathbf{D}^u_\pm(z) \mathbf{K}_n^{-1} = \begin{pmatrix} 1 & -\frac{i\gamma}{n\pi} \frac{w_n(z) e^{\pm \frac{in\pi z}{2\gamma}}}{\Pi(z)} \\ 0 & 1 \end{pmatrix} \\ \qquad \text{when} \quad z \in (-\infty, \alpha') \cup (\beta', \infty) \pm i\varepsilon \\ \mathbf{K}_n \mathbf{D}^l_\pm(z) \mathbf{K}_n^{-1} = \begin{pmatrix} \Pi(z)^{-1} & 0 \\ \frac{in\pi}{\gamma} \frac{e^{\pm \frac{in\pi z}{2\gamma}}}{w_n(z)} & \Pi(z) \end{pmatrix} \quad \text{when} \quad z \in (\alpha', \beta') \pm i\varepsilon \\ \mathbf{K}_n \mathbf{D}^l_\pm(z)^{-1} \mathbf{D}^u_\pm(z) \mathbf{K}_n^{-1} = \begin{pmatrix} \Pi(z) & \frac{\gamma}{n\pi i} w_n(z) e^{\pm \frac{in\pi z}{2\gamma}} \\ -\frac{n\pi i}{\gamma} w_n(z)^{-1} e^{\pm \frac{in\pi z}{2\gamma}} & \mp \frac{n\pi i}{\gamma} e^{\pm \frac{in\pi z}{2\gamma}} \end{pmatrix} \\ \qquad \text{when} \quad z \in (0, \pm i\varepsilon) + \alpha' \quad \text{or} \quad z \in (0, \pm i\varepsilon) + \beta' \\ \mathbf{K}_n \mathbf{D}^0_\pm(z) \mathbf{K}_n^{-1} \quad \text{when} \quad z \in (0, \pm i\varepsilon) \end{cases} \tag{4.72}$$

and

$$\mathbf{D}^0_\pm(z) = \begin{pmatrix} 1 & 0 \\ -\frac{2\sinh(nz) e^{-n\zeta z} e^{\pm \frac{in\pi z}{2\gamma}}}{\Pi(z)} & 1 \end{pmatrix}.$$

Notice that the jumps on vertical contours close to the origin, $\mathbf{D}^0_\pm(z)$, are exponentially close to the identity matrix.

4.11 First transformation of the RHP

Define the matrix function $\mathbf{T}_n(z)$ as follows from the equation,

$$\mathbf{R}_n(z) = e^{\frac{nl}{2}\sigma_3}\mathbf{T}_n(z)e^{n(g(z)-\frac{l}{2})\sigma_3},$$

where the Lagrange multiplier l and the function $g(z)$ are as described in Section 4.3 and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the third Pauli matrix. Then $\mathbf{T}_n(z)$ satisfies the following Riemann-Hilbert Problem:

1. $\mathbf{T}_n(z)$ is analytic in $\mathbb{C} \setminus \Sigma_R$.

2. $\mathbf{T}_{n+}(z) = \mathbf{T}_{n-}(z)j_T(z)$ for $z \in \Sigma_R$, where

$$j_T(z) = \begin{cases} e^{n(g_-(z)-\frac{l}{2})\sigma_3}j_R(z)e^{-n(g_+(z)-\frac{l}{2})\sigma_3} & \text{for} \quad z \in \mathbb{R} \\ e^{n(g(z)-\frac{l}{2})\sigma_3}j_R(z)e^{-n(g(z)-\frac{l}{2})\sigma_3} & \text{for} \quad z \in \Sigma_R \setminus \mathbb{R}. \end{cases} \tag{4.73}$$

3. As $z \to \infty$,

$$\mathbf{T}_n(z) \sim I + \frac{\mathbf{T}_1}{z} + \frac{\mathbf{T}_2}{z^2} + \dots .$$

From (4.27) we have that

$$g(z) = \log z + O(z^{-1}) \quad \text{as} \quad z \to \infty.$$

This implies that

$$[\mathbf{T}_1]_{12} = e^{-nl}[\mathbf{R}_1]_{12}. \tag{4.74}$$

Let's take a closer look at the behavior of the jump matrix j_T described in (4.73) on the horizontal segments of Σ_R. We have that

$$j_T(z) = \begin{cases} \begin{pmatrix} e^{-nG(z)} & e^{n(g_+(z)+g_-(z)-V(z)-l)} \\ 0 & e^{nG(z)} \end{pmatrix} & \text{when} \quad z \in (-\infty, \alpha') \cup (\beta', \infty) \\ \begin{pmatrix} e^{-nG(z)} & 0 \\ -(\frac{n\pi}{\gamma})^2 e^{-n(g_+(z)+g_-(z)-V(z)-l)} & e^{nG(z)} \end{pmatrix} & \text{when} \quad z \in (\alpha', \beta') \\ \begin{pmatrix} 1 & \pm\frac{e^{\pm nG(z)}}{1-e^{\frac{\mp in\pi}{\gamma}} e^{\frac{\varepsilon n\pi x}{\gamma}}} \\ 0 & 1 \end{pmatrix} & \text{when} \quad z = x \pm i\varepsilon \in (\alpha, \alpha') \cup (\beta', \beta) \pm i\varepsilon \\ \begin{pmatrix} 1 & \pm\frac{e^{n(2g(z)-l-V(z))}}{1-e^{\frac{\mp in\pi x}{\gamma}} e^{\frac{\varepsilon n\pi}{\gamma}}} \\ 0 & 1 \end{pmatrix} & \text{when} \quad z = x \pm i\varepsilon \in (-\infty, \alpha) \cup (\beta, \infty) \pm i\varepsilon \\ \begin{pmatrix} \Pi(z)^{-1} & 0 \\ \frac{in\pi}{\gamma} e^{\pm\frac{in\pi x}{2\gamma}} e^{-n(2g(z)-V(z)-l)} & \Pi(z) \end{pmatrix} & \text{when} \quad z \in (\alpha', \beta') \pm i\varepsilon. \end{cases}$$

According to the properties of the g-function, we have the following proposition:

Proposition 4.11.1 *The jump function j_T has the following large n asymptotics:*

$$j_T(z) = \begin{cases} \begin{pmatrix} e^{-nG(z)} & 0 \\ O(e^{-nC(z)}) & e^{nG(z)} \end{pmatrix} & \textit{for} \quad z \in (\alpha', \beta') \\ \begin{pmatrix} e^{-nG(z)} & 1 \\ 0 & e^{nG(z)} \end{pmatrix} & \textit{for} \quad z \in (\alpha, \alpha') \cup (\beta', \beta) \\ \begin{pmatrix} 1 & O(e^{-nC(z)}) \\ 0 & 1 \end{pmatrix} & \textit{for} \quad z \in (-\infty, \alpha) \cup (\beta, \infty) \\ \begin{pmatrix} 1 & e^{\pm nG(z)} O(e^{-\frac{\varepsilon n\pi}{\gamma}}) \\ 0 & 1 \end{pmatrix} & \textit{for} \quad z \in (\alpha, \alpha') \cup (\beta', \beta) \pm i\varepsilon, \end{cases}$$

where $C(z)$ is a positive continuous function on any subset of the given interval which is bounded away from the endpoints of each interval and satisfies

$$C(z) > c(|z| + 1) \quad \textit{for some } c > 0. \tag{4.75}$$

4.12 Second transformation of the RHP

The second transformation is based on two observations. The first is the well known "opening of the lenses" in a neighborhood of the unconstrained support of the equilibrium measure. Namely, notice that, for $x \in (\alpha, \alpha') \cup (\beta', \beta)$, the jump matrix $j_T(x)$ factorizes as

$$j_T(x) = \begin{pmatrix} e^{-nG(z)} & 1 \\ 0 & e^{nG(z)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ e^{nG(x)} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{-nG(x)} & 1 \end{pmatrix} \tag{4.76}$$
$$= j_-(x) j_M j_+(x),$$

which allows us to reduce the jump matrix j_T to the one j_M plus asymptotically small jumps on the lens boundaries. The second observation consists of two facts. Firstly, the jumps on the segments $[\alpha', \beta'] \pm i\varepsilon$ behave, for large n, as $\pm e^{\pm \frac{in\pi z}{2\gamma}}$. Secondly, note that, for $x \in [\alpha', \beta']$, $G(x)$ is a linear function with slope $-\frac{\pi i}{\gamma}$. With these facts in mind, we make the second transformation of the RHP. Let

$$\mathbf{S}_n(z) = \begin{cases} \mathbf{T}_n(z) j_+(z)^{-1} \quad \text{for} \quad z \in \{(\alpha, \alpha') \cup (\beta', \beta)\} \times (0, i\varepsilon) \\ \mathbf{T}_n(z) j_-(z) \quad \text{for} \quad z \in \{(\alpha, \alpha') \cup (\beta', \beta)\} \times (0, -i\varepsilon) \\ \mathbf{T}_n(z) \begin{pmatrix} -\frac{\gamma}{n\pi i} e^{-\frac{in\pi z}{2\gamma}} & 0 \\ 0 & -\frac{n\pi i}{\gamma} e^{\frac{in\pi z}{2\gamma}} \end{pmatrix} \quad \text{for} \quad z \in (\alpha', \beta') \times (0, i\varepsilon) \\ \mathbf{T}_n(z) \begin{pmatrix} \frac{\gamma}{n\pi i} e^{\frac{in\pi z}{2\gamma}} & 0 \\ 0 & \frac{n\pi i}{\gamma} e^{-\frac{in\pi z}{2\gamma}} \end{pmatrix} \quad \text{for} \quad z \in (\alpha', \beta') \times (0, -i\varepsilon) \\ \mathbf{T}_n(z) \quad \text{otherwise.} \end{cases} \tag{4.77}$$

This function satisfies a similar RHP to $\mathbf{T}$, but jumps now occur on a new contour, Σ_S, which is obtained from Σ_R by adding the two segments $(\alpha - i\varepsilon, \alpha + i\varepsilon)$ and $(\beta - i\varepsilon, \beta + i\varepsilon)$, see Figure 4.6.

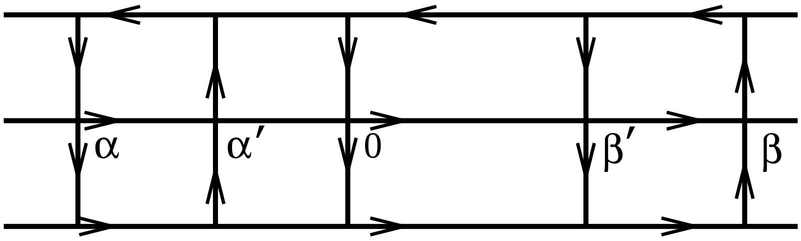

Fig. 4.6. The contour Σ_S.

On the horizontal segments for which the jump function j_S differs from j_T , we have that, as $n \to \infty$,

$$j_S(z) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{for} \quad z \in (\alpha, \alpha') \cup (\beta', \beta) \\ \begin{pmatrix} 1 + O(e^{-\varepsilon n\pi/\gamma}) & O(e^{n[G(z) - \varepsilon\pi/\gamma]}) \\ -e^{-nG(z)} & 1 \end{pmatrix} & \text{for} \quad z - i\varepsilon \in (\alpha, \alpha') \cup (\beta', \beta) \\ \begin{pmatrix} 1 + O(e^{-\varepsilon n\pi/\gamma}) & O(e^{n[-G(z) - \varepsilon n\pi/\gamma]}) \\ e^{nG(z)} & 1 \end{pmatrix} & \text{for} \quad z + i\varepsilon \in (\alpha, \alpha') \cup (\beta', \beta) \\ \begin{pmatrix} 1 + O(e^{-\varepsilon n\pi/\gamma}) & 0 \\ \frac{n\pi i}{\gamma} e^{-n(2g(z) - l - V(z))} & 1 + O(e^{-\varepsilon n\pi/\gamma}) \end{pmatrix} & \text{for} \quad z \in [\alpha', \beta'] \pm i\varepsilon \\ \begin{pmatrix} -e^{-n\pi i(1+\zeta)} & 0 \\ e^{-n(g_+(z) + g_-(z) - l - V(z))} & -e^{n\pi i(1+\zeta)} \end{pmatrix} & \text{for} \quad z \in [\alpha', \beta']. \end{cases}$$

By formula (4.31) for the G-function and the upper constraint (4.17) on the density ρ, we obtain that, for sufficiently small $\varepsilon > 0$ and $x \in (\alpha, \alpha') \cup (\beta', \beta)$,

$$0 < \mp \operatorname{Re} G(x \pm i\varepsilon) = 2\pi\rho(x) + O(\varepsilon^2) < \frac{\pi\varepsilon}{\gamma} + O(\varepsilon^2).$$

This, combined with property (4.30) of the g-function, imply that all jumps on horizontal segments are exponentially close to the identity matrix, provided that they are bounded away from the interval $[\alpha, \beta]$. For what follows we denote

$$\Omega_n = \pi + n2\pi \int_0^\beta \rho(x)\, dx = \pi + n\pi(1 + \zeta), \tag{4.78}$$

so that

$$-e^{-n\pi i(1+\zeta)} = e^{-i\Omega_n}.$$

4.13 Model RHP

The model RHP appears when we drop in the jump matrix $j_S(z)$ the terms that vanish as $n \to \infty$:

1. $\mathbf{M}(z)$ is analytic in $\mathbb{C} \setminus [\alpha, \beta]$.

2. $\mathbf{M}_+(z) = \mathbf{M}_-(z) j_M(z)$ for $z \in [\alpha, \beta]$, where

$$j_M(z) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{for} \quad z \in [\alpha, \alpha'] \cup [\beta, \beta'] \\ e^{-i\Omega_n \sigma_3} & \text{for} \quad z \in [\alpha', \beta']. \end{cases} \tag{4.79}$$

3. As $z \to \infty$,

$$\mathbf{M}(z) \sim I + \frac{\mathbf{M}_1}{z} + \frac{\mathbf{M}_2}{z^2} + \dots. \tag{4.80}$$

This model problem was first solved, in the general multi-cut case, in [16] (see also [15]), and is solved in two steps. In the first step, we solve the following auxiliary RHP:

1. $\mathbf{Q}(z)$ is analytic in $\mathbb{C} \setminus [\alpha, \alpha'] \cup [\beta', \beta]$.

2. $\mathbf{Q}_+(z) = \mathbf{Q}_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ for $z \in [\alpha, \alpha'] \cup [\beta', \beta]$.

3. $\mathbf{Q}(z) = I + O(z^{-1})$ as $z \to \infty$.

This RHP has the unique solution (see [16])

$$\mathbf{Q}(z) = \begin{pmatrix} \frac{\gamma(z)+\gamma^{-1}(z)}{2} & \frac{\gamma(z)-\gamma^{-1}(z)}{-2i} \\ \frac{\gamma(z)-\gamma^{-1}(z)}{2i} & \frac{\gamma(z)+\gamma^{-1}(z)}{2} \end{pmatrix} \tag{4.81}$$

where

$$\gamma(z) = \left(\frac{(z-\alpha)(z-\beta')}{(z-\alpha')(z-\beta)} \right)^{1/4} \tag{4.82}$$

with cuts on $[\alpha, \alpha'] \cup [\beta', \beta]$.

To solve the model RHP described in (4.79) and (4.80), we again use elliptic functions. Define the function

$$f(s) = \frac{\vartheta_3(s+d+c)}{\vartheta_3(s+d)}$$

where ϑ_3 is as defined in (4.8) with elliptic nome $q = e^{i\pi\tau} = e^{\frac{-\pi^2}{2\gamma}}$ ($\tau = \frac{i\pi}{2\gamma}$), and d and c are arbitrary complex numbers. Notice that f has the periodic properties

$$f(s+\pi) = f(s)\,, \quad f(s+\pi\tau) = e^{-2ic} f(s)\,, \tag{4.83}$$

and that f is an even function of its argument. Now let

$$\tilde{u}(z) = \frac{\pi}{2K} u(z) = \frac{\pi}{2} \int_{\beta}^{z} \frac{dz'}{\sqrt{R(z')}} \tag{4.84}$$

where u is as defined in (4.34). Then $\tilde{u}$ is two-valued on $[\alpha, \beta]$ and satisfies

$$\tilde{u}_+(x) - \tilde{u}_-(x) = \pi\tau \quad \text{for} \quad x \in [\alpha', \beta']. \tag{4.85}$$

Also,

$$\tilde{u}_\pm(\alpha) = \frac{\pi}{2}\,, \quad \tilde{u}_\pm(\alpha') = \frac{\pi}{2} \pm \frac{\pi\tau}{2}\,, \quad \tilde{u}_\pm(\beta') = \pm\frac{\pi\tau}{2}\,, \quad \tilde{u}_\pm(\beta) = 0\,,$$

see Figure 4.3. Because $\sqrt{R(x)}_+ = -\sqrt{R(x)}_-$ for $x \in [\alpha, \alpha'] \cup [\beta', \beta]$, it immediately follows that

$$\tilde{u}_+(x) + \tilde{u}_-(x) = 0 \quad \text{for} \quad x \in [\beta', \beta], \tag{4.86}$$

and that

$$\tilde{u}_+(x) + \tilde{u}_-(x) = \tilde{u}_+(\alpha') - \tilde{u}_+(\beta') + \tilde{u}_-(\alpha') - \tilde{u}_-(\beta') = \pi \quad \text{for} \quad x \in [\alpha, \alpha']. \tag{4.87}$$

We now define

$$f_1(z) = \frac{\vartheta_3(\tilde{u}(z) + d + \frac{\Omega_n}{2})}{\vartheta_3(\tilde{u}(z) + d)}\,, \quad f_2(z) = \frac{\vartheta_3(-\tilde{u}(z) + d + \frac{\Omega_n}{2})}{\vartheta_3(-\tilde{u}(z) + d)} \quad \text{for} \quad z \in \mathbb{C} \setminus [\alpha, \beta],$$

where d is an arbitrary complex number. It then follows from (4.83) and(4.85) that

$$f_{1+}(x) = e^{-i\Omega_n} f_{1-}(x) \quad \text{and} \quad f_{2+}(x) = e^{i\Omega_n} f_{2-}(x) \quad \text{for} \quad x \in [\alpha', \beta'], \tag{4.88}$$

and from (4.83), (4.86), and (4.87) that

$$f_{1+}(x) = f_{2-}(x) \quad \text{and} \quad f_{2+}(x) = f_{1-}(x) \quad \text{for} \quad x \in [\alpha, \alpha'] \cup [\beta'\beta]. \tag{4.89}$$

Define the matrix valued function

$$\mathbf{F}(z) = \begin{pmatrix} \frac{\vartheta_3(\tilde{u}(z)+d_1+\frac{\Omega_n}{2})}{\vartheta_3(\tilde{u}(z)+d_1)} & \frac{\vartheta_3(-\tilde{u}(z)+d_1+\frac{\Omega_n}{2})}{\vartheta_3(-\tilde{u}(z)+d_1)} \\ \frac{\vartheta_3(\tilde{u}(z)+d_2+\frac{\Omega_n}{2})}{\vartheta_3(\tilde{u}(z)+d_2)} & \frac{\vartheta_3(-\tilde{u}(z)+d_2+\frac{\Omega_n}{2})}{\vartheta_3(-\tilde{u}(z)+d_2)} \end{pmatrix} \tag{4.90}$$

where d_1 and d_2 are yet undetermined complex constants. Then, from (4.88) and(4.89) we have that

$$\mathbf{F}_+(x) = \mathbf{F}_-(x) \begin{pmatrix} e^{-i\Omega_n} & 0 \\ 0 & e^{i\Omega_n} \end{pmatrix} \qquad \text{for} \quad x \in [\alpha', \beta'],$$

$$\mathbf{F}_+(x) = \mathbf{F}_-(x) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{for} \quad x \in [\alpha, \alpha'] \cup [\beta', \beta].$$

We can now combine (4.81) and (4.90) to obtain

$$\mathbf{M}(z) = \mathbf{F}(\infty)^{-1} \begin{pmatrix} \frac{\gamma(z)+\gamma^{-1}(z)}{2} \frac{\vartheta_3(\tilde{u}(z)+d_1+\frac{\Omega_n}{2})}{\vartheta_3(\tilde{u}(z)+d_1)} & \frac{\gamma(z)-\gamma^{-1}(z)}{-2i} \frac{\vartheta_3(-\tilde{u}(z)+d_1+\frac{\Omega_n}{2})}{\vartheta_3(-\tilde{u}(z)+d_1)} \\ \frac{\gamma(z)-\gamma^{-1}(z)}{2i} \frac{\vartheta_3(\tilde{u}(z)+d_2+\frac{\Omega_n}{2})}{\vartheta_3(\tilde{u}(z)+d_2)} & \frac{\gamma(z)+\gamma^{-1}(z)}{2} \frac{\vartheta_3(-\tilde{u}(z)+d_2+\frac{\Omega_n}{2})}{\vartheta_3(-\tilde{u}(z)+d_2)} \end{pmatrix} \tag{4.91}$$

where

$$\mathbf{F}(\infty) = \begin{pmatrix} \frac{\vartheta_3(\tilde{u}_\infty+d_1+\frac{\Omega_n}{2})}{\vartheta_3(\tilde{u}_\infty+d_1)} & 0 \\ 0 & \frac{\vartheta_3(-\tilde{u}_\infty+d_2+\frac{\Omega_n}{2})}{\vartheta_3(-\tilde{u}_\infty+d_2)} \end{pmatrix}. \tag{4.92}$$

and $\tilde{u}_\infty \equiv \tilde{u}(\infty)$. This matrix satisfies conditions (4.79) and (4.80) of the model RHP, but may not be analytic on $\mathbb{C} \setminus [\alpha, \beta]$, as it may have some poles at the zeroes of $\vartheta_3(\pm\tilde{u}(z) + d_{1,2})$. However, we can choose the constants d_1 and d_2 such that these zeroes coincide with the zeroes of $\gamma(z) \pm \gamma^{-1}(z)$ and are thus cancelled in the product.

First consider the zeroes of $\gamma(z) \pm \gamma^{-1}(z)$. These are the zeroes of $\gamma^2(z) \pm 1$ and thus of $\gamma^4(z) - 1$. Thus there is only one zero, which uniquely solves the equation

$$p(z) \equiv \frac{(z-\alpha)(z-\beta')}{(z-\alpha')(z-\beta)} = 1,$$

which is

$$x_0 = \frac{\beta\alpha' - \alpha\beta'}{(\alpha' - \alpha) + (\beta - \beta')} \in (\alpha', \beta').$$

It is easy to check that $\gamma(x_0) = 1$, thus x_0 is the unique zero of $\gamma(z) - \gamma^{-1}(z)$, whereas there are no zeroes of $\gamma(z) + \gamma^{-1}(z)$ on the specified sheet. We use here the change of variables v defined in (4.35). Notice that, by (4.46),

$$v(x_0) = \frac{\beta' - \alpha}{\beta' - \alpha'} = \frac{\mathrm{dn}^2(u_\infty)}{k^2\mathrm{cn}^2(u_\infty)}$$

implying that

$$\mathrm{sn}^2\big(u(x_0)\big) = \frac{\mathrm{dn}^2(u_\infty)}{k^2\mathrm{cn}^2(u_\infty)}. \tag{4.93}$$

Since $x_0 \in (\alpha', \beta')$, we must have $u(x_0) \in (iK', K + iK')$ (if we choose to take u_+). Since sn^2 is a one-to-one function on this interval, there is a unique point $u_0 \in (iK', K + iK')$ such that $\mathrm{sn}^2(u_0) = \frac{\mathrm{dn}^2(u_\infty)}{k^2\mathrm{cn}^2(u_\infty)}$. The simple period identity

$$\mathrm{sn}(u + K + iK') = \frac{\mathrm{dn}(u)}{k\mathrm{cn}(u)},$$

along with (4.93), gives that we must have

$$u_0 = u(x_0) = K - u_\infty + iK'.$$

Thus,

$$\tilde{u}(x_0) = \frac{\pi}{2K}(K - u_\infty + iK') = \frac{\tau\pi}{2} + \frac{\pi}{2} - \tilde{u}_\infty.$$

We now consider zeroes of the function $\vartheta_3(\tilde{u}(z) - d) \equiv \vartheta_3(-\tilde{u}(z) + d)$. The zeroes of this function are the solutions to the equation

$$\tilde{u}(z) - d = (2m + 1)\frac{\pi}{2} + (2k + 1)\frac{\tau\pi}{2}$$

for any $m, k \in \mathbb{Z}$. Because $\tilde{u}$ maps the first sheet of X to the rectangular domain $[0, \frac{\pi}{2}] \times [-\frac{\tau\pi}{2}, \frac{\tau\pi}{2}]$, it is clear that this equation can have at most one solution, and without any loss of generality we may take $m = k = 0$. Then, if we want the solution of this equation to be x_0, we need to let

$$d = \tilde{u}(x_0) - \frac{\pi}{2}(1 + \tau) = -\tilde{u}_\infty.$$

This choice of d also ensures that $\vartheta_3(\tilde{u}(z) + d) \equiv \vartheta_3(-\tilde{u}(z) - d)$ has no zeroes on the first sheet of X. We can then let

$$d_1 = d, \quad d_2 = -d$$

so that (4.91) and (4.92) become

$$\mathbf{M}(z) = \mathbf{F}(\infty)^{-1}\begin{pmatrix} \frac{\gamma(z)+\gamma^{-1}(z)}{2}\frac{\vartheta_3(\tilde{u}(z)+d+\frac{\Omega_n}{2})}{\vartheta_3(\tilde{u}(z)+d)} & \frac{\gamma(z)-\gamma^{-1}(z)}{-2i}\frac{\vartheta_3(-\tilde{u}(z)+d+\frac{\Omega_n}{2})}{\vartheta_3(-\tilde{u}(z)+d)} \\ \frac{\gamma(z)-\gamma^{-1}(z)}{2i}\frac{\vartheta_3(\tilde{u}(z)-d+\frac{\Omega_n}{2})}{\vartheta_3(\tilde{u}(z)-d)} & \frac{\gamma(z)+\gamma^{-1}(z)}{2}\frac{\vartheta_3(-\tilde{u}(z)-d+\frac{\Omega_n}{2})}{\vartheta_3(-\tilde{u}(z)-d)} \end{pmatrix}$$
$$= \mathbf{F}(\infty)^{-1}\begin{pmatrix} \frac{\gamma(z)+\gamma^{-1}(z)}{2}\frac{\vartheta_3(\tilde{u}(z)+(n+\frac{1}{2})\omega)}{\vartheta_4(\tilde{u}(z)+\frac{\omega}{2})} & \frac{\gamma(z)-\gamma^{-1}(z)}{-2i}\frac{\vartheta_3(\tilde{u}(z)-(n+\frac{1}{2})\omega)}{\vartheta_4(\tilde{u}(z)-\frac{\omega}{2})} \\ \frac{\gamma(z)-\gamma^{-1}(z)}{2i}\frac{\vartheta_3(\tilde{u}(z)+(n-\frac{1}{2})\omega)}{\vartheta_4(\tilde{u}(z)-\frac{\omega}{2})} & \frac{\gamma(z)+\gamma^{-1}(z)}{2}\frac{\vartheta_3(\tilde{u}(z)-(n-\frac{1}{2})\omega)}{\vartheta_4(\tilde{u}(z)+\frac{\omega}{2})} \end{pmatrix}, \tag{4.94}$$

where

$$\mathbf{F}(\infty) = \begin{pmatrix} \frac{\vartheta_3(\frac{\Omega_n}{2})}{\vartheta_3(0)} & 0 \\ 0 & \frac{\vartheta_3(\frac{\Omega_n}{2})}{\vartheta_3(0)} \end{pmatrix} = \begin{pmatrix} \frac{\vartheta_4(n\omega)}{\vartheta_3(0)} & 0 \\ 0 & \frac{\vartheta_4(n\omega)}{\vartheta_3(0)} \end{pmatrix}, \tag{4.95}$$

solving the model RHP. The asymptotics at infinity are

$$\mathbf{M}(z) = I + \frac{\mathbf{M}_1}{z} + O(z^{-2}),$$

where the matrix $\mathbf{M}_1$ has the form

$$\mathbf{M}_1 = \begin{pmatrix} * & \frac{\vartheta_3(-\tilde{u}_\infty+d+\frac{\Omega_n}{2})\vartheta_3(\tilde{u}_\infty+d)}{\vartheta_3(\tilde{u}_\infty+d+\frac{\Omega_n}{2})\vartheta_3(-\tilde{u}_\infty+d)}\frac{(\beta-\beta')+(\alpha'-\alpha)}{-4i} \\ \frac{\vartheta_3(\tilde{u}_\infty-d+\frac{\Omega_n}{2})\vartheta_3(-\tilde{u}_\infty-d)}{\vartheta_3(-\tilde{u}_\infty-d+\frac{\Omega_n}{2})\vartheta_3(\tilde{u}_\infty-d)}\frac{(\beta-\beta')+(\alpha'-\alpha)}{4i} & * \end{pmatrix}. \tag{4.96}$$

The matrix $\mathbf{M}_1$ can be written in a cleaner fashion and in terms of the original parameters as follows.

Proposition 4.13.1 *We have that*

$$[\mathbf{M}_1]_{12} = \frac{iA(\omega)\vartheta_4\big((n+1)\omega\big)}{\vartheta_4(n\omega)}, \qquad [\mathbf{M}_1]_{21} = \frac{A(\omega)\vartheta_4(n\omega)}{i\vartheta_4\big((n-1)\omega\big)}. \tag{4.97}$$

where

$$\omega = \frac{\pi(1+\zeta)}{2}, \qquad A(\omega) = \frac{\pi\vartheta_1'(0)}{2\vartheta_1(\omega)}.$$

For a proof of this proposition see Appendix D. Notice that since $\mathbf{M}$ solves the model RHP, we have that

$$\det \mathbf{M}(z) = 1, \qquad z \in \mathbb{C}.$$

4.14 Parametrix at outer turning points

We now consider small disks $D(\alpha, \varepsilon)$ and $D(\beta, \varepsilon)$ centered at the outer turning points. Denote $D = D(\alpha, \varepsilon) \cup D(\beta, \varepsilon)$. We will seek a local parametrix $\mathbf{U}_n(z)$ defined on D such that

1. $\mathbf{U}_n(z)$ is analytic on $D \setminus \Sigma_S$.

2. $\mathbf{U}_{n+}(z) = \mathbf{U}_{n-}(z) j_S(z) \quad \text{for} \quad z \in D \cap \Sigma_S$.

3. $\mathbf{U}_n(z) = \mathbf{M}(z)\big(I + O(n^{-1})\big) \quad$ uniformly for $z \in \partial D$.

We first construct the parametrix near β. The jumps j_S are given by

$$j_S(z) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{for } z \in (\beta - \varepsilon, \beta) \\ \begin{pmatrix} 1 & 0 \\ -e^{-nG(z)} & 1 \end{pmatrix} & \text{for } z \in (\beta, \beta + i\varepsilon) \\ \begin{pmatrix} 1 & 0 \\ e^{nG(z)} & 1 \end{pmatrix} & \text{for } z \in (\beta, \beta - i\varepsilon) \\ \begin{pmatrix} e^{-nG(z)} & e^{n(g_+(z)+g_-(z)-V(z)-l)} \\ 0 & e^{nG(z)} \end{pmatrix} & \text{for } z \in (\beta, \beta + \varepsilon). \end{cases}$$

If we let

$$\mathbf{U}_n(z) = \mathbf{Q}_n(z) e^{-n(g(z) - \frac{V(z)}{2} - \frac{l}{2})\sigma_3},$$

then the jump conditions on $\mathbf{Q}_n$ become

$$\mathbf{Q}_{n+}(z) = \mathbf{Q}_{n-}(z) j_Q(z)$$

where

$$j_Q(z) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{for } z \in (\beta - \varepsilon, \beta) \\ \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} & \text{for } z \in (\beta, \beta + i\varepsilon) \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} & \text{for } z \in (\beta, \beta - i\varepsilon) \\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \text{for } z \in (\beta, \beta + \varepsilon), \end{cases} \tag{4.98}$$

and orientation is from left to right on horizontal contours, and down to up on vertical contours, according to Figure 4.6.

$\mathbf{Q}_n$ can be constructed using Airy functions. The Airy function solves the differential equation $y'' = zy$, and has the following asymptotics at infinity (see, e.g. [40]):

$$\begin{aligned} \mathrm{Ai}(z) &= \frac{1}{2\sqrt{\pi} z^{1/4}} e^{-\frac{2}{3} z^{3/2}} \left(1 - \frac{5}{48} z^{-3/2} + O(z^{-3}) \right), \\ \mathrm{Ai}'(z) &= -\frac{1}{2\sqrt{\pi}} z^{1/4} e^{-\frac{2}{3} z^{3/2}} \left(1 + \frac{7}{48} z^{-3/2} + O(z^{-3}) \right), \end{aligned} \tag{4.99}$$

as $z \to \infty$ with $\arg z \in (-\pi + \varepsilon, \pi - \varepsilon)$ for any $\varepsilon > 0$. If we let

$$y_0(z) = \mathrm{Ai}(z), \quad y_1(z) = \omega \mathrm{Ai}(\omega z), \quad y_2(z) = \omega^2 \mathrm{Ai}(\omega^2 z)$$

where $\omega = e^{\frac{2\pi i}{3}}$, then the functions y_0, y_1, and y_2 satisfy the relation

$$y_0(z) + y_1(z) + y_2(z) = 0.$$

If we take

$$\Phi_\beta(z) = \begin{cases} \begin{pmatrix} y_0(z) & -y_2(z) \\ y_0'(z) & -y_2'(z) \end{pmatrix} & \text{for } \arg z \in \left(0, \frac{\pi}{2}\right) \\ \begin{pmatrix} -y_1(z) & -y_2(z) \\ -y_1'(z) & -y_2'(z) \end{pmatrix} & \text{for } \arg z \in \left(\frac{\pi}{2}, \pi\right) \\ \begin{pmatrix} -y_2(z) & y_1(z) \\ -y_2'(z) & y_1'(z) \end{pmatrix} & \text{for } \arg z \in \left(-\pi, -\frac{\pi}{2}\right) \\ \begin{pmatrix} y_0(z) & y_1(z) \\ y_0'(z) & y_1'(z) \end{pmatrix} & \text{for } \arg z \in \left(-\frac{\pi}{2}, 0\right), \end{cases}$$

then Φ_β satisfies jump conditions similar to (4.98), but for jumps on rays emanating from the origin rather than from β. We thus need to map the disk $D(\beta, \varepsilon)$ onto some convex neighborhood of the origin in order to take advantage of the function Φ_β. Our mapping should match the asymptotics of the Airy function in order to have the matching property (3) of the RHP.

To this end notice that, by (4.23), for $t \in [\beta', \beta]$, as $t \to \beta$,

$$\rho(t) = C(\beta - t)^{1/2} + O\big((\beta - t)^{3/2}\big), \quad C > 0.$$

It follows that, as $z \to \beta$ for $z \in (\beta', \beta)$,

$$\int_z^\beta \rho(t)dt = C_0(\beta - z)^{3/2} + O\big((\beta - z)^{5/2}\big), \quad C_0 = \frac{2}{3}C.$$

Thus

$$\psi_\beta(z) = -\left\{\frac{3\pi}{2}\int_z^\beta \rho(t)dt\right\}^{2/3}$$

is analytic at β, and so extends to a conformal map from $D(\beta, \varepsilon)$ (for small enough ε) onto a convex neighborhood of the origin. Furthermore,

$$\psi_\beta(\beta) = 0\,, \quad \psi_\beta'(\beta) > 0\,;$$

therefore ψ_β is real negative on $(\beta - \varepsilon, \beta)$ and real positive on $(\beta, \beta + \varepsilon)$. Also, we can slightly deform the vertical pieces of the contour Σ_S close to β, so that

$$\psi_\beta\{D(\beta, \varepsilon) \cap \Sigma_S\} = (-\varepsilon, \varepsilon) \cup (-i\varepsilon, i\varepsilon).$$

We now set

$$\mathbf{Q}_n(z) = \mathbf{E}_n^{\beta}(z)\Phi_\beta\big(n^{2/3}\psi_\beta(z)\big)$$

so that

$$\mathbf{U}_n(z) = \mathbf{E}_n^{\beta}(z)\Phi_\beta\big(n^{2/3}\psi_\beta(z)\big)e^{-n(g(z)-\frac{V(z)}{2}-\frac{l}{2})\sigma_3}$$

where

$$\mathbf{E}_n^{\beta}(z) = \mathbf{M}(z)\mathbf{L}_n^{\beta}(z)^{-1}, \quad \mathbf{L}_n^{\beta}(z) = \frac{1}{2\sqrt{\pi}}\begin{pmatrix} n^{-1/6}\psi_\beta^{-1/4}(z) & 0 \\ 0 & n^{1/6}\psi_\beta^{1/4}(z)\end{pmatrix}\begin{pmatrix} 1 & i \\ -1 & i\end{pmatrix},$$

and we take the branch of $\psi_\beta^{1/4}$ which is positive on $(\beta, \beta+\varepsilon)$ and has a cut on $(\beta-\varepsilon, \beta)$.

We claim that $\mathbf{E}_n^{\beta}(z)$ is analytic in $D(\beta, \varepsilon)$, thus $\mathbf{U}(z)$ has the jump conditions j_S. This is clear, as both $\mathbf{M}$ and $\mathbf{L}_n^{\beta}$ have jump the same constant jump, $\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$, on the interval $(\beta-\varepsilon, \beta]$, and are analytic elsewhere. The only other possible singularity for either $\mathbf{M}$ or $\mathbf{L}_n^{\beta}$ is the isolated singularity at β, and this is at most a fourth-root singularity, thus removable. It follows that $\mathbf{E}_n^{\beta}(z) = \mathbf{M}(z)\mathbf{L}_n^{\beta}(z)^{-1}$ is analytic on $D(\beta, \varepsilon)$, thus $\mathbf{U}_n$ has the prescribed jumps in $D(\beta, \varepsilon)$.

We are left only to prove the matching condition (3). Using (4.99), it is straightforward to check that, for z in each of the sectors of analyticity, $\Phi_\beta(n^{2/3}\psi_\beta(z))$ satisfies the following asymptotics as $n \to \infty$:

$$\Phi_\beta\big(n^{2/3}\psi_\beta(z)\big) = \frac{1}{2\sqrt{\pi}}n^{\frac{1}{6}\sigma_3}\psi_\beta(z)^{-\frac{1}{4}\sigma_3}\left[\begin{pmatrix} 1 & i \\ -1 & i\end{pmatrix} + \frac{\psi_\beta(z)^{-3/2}}{48n}\begin{pmatrix} -5 & 5i \\ -7 & -7i\end{pmatrix} + O(n^{-2})\right] \times e^{-\frac{2}{3}n\psi_\beta(z)^{3/2}\sigma_3}, \tag{4.100}$$

where we always take the principal branch of $\psi_\beta(z)^{3/2}$. As such, $\psi_\beta(z)^{3/2}$ is two-valued for $z \in (\beta-\varepsilon, \beta)$, so that

$$\left[\frac{2}{3}\psi_\beta(x)^{3/2}\right]_\pm = \mp\pi i\int_x^\beta \rho(t)dt. \tag{4.101}$$

Notice that, by (4.30) and (4.32), for $x \in (\beta-\varepsilon, \beta)$,

$$2g_\pm(x) - V(x) = l \pm 2\pi i\int_x^\beta \rho(t)dt.$$

This implies that

$$[2g_\pm(\beta) - V(\beta)] - [2g_\pm(x) - V(x)] = \mp 2\pi i \int_x^\beta \rho(t)dt\,.$$

Combining these equations with (4.101) gives

$$\left[\frac{2}{3}\psi_\beta(x)^{3/2}\right]_\pm = \frac{1}{2}\left[\big(2g_\pm(\beta) - V(\beta)\big) - \big(2g_\pm(x) - V(x)\big)\right]. \tag{4.102}$$

This equation can be extended into the upper and lower planes, respectively, giving

$$\frac{2}{3}\psi_\beta(z)^{3/2} = \frac{1}{2}\left[\big(2g_\pm(\beta) - V(\beta)\big) - \big(2g(z) - V(z)\big)\right] \quad \text{for } \pm \operatorname{Im} z > 0.$$

Since, by (4.14), $2g_\pm(\beta) - V(\beta) = l$, we get that

$$\frac{2}{3}\psi_\beta(z)^{3/2} = -g(z) + \frac{V(z)}{2} + \frac{l}{2}$$

for z throughout $D(\beta, \varepsilon)$. Plugging (4.100) and (4.14) into (4.14), we get, as $n \to \infty$,

$$\begin{aligned}
\mathbf{U}_n(z) &= \mathbf{M}(z)\mathbf{L}_n^\beta(z)^{-1}\frac{1}{2\sqrt{\pi}}n^{-\frac{1}{6}\sigma_3}\psi_\beta(z)^{-\frac{1}{4}\sigma_3}\left[\begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix} + \frac{\psi_\beta(z)^{-3/2}}{48n}\begin{pmatrix} -5 & 5i \\ -7 & -7i \end{pmatrix}\right. \\
&\quad \left. + O(n^{-2})\right] e^{n(g(z) - \frac{V(z)}{2} - \frac{l}{2})\sigma_3} e^{-n(g(z) - \frac{V(z)}{2} - \frac{l}{2})\sigma_3} \\
&= \mathbf{M}(z)\left[I + \frac{\psi_\beta(z)^{-3/2}}{48n}\begin{pmatrix} 1 & 6i \\ 6i & -1 \end{pmatrix} + O(n^{-2})\right].
\end{aligned}$$

Thus we have that $\mathbf{U}_n$ satisfies conditions (1), (2), and (3) of the RHP.

A similar construction gives the parametrix at the α. Namely, if we let

$$\psi_\alpha(z) = -\left\{\frac{3\pi}{2}\int_\alpha^z \rho(t)dt\right\}^{2/3}, \tag{4.103}$$

then ψ_α is analytic throughout $D(\alpha,\varepsilon)$, real valued on the real line, and has negative derivative at α. Close to α, the jumps j_Q become

$$j_Q(z) = \begin{cases} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \text{for } z \in (\alpha-\varepsilon, \alpha) \\ \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} & \text{for } z \in (\alpha, \alpha+i\varepsilon) \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} & \text{for } z \in (\alpha, \alpha-i\varepsilon) \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{for } z \in (\alpha, \alpha+\varepsilon), \end{cases}$$

where orientation is taken left to right on horizontal contours, and up to down on vertical contours according to Figure 4.6. After the change of variables ψ_α (and a slight deformation of vertical contours), these jumps become the following jumps close to the origin:

$$j_Q\big(\psi_\alpha(z)\big) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{for } \psi_\alpha(z) \in (-\varepsilon, 0) \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} & \text{for } \psi_\alpha(z) \in (0, i\varepsilon) \\ \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} & \text{for } \psi_\alpha(z) \in (0, -i\varepsilon) \\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \text{for } \psi_\alpha(z) \in (0, \varepsilon), \end{cases}$$

where orientation is taken right to left on horizontal contours, and down to up on vertical contours. These jump conditions are satisfied by the function

$$\Phi_\alpha(z) = \Phi_\beta(z) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then we can take

$$\mathbf{U}_n(z) = \mathbf{E}_n^\alpha(z)\Phi_\alpha\big(n^{2/3}\psi_\alpha(z)\big)e^{-n(g(z)-\frac{V(z)}{2}-\frac{l}{2})\sigma_3} \tag{4.104}$$

for $z \in D(\alpha,\varepsilon)$, where

$$\mathbf{E}_n^\alpha(z) = \mathbf{M}(z)\mathbf{L}_n^\alpha(z)^{-1},$$
$$\mathbf{L}_n^\alpha(z) = \frac{1}{2\sqrt{\pi}}\begin{pmatrix} n^{-1/6}\psi_\alpha^{-1/4}(z) & 0 \\ 0 & n^{1/6}\psi_\alpha^{1/4}(z)\end{pmatrix}\begin{pmatrix} -1 & i \\ 1 & i\end{pmatrix}.$$

Similar to (4.100), we have that in each sector of analyticity, $\Phi_\alpha\big(n^{2/3}\psi_\alpha(z)\big)$ satisfies

$$\Phi_\alpha\big(n^{2/3}\psi_\alpha(z)\big) = \frac{1}{2\sqrt{\pi}}n^{-\frac{1}{6}\sigma_3}\psi_\alpha(z)^{-\frac{1}{4}\sigma_3}\left[\begin{pmatrix} 1 & -i \\ -1 & -i\end{pmatrix} + \frac{\psi_\alpha(z)^{-3/2}}{48n}\begin{pmatrix} -5 & -5i \\ -7 & 7i\end{pmatrix}\right.$$
$$\left. + O(n^{-2})\right] \times e^{-\frac{2}{3}n\psi_\alpha(z)^{3/2}\sigma_3}. \tag{4.105}$$

Once again, we have that, for $x \in (\alpha, \alpha+\varepsilon)$, $\psi_\alpha(x)^{3/2}$ takes limiting values from above and below, so that

$$\left[\frac{2}{3}\psi_\alpha(x)^{3/2}\right]_\pm = \pm\pi i\int_\alpha^x \rho(t)dt.$$

In analogue to (4.102), we have

$$\frac{2}{3}\psi_\alpha(z)^{3/2} = \frac{1}{2}\Big[\big(2g_\pm(\alpha) - V(\alpha)\big) - \big(2g(z) - V(z)\big)\Big] \quad \text{for } \pm\operatorname{Im} z > 0.$$

Since, by (4.14), $2g_\pm(\alpha) - V(\alpha) = l \pm \pi i$, we get that

$$\frac{2}{3}\psi_\alpha(z)^{3/2} = -g(z) + \frac{V(z)}{2} + \frac{l}{2} \pm \pi i \quad \text{for} \quad \pm\operatorname{Im} z > 0. \tag{4.106}$$

Plugging (4.106) into (4.104) and (4.105) gives, as $n \to \infty$,

$$\begin{aligned}
\mathbf{U}_n(z) &= \mathbf{M}(z)\mathbf{L}_n^{\alpha}(z)^{-1}\frac{1}{2\sqrt{\pi}}n^{\frac{1}{6}\sigma_3}\psi_\alpha(z)^{-\frac{1}{4}\sigma_3}\left[\begin{pmatrix} 1 & -i \\ -1 & -i \end{pmatrix} + \frac{\psi_\alpha(z)^{-3/2}}{48n}\begin{pmatrix} -5 & -5i \\ -7 & 7i \end{pmatrix}\right. \\
&\quad \left. + O(n^{-2})\right] e^{n(g(z)-\frac{V(z)}{2}-\frac{l}{2})\sigma_3}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} e^{-n(g(z)-\frac{V(z)}{2}-\frac{l}{2})\sigma_3} \\
&= \mathbf{M}(z)\begin{pmatrix} -1 & i \\ 1 & i \end{pmatrix}^{-1}\left[\begin{pmatrix} -1 & i \\ 1 & i \end{pmatrix} + \frac{\psi_\alpha(z)^{-3/2}}{48n}\begin{pmatrix} 5 & 5i \\ 7 & -7i \end{pmatrix} + O(n^{-2})\right] \\
&= \mathbf{M}(z)\left[I + \frac{\psi_\alpha(z)^{-3/2}}{48n}\begin{pmatrix} 1 & -6i \\ -6i & -1 \end{pmatrix} + O(n^{-2})\right].
\end{aligned}$$

4.15 Parametrix at the inner turning points

We now consider small disks $D(\alpha', \varepsilon)$ and $D(\beta', \varepsilon)$ centered at the inner turning points. Denote $\tilde{D} = D(\alpha', \varepsilon) \cup D(\beta', \varepsilon)$. We will seek a local parametrix $\mathbf{U}_n(z)$ defined on $\tilde{D}$ such that

1. $\mathbf{U}_n(z)$ is analytic on $\tilde{D} \setminus \Sigma_S$.
2. $\mathbf{U}_{n+}(z) = \mathbf{U}_{n-}(z) j_S(z) \quad \text{for} \quad z \in \tilde{D} \cap \Sigma_S$.
3. $\mathbf{U}_n(z) = \mathbf{M}(z)\big(I + O(n^{-1})\big) \quad$ uniformly for $z \in \partial\tilde{D}$.

We first construct the parametrix near α'. Let

$$\mathbf{U}_n(z) = \tilde{\mathbf{Q}}_n(z) e^{\mp\frac{in\pi z}{2\gamma}\sigma_3} e^{-n(g(z)-\frac{V(z)}{2}-\frac{l}{2})\sigma_3} \quad \text{for} \quad \pm \operatorname{Im} z > 0\,. \tag{4.107}$$

Then the jumps for $\tilde{\mathbf{Q}}_n$ are

$$j_{\tilde{Q}}(z) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{for } z \in (\alpha' - \varepsilon, \alpha') \\ \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} & \text{for } z \in (\alpha', \alpha' + \varepsilon) \\ \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} & \text{for } z \in (\alpha', \alpha' + i\varepsilon) \\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \text{for } z \in (\alpha', \alpha' - i\varepsilon), \end{cases} \tag{4.108}$$

where orientation is taken from left to right on horizontal contours, and down to up on vertical contours according to Figure 4.6. A proof of this statement is given in Appendix C. We now take

$$\Phi_{\alpha'}(z) = \begin{cases} \begin{pmatrix} y_2(z) & -y_0(z) \\ y_2'(z) & -y_0'(z) \end{pmatrix} & \text{for} \quad \arg z \in \left(0, \frac{\pi}{2}\right) \\ \begin{pmatrix} y_2(z) & y_1(z) \\ y_2'(z) & y_1'(z) \end{pmatrix} & \text{for} \quad \arg z \in \left(\frac{\pi}{2}, \pi\right) \\ \begin{pmatrix} y_1(z) & -y_2(z) \\ y_1'(z) & -y_2'(z) \end{pmatrix} & \text{for} \quad \arg z \in \left(-\pi, -\frac{\pi}{2}\right) \\ \begin{pmatrix} y_1(z) & y_0(z) \\ y_1'(z) & y_0'(z) \end{pmatrix} & \text{for} \quad \arg z \in \left(-\frac{\pi}{2}, 0\right). \end{cases}$$

Then $\Phi_{\alpha'}(z)$ solves a RHP similar to that of $\tilde{\mathbf{Q}}_n$, but for jumps emanating from the origin rather than from α'.

Notice that, by (4.23), for $t \in [\alpha, \alpha']$, as $t \to \alpha'$,

$$\rho(t) = \frac{1}{2\gamma} - C(\alpha' - t)^{1/2} + O\big((\alpha' - t)^{3/2}\big), \quad C > 0.$$

It follows that, as $z \to \alpha'$, for $z \in (\alpha, \alpha')$,

$$\int_z^{\alpha'} \left(\frac{1}{2\gamma} - \rho(t)\right) dt = C_0(\alpha' - z)^{3/2} + O((\alpha' - z)^{5/2}), \quad C_0 = \frac{2}{3}C.$$

Thus,

$$\psi_{\alpha'}(z) = -\left\{ \frac{3\pi}{2} \int_z^{\alpha'} \left(\frac{1}{2\gamma} - \rho(t) \right) dt \right\}^{2/3} \tag{4.109}$$

is analytic at α', and so extends to a conformal map from $D(\alpha', \varepsilon)$ onto a convex neighborhood of the origin. Furthermore,

$$\psi_{\alpha'}(\alpha') = 0\,, \quad \psi'_{\alpha'}(\alpha') > 0;$$

consequently, $\psi_{\alpha'}$ is real negative on $(\alpha' - \varepsilon, \alpha')$, and real positive on $(\alpha', \alpha' + \varepsilon)$. Again, we can slightly deform the vertical pieces of the contour Σ_S close to α', so that

$$\psi_{\alpha'}\big\{D(\alpha', \varepsilon) \cap \Sigma_S\big\} = (-\varepsilon, \varepsilon) \cup (-i\varepsilon, i\varepsilon).$$

We now take

$$\tilde{\mathbf{Q}}_n(z) = \mathbf{E}_n^{\alpha'}(z) \Phi_{\alpha'}\big(n^{2/3}\psi_{\alpha'}(z)\big)$$

where

$$\mathbf{E}_n^{\alpha'}(z) = \mathbf{M}(z) e^{\pm \frac{i\Omega_n}{2}\sigma_3} \tilde{\mathbf{L}}_n(z)^{-1} \quad \text{for} \quad \pm \operatorname{Im} z \geq 0,$$

$$\mathbf{L}_n^{\alpha'}(z) = \frac{1}{2\sqrt{\pi}} \begin{pmatrix} n^{-1/6}\psi_{\alpha'}^{-1/4}(z) & 0 \\ 0 & n^{1/6}\psi_{\alpha'}^{1/4}(z) \end{pmatrix} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix},$$

and we take the branch of $\psi_{\alpha'}^{1/4}$ which is positive on $(\alpha', \alpha' + \varepsilon)$ and has a cut on $(\alpha' - \varepsilon, \alpha')$. $\mathbf{U}_n$ then becomes

$$\mathbf{U}_n(z) = \mathbf{M}(z) e^{\pm \frac{i\Omega_n}{2}\sigma_3} \mathbf{L}_n^{\alpha'}(z)^{-1} \Phi_{\alpha'}\big(n^{2/3}\psi_{\alpha'}(z)\big) e^{\mp \frac{in\pi z}{2\gamma}\sigma_3} e^{-n(g(z) - \frac{V(z)}{2} - \frac{l}{2})\sigma_3} \quad \text{for} \quad \pm \operatorname{Im} z > 0. \tag{4.110}$$

The function $\Phi_{\alpha'}(n^{2/3}\psi_{\alpha'}(z))$ has the jumps j_S, and we claim that the prefactor $\mathbf{E}_n^{\alpha'}$ is analytic in $D(\alpha', \varepsilon)$, thus does not change these jumps. This can be seen, as

$$\mathbf{M}_+(z) e^{\frac{i\Omega_n}{2}\sigma_3} = \mathbf{M}_-(z) e^{-\frac{i\Omega_n}{2}\sigma_3} e^{\frac{i\Omega_n}{2}\sigma_3} j_M e^{\frac{i\Omega_n}{2}\sigma_3}$$

thus the jump for the function $\mathbf{M}(z) e^{\pm \frac{i\Omega_n}{2}\sigma_3}$ is

$$e^{\frac{i\Omega_n}{2}\sigma_3} j_M e^{\frac{i\Omega_n}{2}\sigma_3} = \begin{cases} e^{\frac{i\Omega_n}{2}\sigma_3} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} e^{\frac{i\Omega_n}{2}\sigma_3} & \text{for } z \in (\alpha' - \varepsilon, \alpha') \\ e^{\frac{i\Omega_n}{2}\sigma_3} e^{-i\Omega_n\sigma_3} e^{\frac{i\Omega_n}{2}\sigma_3} & \text{for } z \in (\alpha', \alpha' + \varepsilon), \end{cases}$$

or equivalently,

$$e^{\frac{i\Omega_n}{2}\sigma_3} j_M e^{\frac{i\Omega_n}{2}\sigma_3} = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{for} \quad z \in (\alpha' - \varepsilon, \alpha') \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{for} \quad z \in (\alpha', \alpha' + \varepsilon), \end{cases}$$

which is exactly the same as the jump conditions for $\mathbf{L}_n^{\alpha'}$. Thus $\mathbf{E}_n^{\alpha'}(z)$ has no jumps in $D(\alpha', \varepsilon)$. The only other possible singularity for $\mathbf{E}_n^{\alpha'}$ is at α', and this singularity is at most a fourth root singularity, thus removable. Thus, $\mathbf{E}_n^{\alpha'}$ is analytic in $D(\alpha', \varepsilon)$, and $\tilde{\mathbf{Q}}_n$ has the prescribed jumps.

We are left check that $\mathbf{U}_n$ satisfies the matching condition (3). The large n asymptotics of $\Phi_{\alpha'}\big(n^{2/3}\psi_{\alpha'}(z)\big)$ are given in the different regions of analyticity as follows:

$$\begin{aligned} \Phi_{\alpha'}\big(n^{2/3}\psi_{\alpha'}(z)\big) = \frac{1}{2\sqrt{\pi}} n^{-\frac{1}{6}\sigma_3} \psi_{\alpha'}(z)^{-\frac{1}{4}\sigma_3} \Bigg[\mp \begin{pmatrix} i & 1 \\ i & -1 \end{pmatrix} \pm \frac{\psi_{\alpha'}(z)^{-3/2}}{48n} \begin{pmatrix} -5i & 5 \\ 7i & 7 \end{pmatrix} \\ + O(n^{-2}) \Bigg] e^{\frac{2}{3} n \psi_{\alpha'}(z)^{3/2}\sigma_3} \qquad \text{for} \quad \pm \operatorname{Im} z > 0, \end{aligned} \tag{4.111}$$

where we always take the principal branch of $\psi_{\alpha'}(z)^{3/2}$. As such, $\psi_{\alpha'}(z)^{3/2}$ is two-valued for $x \in (\alpha' - \varepsilon, \alpha)$, so that

$$\left[\frac{2}{3}\psi_{\alpha'}(x)^{3/2}\right]_{\pm} = \mp \pi i \int_x^{\alpha'} \left(\frac{1}{2\gamma} - \rho(t)\right) dt = \mp \frac{\pi i}{2\gamma}(\alpha' - x) \pm \pi i \int_x^{\alpha'} \rho(t) dt. \tag{4.112}$$

From (4.30) and (4.32), we have that

$$2g_+(x) - V(x) = l + 2\pi i \int_x^{\beta} \rho(t) dt\,, \quad 2g_-(x) - V(x) = l - 2\pi i \int_x^{\beta} \rho(t) dt \tag{4.113}$$

for $x \in (\alpha' - \varepsilon, \alpha')$. These equations imply that

$$\big(2g_\pm(x) - V(x)\big) - \big(2g_\pm(\alpha') - V(\alpha')\big) = \pm 2\pi i \int_x^{\alpha'} \rho(t) dt.$$

We can therefore write (4.112) as

$$\left[\frac{2}{3}\psi_{\alpha'}(x)^{3/2}\right]_{\pm} = \mp \frac{\pi i}{2\gamma}(\alpha' - x) + \frac{1}{2}\Big[\big(2g_\pm(x) - V(x)\big) - \big(2g_\pm(\alpha') - V(\alpha')\big)\Big].$$

We can extend these equations into the upper and lower half-plane, respectively, obtaining

$$\frac{2}{3}\psi_{\alpha'}(z)^{3/2} = \mp\frac{\pi i}{2\gamma}(\alpha'-z)+\frac{1}{2}\Big[\big(2g(z)-V(z)\big)-\big(2g_\pm(\alpha')-V(\alpha')\big)\Big] \quad \text{for} \quad \pm\mathrm{Im}\, z>0\,.$$

Using (4.113) at $x=\alpha'$, we can write

$$\frac{2}{3}\psi_{\alpha'}(z)^{3/2} = \mp\frac{\pi i}{2\gamma}(\alpha'-z)+g(z)-\frac{V(z)}{2}-\frac{l}{2}\mp\pi i\int_{\alpha'}^{\beta}\rho(t)dt \quad \text{for} \quad \pm\mathrm{Im}\, z>0\,, \tag{4.114}$$

or equivalently,

$$\frac{2}{3}\psi_{\alpha'}(z)^{3/2} = g(z)-\frac{V(z)}{2}-\frac{l}{2}\pm\frac{\pi i z}{2\gamma}\mp\frac{i(\Omega_n-\pi)}{2n} \quad \text{for} \quad \pm\,\mathrm{Im}\, z>0\,.$$

Plugging (4.111) and (4.114) into (4.110) gives, as $n\to\infty$,

$$\begin{aligned}
\mathbf{U}_n(z) &= \mathbf{M}(z)e^{\pm\frac{i\Omega_n}{2}\sigma_3}\mathbf{L}_n^{\alpha'}(z)^{-1}\frac{1}{2\sqrt{\pi}}n^{-\frac{1}{6}\sigma_3}\psi_{\alpha'}(z)^{-\frac{1}{4}\sigma_3}\\
&\quad\times\left[\mp\begin{pmatrix} i & 1\\ i & -1\end{pmatrix}\pm\frac{\psi_{\alpha'}(z)^{-3/2}}{48n}\begin{pmatrix}-5i & 5\\ 7i & 7\end{pmatrix}+O(n^{-2})\right]\\
&\quad\times e^{n(g(z)-\frac{l}{2}-\frac{V(z)}{2})\sigma_3}e^{\mp\frac{i\Omega_n}{2}\sigma_3}e^{\pm\frac{i\pi}{2}\sigma_3}e^{\pm\frac{in\pi z}{2\gamma}\sigma_3}e^{\mp\frac{in\pi z}{2\gamma}\sigma_3}e^{-n(g(z)-\frac{V(z)}{2}-\frac{l}{2})\sigma_3}\\
&= \mathbf{M}(z)e^{\pm\frac{i\Omega_n}{2}\sigma_3}\mathbf{L}_n^{\alpha'}(z)^{-1}\frac{1}{2\sqrt{\pi}}n^{-\frac{1}{6}\sigma_3}\psi_{\alpha'}(z)^{-\frac{1}{4}\sigma_3}\\
&\quad\times\left[\begin{pmatrix} 1 & i\\ 1 & -i\end{pmatrix}+\frac{\psi_{\alpha'}(z)^{-3/2}}{48n}\begin{pmatrix}5 & -5i\\ -7 & -7i\end{pmatrix}+O(n^{-2})\right]e^{\mp\frac{i\Omega_n}{2}\sigma_3}\\
&= \mathbf{M}(z)\left[I+\frac{\psi_{\alpha'}(z)^{-3/2}}{48n}e^{\pm i\frac{\Omega_n}{2}\sigma_3}\begin{pmatrix}-1 & -6i\\ -6i & 1\end{pmatrix}e^{\mp i\frac{\Omega_n}{2}\sigma_3}+O(n^{-2})\right]\\
&= \mathbf{M}(z)\left[I+\frac{\psi_{\alpha'}(z)^{-3/2}}{48n}\begin{pmatrix}-1 & -6ie^{\pm i\Omega_n}\\ -6ie^{\mp i\Omega_n} & 1\end{pmatrix}+O(n^{-2})\right]\\
&\qquad\qquad\qquad\qquad\qquad\text{for }\ \pm\,\mathrm{Im}\,(z)>0.
\end{aligned}$$

We can make a similar construction near β'. Let

$$\psi_{\beta'}(z) = -\left\{\frac{3\pi}{2}\int_{\beta'}^{z}\left(\frac{1}{2\gamma}-\rho(t)dt\right)\right\}^{2/3}. \tag{4.115}$$

This function is analytic in $D(\beta',\varepsilon)$ and has negative derivative at β', thus $\operatorname{Im} z$ and $\operatorname{Im}\psi_{\beta'}(z)$ have opposite signs for $z \in D(\beta',\varepsilon)$. Then the jumps for $\tilde{\mathbf{Q}}_n$ are

$$j_{\tilde{Q}}(z) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{for } z \in (\beta', \beta'+\varepsilon) \\ \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} & \text{for } z \in (\beta'-\varepsilon, \beta') \\ \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} & \text{for } z \in (\beta', \beta'+i\varepsilon) \\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \text{for } z \in (\beta', \beta'-i\varepsilon), \end{cases}$$

where the contour is oriented from left to right on horizontal segments and up to down on vertical segments according to Figure 4.6. After a slight deformation of the vertical contours and the change of variables $\psi_{\beta'}$, these jumps become the following jumps close to the origin:

$$j_{\tilde{Q}}(\psi_{\beta'}(z)) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{for } \psi_{\beta'}(z) \in (-\varepsilon, 0) \\ \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} & \text{for } \psi_{\beta'}(z) \in (0, \varepsilon) \\ \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} & \text{for } \psi_{\beta'}(z) \in (-i\varepsilon, 0) \\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \text{for } \psi_{\beta'}(z) \in (0, i\varepsilon), \end{cases}$$

where the contour is oriented from right to left on horizontal segments and down to up on vertical segments. These jump conditions are satisfied by the function

$$\Phi_{\beta'}(z) = \Phi_{\alpha'}(z) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then we can take for $z \in D(\beta', \varepsilon)$,

$$\mathbf{U}_n(z) = \mathbf{M}(z) e^{\frac{\pm i\Omega_n}{2}\sigma_3} \mathbf{L}_n^{\beta'}(z)^{-1} \Phi_{\beta'}(n^{2/3}\psi_{\beta'}(z)) e^{\mp\frac{in\pi z}{2\gamma}\sigma_3} e^{-n(g(z)-\frac{V(z)}{2}-\frac{l}{2})\sigma_3} \quad \text{for} \quad \pm \operatorname{Im} z > 0, \tag{4.116}$$

where

$$\mathbf{L}_n^{\beta'}(z) = \frac{1}{2\sqrt{\pi}} \begin{pmatrix} n^{-1/6}\psi_{\beta'}^{-1/4}(z) & 0 \\ 0 & n^{1/6}\psi_{\beta'}^{1/4}(z) \end{pmatrix} \begin{pmatrix} -1 & i \\ -1 & -i \end{pmatrix}.$$

We once again have, as $n \to \infty$,

$$\Phi_{\beta'}\big(n^{2/3}\psi_{\beta'}(z)\big) = \frac{1}{2\sqrt{\pi}} n^{-\frac{1}{6}\sigma_3} \psi_{\beta'}(z)^{-\frac{1}{4}\sigma_3} \left[\mp \begin{pmatrix} i & -1 \\ i & 1 \end{pmatrix} \mp \frac{\psi_{\beta'}(z)^{-3/2}}{48n} \begin{pmatrix} 5i & 5 \\ -7i & 7 \end{pmatrix} + O(n^{-2}) \right] e^{\frac{2}{3}n\psi_{\beta'}(z)^{3/2}\sigma_3} \quad \text{for } \pm \operatorname{Im}\psi_{\beta'}(z) > 0 \text{ (so } \pm \operatorname{Im} z < 0), \tag{4.117}$$

and for $z \in D(\beta', \varepsilon)$,

$$\frac{2}{3}\psi_{\beta'}^{3/2}(z) = \pm\frac{\pi i z}{2\gamma} + g(z) - \frac{V(z)}{2} - \frac{l}{2} \mp \frac{i(\Omega_n - \pi)}{2n} \quad \text{for} \quad \pm \operatorname{Im} z > 0. \tag{4.118}$$

Combining (4.116), (4.117), and (4.118) gives, as $n \to \infty$,

$$\begin{aligned} \mathbf{U}_n(z) &= \mathbf{M}(z) e^{\pm\frac{i\Omega_n}{2}\sigma_3} \mathbf{L}_n^{\beta'}(z)^{-1} \frac{1}{2\sqrt{\pi}} n^{-\frac{1}{6}\sigma_3} \psi_{\beta'}(z)^{-\frac{1}{4}\sigma_3} \\ &\quad \times \left[\pm \begin{pmatrix} i & -1 \\ i & 1 \end{pmatrix} \pm \frac{\psi_{\beta'}(z)^{-3/2}}{48n} \begin{pmatrix} 5i & 5 \\ -7i & 7 \end{pmatrix} + O(n^{-2}) \right] \\ &\quad \times e^{\frac{\pm in\pi z}{2\gamma}\sigma_3} e^{n(g(z)-\frac{V(z)}{2}-\frac{l}{2})\sigma_3} e^{\mp\frac{i\Omega_n}{2}\sigma_3} e^{\pm\frac{i\pi}{2}\sigma_3} e^{\mp\frac{in\pi z}{2\gamma}\sigma_3} e^{-n(g(z)-\frac{V(z)}{2}-\frac{l}{2})\sigma_3} \\ &= \mathbf{M}(z) e^{\pm\frac{i\Omega_n}{2}\sigma_3} \left[I + \frac{\psi_{\beta'}(z)^{-3/2}}{48n} \begin{pmatrix} -1 & 6i \\ 6i & 1 \end{pmatrix} + O(n^{-2}) \right] e^{\mp\frac{i\Omega_n}{2}\sigma_3} \\ &= \mathbf{M}(z) \left[I + \frac{\psi_{\beta'}(z)^{-3/2}}{48n} \begin{pmatrix} -1 & 6ie^{\pm i\Omega_n} \\ 6ie^{\mp i\Omega_n} & 1 \end{pmatrix} + O(n^{-2}) \right] \quad \text{for } \pm \operatorname{Im} z > 0. \end{aligned}$$

4.16 The third and final transformation of the RHP

We now consider the contour Σ_X, which consists of the circles $\partial D(\alpha, \varepsilon)$, $\partial D(\alpha', \varepsilon)$, $\partial D(\beta', \varepsilon)$, and $\partial D(\beta, \varepsilon)$, all oriented counterclockwise, together with the parts of

$\Sigma_S \setminus \big([\alpha,\alpha'] \cup [\beta',\beta]\big)$ which lie outside of the disks $D(\alpha,\varepsilon),\ D(\alpha',\varepsilon),\ D(\beta',\varepsilon)$, and $D(\beta,\varepsilon)$, see Figure 4.7.

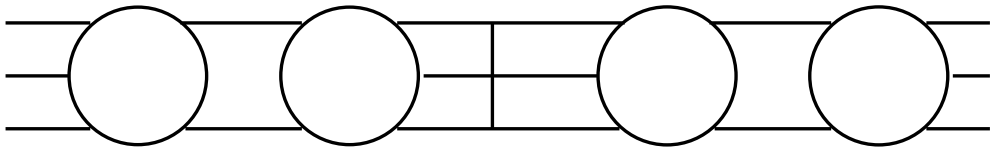

Fig. 4.7. The contour Σ_X.

We let

$$\mathbf{X}_n(z) = \begin{cases} \mathbf{S}_n(z)\mathbf{M}(z)^{-1} \text{ for } z \text{ outside the disks } D(\alpha,\varepsilon),\ D(\alpha',\varepsilon),\ D(\beta',\varepsilon),\ D(\beta,\varepsilon) \\ \mathbf{S}_n(z)\mathbf{U}_n(z)^{-1} \text{ for } z \text{ inside the disks } D(\alpha,\varepsilon),\ D(\alpha',\varepsilon),\ D(\beta',\varepsilon),\ D(\beta,\varepsilon). \end{cases} \tag{4.119}$$

Then $\mathbf{X}_n(z)$ solves the following RHP:

1. $\mathbf{X}_n(z)$ is analytic on $\mathbb{C} \setminus \Sigma_X$.

2. $\mathbf{X}_n(z)$ has the jump properties

$$\mathbf{X}_{n+}(x) = \mathbf{X}_{n-}(z) j_X(z)$$

where

$$j_X(z) = \begin{cases} \mathbf{M}(z)\mathbf{U}_n(z)^{-1} & \text{for } z \text{ on the circles} \\ \mathbf{M}(z) j_S \mathbf{M}(z)^{-1} & \text{otherwise.} \end{cases}$$

3. As $z \to \infty$,

$$\mathbf{X}_n(z) \sim I + \frac{\mathbf{X}_1}{z} + \frac{\mathbf{X}_2}{z^2} + \dots \tag{4.120}$$

Additionally, we have that $j_X(z)$ is uniformly close to the identity in the following sense:

$$j_X(z) = \begin{cases} I + O(n^{-1}) & \text{uniformly on the circles} \\ I + O(e^{-C(z)n}) & \text{on the rest of } \Sigma_X, \end{cases} \tag{4.121}$$

where $C(z)$ is a positive, continuous function satisfying (4.75). If we set

$$j_X^0(z) = j_X(z) - I,$$

then (4.121) becomes

$$j_X^0(z) = \begin{cases} O(n^{-1}) & \text{uniformly on the circles} \\ O(e^{-C(z)n}) & \text{on the rest of } \Sigma_X. \end{cases} \tag{4.122}$$

The solution to the RHP for $\mathbf{X}_n$ is based on the following lemma:

Lemma 4.16.1 *Suppose v(z) is a function on Σ_X solving the equation*

$$v(z) = I - \frac{1}{2\pi i}\int_{\Sigma_X} \frac{v(u)j_X^0(u)}{z_- - u}du \quad \textit{for } z \in \Sigma_X \tag{4.123}$$

where z_- means the value of the integral on the minus side of Σ_X. Then

$$\mathbf{X}_n(z) = I - \frac{1}{2\pi i}\int_{\Sigma_X} \frac{v(u)j_X^0(u)}{z - u}du \quad \textit{for } z \in \mathbb{C} \setminus \Sigma_X \tag{4.124}$$

solves the RHP for $\mathbf{X}_n$.

The proof of this lemma is immediate from the jump property of the Cauchy transform. By assumption

$$\mathbf{X}_{n-}(z) = v(z)$$

and the additive jump of the Cauchy transform gives

$$\mathbf{X}_{n+}(z) - \mathbf{X}_{n-}(z) = v(z)j_X^0(z) = \mathbf{X}_{n-}(z)j_X^0(z),$$

thus $\mathbf{X}_{n+}(z) = \mathbf{X}_{n-}(z)j_X(z)$. Asymptotics at infinity are given by (4.124).

The solution to equation (4.123) is given by a series of perturbation theory. Namely, the solution is

$$v(z) = I + \sum_{k=1}^{\infty} v_k(z) \tag{4.125}$$

where

$$v_k(z) = -\frac{1}{2\pi i}\int_{\Sigma_X} \frac{v_{k-1}(u)j_X^0(u)}{z_- - u}du\,, \quad v_0(z) = I.$$

This function clearly solves (4.123) provided the series converges, which it does, for sufficiently large n. Indeed it does, as it is bounded from above by a convergent geometric series, and is therefore absolutely convergent (see e.g. [17]). This in turn gives

$$\mathbf{X}_n(z) = I + \sum_{k=1}^{\infty} \mathbf{X}_{n,k}(z)$$

where

$$\mathbf{X}_{n,k}(z) = -\frac{1}{2\pi i}\int_{\Sigma_X} \frac{v_{k-1}(u) j_X^0(u)}{z-u} du.$$

In particular, (see [17])

$$v_k(x) = O\left(\frac{1}{n^k(|z|+1)}\right), \quad \mathbf{X}_n = I + O\left(\frac{1}{n(|z|+1)}\right) \quad \text{as} \quad n \to \infty. \tag{4.126}$$

We will need to compute

$$\mathbf{X}_{n,1}(z) = -\frac{1}{2\pi i}\int_{\Sigma_X} \frac{j_X^0(u)}{z-u} du.$$

4.17 Evaluation of $\mathbf{X}_1$

We are interested in the matrix $\mathbf{X}_1$, which gives the $\frac{1}{z}$-term of $\mathbf{X}_n(z)$ at infinity, see (4.120). By (4.124),

$$\mathbf{X}_1 = -\frac{1}{2\pi i}\int_{\Sigma_X} v(u) j_X^0(u)\, du,$$

hence by (4.125), (4.126),

$$\mathbf{X}_1 = -\frac{1}{2\pi i}\int_{\Sigma_X} j_X^0(u)\, du + O(n^{-2}).$$

We would like to evaluate the integral,

$$-\frac{1}{2\pi i}\int_{\Sigma_X} j_X^0(u) du$$

with an error of the order of n^{-2}. By (4.122), it is enough to evaluate this integral over the circles $\partial D(\alpha, \varepsilon)$, $\partial D(\alpha', \varepsilon)$, $\partial D(\beta', \varepsilon)$, and $\partial D(\beta, \varepsilon)$. As we will see in the next section, the matrix-valued function $j_X^0(z)$ is analytic in the punctured disks, hence

$$\mathbf{X}_1 = -\left(\operatorname*{Res}_{z=\alpha} + \operatorname*{Res}_{z=\alpha'} + \operatorname*{Res}_{z=\beta'} + \operatorname*{Res}_{z=\beta}\right) j_X^0(z) + O(n^{-2}).$$

We will be especially interested in evaluation of the [12] element of the matrix $\mathbf{X}_1$, and we will prove the following asymptotic formula. Introduce the numbers

$$\eta_\alpha = \left[5\frac{\vartheta_4''(n\omega+\frac{\omega}{2})}{\vartheta_4(n\omega+\frac{\omega}{2})} - 5\frac{\vartheta_3''(\frac{\omega}{2})}{\vartheta_3(\frac{\omega}{2})} + 7\left(\frac{\vartheta_4'(n\omega+\frac{\omega}{2})}{\vartheta_4(n\omega+\frac{\omega}{2})}\right)^2 + 17\left(\frac{\vartheta_3'(\frac{\omega}{2})}{\vartheta_3(\frac{\omega}{2})}\right)^2 \right.$$
$$\left. -24\frac{\vartheta_4'(n\omega+\frac{\omega}{2})\vartheta_3'(\frac{\omega}{2})}{\vartheta_4(n\omega+\frac{\omega}{2})\vartheta_3(\frac{\omega}{2})}\right],$$

$$\eta_{\alpha'} = -\left[5\frac{\vartheta_1''(n\omega+\frac{\omega}{2})}{\vartheta_1(n\omega+\frac{\omega}{2})} - 5\frac{\vartheta_2''(\frac{\omega}{2})}{\vartheta_2(\frac{\omega}{2})} + 7\left(\frac{\vartheta_1'(n\omega+\frac{\omega}{2})}{\vartheta_1(n\omega+\frac{\omega}{2})}\right)^2 + 17\left(\frac{\vartheta_2'(\frac{\omega}{2})}{\vartheta_2(\frac{\omega}{2})}\right)^2 \right.$$
$$\left. -24\frac{\vartheta_1'(n\omega+\frac{\omega}{2})\vartheta_2'(\frac{\omega}{2})}{\vartheta_1(n\omega+\frac{\omega}{2})\vartheta_2(\frac{\omega}{2})}\right],$$

$$\eta_{\beta'} = -\left[5\frac{\vartheta_2''(n\omega+\frac{\omega}{2})}{\vartheta_2(n\omega+\frac{\omega}{2})} - 5\frac{\vartheta_1''(\frac{\omega}{2})}{\vartheta_1(\frac{\omega}{2})} + 7\left(\frac{\vartheta_2'(n\omega+\frac{\omega}{2})}{\vartheta_2(n\omega+\frac{\omega}{2})}\right)^2 + 17\left(\frac{\vartheta_1'(\frac{\omega}{2})}{\vartheta_1(\frac{\omega}{2})}\right)^2 \right.$$
$$\left. -24\frac{\vartheta_2'(n\omega+\frac{\omega}{2})\vartheta_1'(\frac{\omega}{2})}{\vartheta_2(n\omega+\frac{\omega}{2})\vartheta_1(\frac{\omega}{2})}\right],$$

$$\eta_\beta = \left[5\frac{\vartheta_3''(n\omega+\frac{\omega}{2})}{\vartheta_3(n\omega+\frac{\omega}{2})} - 5\frac{\vartheta_4''(\frac{\omega}{2})}{\vartheta_4(\frac{\omega}{2})} + 7\left(\frac{\vartheta_3'(n\omega+\frac{\omega}{2})}{\vartheta_3(n\omega+\frac{\omega}{2})}\right)^2 + 17\left(\frac{\vartheta_4'(\frac{\omega}{2})}{\vartheta_4(\frac{\omega}{2})}\right)^2 \right.$$
$$\left. -24\frac{\vartheta_3'(n\omega+\frac{\omega}{2})\vartheta_4'(\frac{\omega}{2})}{\vartheta_3(n\omega+\frac{\omega}{2})\vartheta_4(\frac{\omega}{2})}\right],$$

and

$$C_\alpha = \frac{7}{2}(\beta'-\alpha) + \frac{3}{2}(\beta-\alpha) + \frac{3}{2}(\alpha'-\alpha) - \frac{(\alpha'-\alpha)(\beta-\alpha)}{(\beta'-\alpha)},$$
$$C_{\alpha'} = -\frac{7}{2}(\beta-\alpha') - \frac{3}{2}(\beta'-\alpha') + \frac{3}{2}(\alpha'-\alpha) - \frac{(\alpha'-\alpha)(\beta'-\alpha')}{(\beta-\alpha')},$$
$$C_{\beta'} = -\frac{7}{2}(\beta'-\alpha) - \frac{3}{2}(\beta'-\alpha') + \frac{3}{2}(\beta-\beta') - \frac{(\beta-\beta')(\beta'-\alpha')}{(\beta'-\alpha)},$$
$$C_\beta = \frac{7}{2}(\beta-\alpha') + \frac{3}{2}(\beta-\alpha) + \frac{3}{2}(\beta-\beta') - \frac{(\beta-\beta')(\beta-\alpha)}{(\beta-\alpha')}.$$

Introduce also the numbers,

$$\Xi_\alpha = \frac{\vartheta_3^2(0)\vartheta_4^2(n\omega+\frac{\omega}{2})}{\vartheta_3^2(\frac{\omega}{2})\vartheta_4^2(n\omega)}, \qquad \Xi_{\alpha'} = \frac{\vartheta_3^2(0)\vartheta_1^2(n\omega+\frac{\omega}{2})}{\vartheta_2^2(\frac{\omega}{2})\vartheta_4^2(n\omega)},$$
$$\Xi_{\beta'} = \frac{\vartheta_3^2(0)\vartheta_2^2(n\omega+\frac{\omega}{2})}{\vartheta_1^2(\frac{\omega}{2})\vartheta_4^2(n\omega)}, \qquad \Xi_\beta = \frac{\vartheta_3^2(0)\vartheta_3^2(n\omega+\frac{\omega}{2})}{\vartheta_4^2(\frac{\omega}{2})\vartheta_4^2(n\omega)},$$

and

$$\xi_\alpha = \frac{\vartheta_3'(\frac{\omega}{2})}{\vartheta_3(\frac{\omega}{2})} - \frac{\vartheta_4'(n\omega+\frac{\omega}{2})}{\vartheta_4(n\omega+\frac{\omega}{2})}, \qquad \xi_{\alpha'} = -\left(\frac{\vartheta_2'(\frac{\omega}{2})}{\vartheta_2(\frac{\omega}{2})} - \frac{\vartheta_1'(n\omega+\frac{\omega}{2})}{\vartheta_1(n\omega+\frac{\omega}{2})}\right),$$

$$\xi_{\beta'} = \frac{\vartheta_1'(\frac{\omega}{2})}{\vartheta_1(\frac{\omega}{2})} - \frac{\vartheta_2'(n\omega+\frac{\omega}{2})}{\vartheta_2(n\omega+\frac{\omega}{2})}, \qquad \xi_{\beta} = -\left(\frac{\vartheta_4'(\frac{\omega}{2})}{\vartheta_4(\frac{\omega}{2})} - \frac{\vartheta_3'(n\omega+\frac{\omega}{2})}{\vartheta_3(n\omega+\frac{\omega}{2})}\right).$$

We have the following lemma.

Lemma 4.17.1 *As $n \to \infty$,*

$$[\mathbf{X}_1]_{12} = \frac{1}{n}\left(X_\alpha + X_{\alpha'} + X_{\beta'} + X_\beta\right) + O(n^{-2}), \tag{4.127}$$

where

$$X_\alpha = \frac{i\Xi_\alpha}{96}\left(C_\alpha + 12\pi\xi_\alpha + \frac{\pi^2\eta_\alpha}{2(\beta'-\alpha)}\right),$$

$$X_{\alpha'} = \frac{i\Xi_\alpha'}{96}\left(C_{\alpha'} + 12\pi\xi_{\alpha'} + \frac{\pi^2\eta_{\alpha'}}{2(\beta-\alpha')}\right),$$

$$X_{\beta'} = \frac{i\Xi_{\beta'}}{96}\left(C_{\beta'} + 12\pi\xi_{\beta'} + \frac{\pi^2\eta_{\beta'}}{2(\beta'-\alpha)}\right),$$

$$X_{\beta} = \frac{i\Xi_{\beta}}{96}\left(C_{\beta} + 12\pi\xi_{\beta} + \frac{\pi^2\eta_{\beta}}{2(\beta-\alpha')}\right).$$

Proof of this lemma is given in the next section.

4.18 Proof of Lemma 4.17.1

We have that

$$j_X(z) = \begin{cases} I - \dfrac{\psi_\alpha(z)^{-3/2}}{48n}\mathbf{M}(z)\begin{pmatrix} 1 & -6i \\ -6i & -1 \end{pmatrix}\mathbf{M}^{-1}(z) + O(n^{-2}) & \text{for } z \in \partial D(\alpha,\varepsilon) \\ I - \dfrac{\psi_\beta(z)^{-3/2}}{48n}\mathbf{M}(z)\begin{pmatrix} 1 & 6i \\ 6i & -1 \end{pmatrix}\mathbf{M}^{-1}(z) + O(n^{-2}) & \text{for } z \in \partial D(\beta,\varepsilon) \\ I - \dfrac{\psi_{\alpha'}(z)^{-3/2}}{48n}\mathbf{M}(z)\begin{pmatrix} -1 & -6ie^{\pm i\Omega_n} \\ -6ie^{\mp i\Omega_n} & 1 \end{pmatrix}\mathbf{M}^{-1}(z) + O(n^{-2}) & \\ \qquad\qquad \text{for } z \in \partial D(\alpha',\varepsilon),\ \pm\operatorname{Im} z > 0 & \\ I - \dfrac{\psi_{\beta'}(z)^{-3/2}}{48n}\mathbf{M}(z)\begin{pmatrix} -1 & 6ie^{\pm i\Omega_n} \\ 6ie^{\mp i\Omega_n} & 1 \end{pmatrix}\mathbf{M}^{-1}(z) + O(n^{-2}) & \\ \qquad\qquad \text{for } z \in \partial D(\beta',\varepsilon),\ \pm\operatorname{Im} z > 0. & \end{cases}$$

Thus

$$j_X^0(z) = \begin{cases} -\dfrac{\psi_\alpha(z)^{-3/2}}{48n}\mathbf{M}(z)\begin{pmatrix} 1 & -6i \\ -6i & -1 \end{pmatrix}\mathbf{M}^{-1}(z) + O(n^{-2}) \quad \text{for } z \in \partial D(\alpha,\varepsilon) \\ -\dfrac{\psi_\beta(z)^{-3/2}}{48n}\mathbf{M}(z)\begin{pmatrix} 1 & 6i \\ 6i & -1 \end{pmatrix}\mathbf{M}^{-1}(z) + O(n^{-2}) \quad \text{for } z \in \partial D(\beta,\varepsilon) \\ -\dfrac{\psi_{\alpha'}(z)^{-3/2}}{48n}\mathbf{M}(z)\begin{pmatrix} -1 & -6ie^{\pm i\Omega_n} \\ -6ie^{\mp i\Omega_n} & 1 \end{pmatrix}\mathbf{M}^{-1}(z) + O(n^{-2}) \\ \qquad \text{for } z \in \partial D(\alpha',\varepsilon) \\ -\dfrac{\psi_{\beta'}(z)^{-3/2}}{48n}\mathbf{M}(z)\begin{pmatrix} -1 & 6ie^{\pm i\Omega_n} \\ 6ie^{\mp i\Omega_n} & 1 \end{pmatrix}\mathbf{M}^{-1}(z) + O(n^{-2}) \\ \qquad \text{for } z \in \partial D(\beta',\varepsilon), \end{cases} \tag{4.128}$$

for $\pm \operatorname{Im} z > 0$. To simplify notation, we will write the model solution given in (4.94) and (4.95) as

$$\mathbf{M}(z) = \frac{1}{2}\begin{pmatrix} \big(\gamma(z)+\gamma^{-1}(z)\big)\vartheta_{11}(z) & i\big(\gamma(z)-\gamma^{-1}(z)\big)\vartheta_{12}(z) \\ -i\big(\gamma(z)-\gamma^{-1}(z)\big)\vartheta_{21}(z) & \big(\gamma(z)+\gamma^{-1}(z)\big)\vartheta_{22}(z) \end{pmatrix},$$

where

$$\begin{aligned} \vartheta_{11}(z) &= \frac{\vartheta_3(\tilde u(z)-\tilde u_\infty+\frac{\Omega_n}{2})\vartheta_3(0)}{\vartheta_3(\tilde u(z)-\tilde u_\infty)\vartheta_3(\frac{\Omega_n}{2})}, & \vartheta_{12}(z) &= \frac{\vartheta_3(\tilde u(z)+\tilde u_\infty-\frac{\Omega_n}{2})\vartheta_3(0)}{\vartheta_3(\tilde u(z)+\tilde u_\infty)\vartheta_3(\frac{\Omega_n}{2})}, \\ \vartheta_{21}(z) &= \frac{\vartheta_3(\tilde u(z)+\tilde u_\infty+\frac{\Omega_n}{2})\vartheta_3(0)}{\vartheta_3(\tilde u(z)+\tilde u_\infty)\vartheta_3(\frac{\Omega_n}{2})}, & \vartheta_{22}(z) &= \frac{\vartheta_3(\tilde u(z)-\tilde u_\infty-\frac{\Omega_n}{2})\vartheta_3(0)}{\vartheta_3(\tilde u(z)-\tilde u_\infty)\vartheta_3(\frac{\Omega_n}{2})}. \end{aligned} \tag{4.129}$$

Notice that each of the functions ϑ_{ij} is analytic throughout the complex plane, except on the intervals (α,α') and (β',β), where they satisfy the relations

$$[\vartheta_{11}]_\pm = [\vartheta_{12}]_\mp\,, \quad [\vartheta_{21}]_\pm = [\vartheta_{22}]_\mp,$$

and on the interval (α',β'), where they satisfy

$$[\vartheta_{11}]_+ = e^{-i\Omega_n}[\vartheta_{11}]_-\ ,\ [\vartheta_{12}]_+ = e^{i\Omega_n}[\vartheta_{12}]_-\ ,\ [\vartheta_{21}]_+ = e^{-i\Omega_n}[\vartheta_{21}]_-\ ,\ [\vartheta_{22}]_+ = e^{i\Omega_n}[\vartheta_{22}]_-.$$

Multiplying out equation (4.128) gives

$$j_X^0(z) = \frac{\psi_\xi(z)^{-3/2}}{48n}\begin{pmatrix} j_{11}^\xi & j_{12}^\xi \\ j_{21}^\xi & j_{22}^\xi \end{pmatrix} + O(n^{-2}) \quad \text{for } z \in \partial D(\xi,\varepsilon), \quad \xi = \alpha, \alpha', \beta', \beta, \tag{4.130}$$

where

$$\begin{aligned}
j_{12}^\alpha &= \frac{i}{2}\left[3\big((\gamma^2(z)+\gamma^{-2}(z)\big)(\vartheta_{11}^2-\vartheta_{12}^2) + \big(\gamma^2(z)-\gamma^{-2}(z)\big)\vartheta_{11}\vartheta_{12} + 6(\vartheta_{11}^2-\vartheta_{12}^2)\right], \\
j_{12}^{\alpha'} &= \frac{i}{2}\left[3\big(\gamma^2(z)+\gamma^{-2}(z)\big)J(z) - \big(\gamma^2(z)-\gamma^{-2}(z)\big)\vartheta_{11}\vartheta_{12} + 6K(z)\right], \\
j_{12}^{\beta'} &= -\frac{i}{2}\left[3\big(\gamma^2(z)+\gamma^{-2}(z)\big)J(z) + \big(\gamma^2(z)-\gamma^{-2}(z)\big)\vartheta_{11}\vartheta_{12} + 6K(z)\right], \\
j_{12}^\beta &= -\frac{i}{2}\left[3\big((\gamma^2(z)+\gamma^{-2}(z)\big)(\vartheta_{11}^2-\vartheta_{12}^2) - \big(\gamma^2(z)-\gamma^{-2}(z)\big)\vartheta_{11}\vartheta_{12} + 6(\vartheta_{11}^2-\vartheta_{12}^2)\right],
\end{aligned} \tag{4.131}$$

and

$$J(z) = \begin{cases} \vartheta_{11}^2 e^{i\Omega_n} + \vartheta_{12}^2 e^{-i\Omega_n} & \text{for} \quad \operatorname{Im} z > 0 \\ \vartheta_{11}^2 e^{-i\Omega_n} + \vartheta_{12}^2 e^{i\Omega_n} & \text{for} \quad \operatorname{Im} z < 0, \end{cases}$$

$$K(z) = \begin{cases} \vartheta_{11}^2 e^{i\Omega_n} - \vartheta_{12}^2 e^{-i\Omega_n} & \text{for} \quad \operatorname{Im} z > 0 \\ \vartheta_{11}^2 e^{-i\Omega_n} - \vartheta_{12}^2 e^{i\Omega_n} & \text{for} \quad \operatorname{Im} z < 0. \end{cases}$$

In order to integrate $j_X^0(z)$, let us examine the behavior of the various functions described above near each of the turning points. Introduce the numbers

$$A_\alpha = \sqrt{(\alpha'-\alpha)(\beta'-\alpha)(\beta-\alpha)}, \qquad A_{\alpha'} = \sqrt{(\alpha'-\alpha)(\beta'-\alpha')(\beta-\alpha')},$$
$$A_{\beta'} = \sqrt{(\beta'-\alpha)(\beta'-\alpha')(\beta-\beta')}, \qquad A_\beta = \sqrt{(\beta-\alpha)(\beta-\alpha')(\beta-\beta')},$$

and

$$\begin{aligned}
B_\alpha &= \frac{1}{\alpha'-\alpha} + \frac{1}{\beta'-\alpha} + \frac{1}{\beta-\alpha}, \qquad & B_{\alpha'} &= -\frac{1}{\alpha'-\alpha} + \frac{1}{\beta'-\alpha'} + \frac{1}{\beta-\alpha'}, \\
B_{\beta'} &= \frac{1}{\beta'-\alpha} + \frac{1}{\beta'-\alpha'} - \frac{1}{\beta-\beta'}, \qquad & B_\beta &= \frac{1}{\beta-\alpha} + \frac{1}{\beta-\alpha'} + \frac{1}{\beta-\beta'}.
\end{aligned} \tag{4.132}$$

For $x \in (\beta, \beta+\varepsilon)$, we have

$$\tilde{u}(x) = \frac{\pi}{2K}u(x) = \frac{\pi}{A_\beta}\sqrt{x-\beta} + O((x-\beta)^{3/2}); \tag{4.133}$$

for $x \in (\alpha-\varepsilon, \alpha)$,

$$\tilde{u}(x) = \frac{\pi}{2K}u(x) = \frac{\pi}{2} - \frac{\pi}{A_\alpha}\sqrt{\alpha-x} + O((\alpha-x)^{3/2}); \tag{4.134}$$

for $x \in (\alpha', \alpha' + \varepsilon)$,

$$\tilde{u}_{\pm}(x) = \frac{\pi}{2K} u_{\pm}(x) = \frac{\pi}{2} \pm \frac{\tau\pi}{2} - \frac{\pi}{A_{\alpha'}} \sqrt{x - \alpha'} + O((x - \alpha')^{3/2}); \tag{4.135}$$

and for $x \in (\beta' - \varepsilon, \beta')$,

$$\tilde{u}_{\pm}(x) = \frac{\pi}{2K} u_{\pm}(x) = \pm \frac{\tau\pi}{2} + \frac{\pi}{A_{\beta'}} \sqrt{\beta' - x} + O((\beta' - x)^{-3/2}). \tag{4.136}$$

Also, from (4.82) we have that

$$\gamma^2(z) \pm \gamma^{-2}(z) = \sqrt{\frac{(z-\alpha)(z-\beta')}{(z-\alpha')(z-\beta)}} \pm \sqrt{\frac{(z-\alpha')(z-\beta)}{(z-\alpha)(z-\beta')}},$$

and from (4.101), (4.103), (4.109), (4.115) , and (4.23) that

$$\begin{aligned}
\psi_\alpha^{-3/2}(z) &= (\alpha - z)^{-3/2} \left[\frac{A_\alpha}{2} + \frac{1}{20} A_\alpha B_\alpha (\alpha - z) + O\big((\alpha - z)^2\big) \right], \\
\psi_{\alpha'}^{-3/2}(z) &= (z - \alpha')^{-3/2} \left[\frac{A_{\alpha'}}{2} - \frac{1}{20} A_{\alpha'} B_{\alpha'} (z - \alpha') + O\big((z - \alpha')^2\big) \right], \\
\psi_{\beta'}^{-3/2}(z) &= (\beta' - z)^{-3/2} \left[\frac{A_{\beta'}}{2} - \frac{1}{20} A_{\beta'} B_{\beta'} (\beta' - z) + O\big((\beta' - z)^2\big) \right], \\
\psi_\beta^{-3/2}(z) &= (z - \beta)^{-3/2} \left[\frac{A_\beta}{2} + \frac{1}{20} A_\beta B_\beta (z - \beta) + O\big((z - \beta)^2\big) \right].
\end{aligned} \tag{4.137}$$

It follows that the functions $(\gamma^2 \pm \gamma^{-2})\psi^{-3/2}(z)$ are meromorphic in a neighborhood of each of the turning points. In particular, at $z = \alpha$, we have

$$\begin{aligned}
(\gamma^2 \pm \gamma^{-2})\psi_\alpha^{-3/2}(z) = \pm \frac{(\alpha' - \alpha)(\beta - \alpha)}{2(\alpha - z)^2} + \frac{1}{(\alpha - z)} \left[\frac{(\beta' - \alpha)}{2} \pm \frac{3}{10}(\beta - \alpha) \right. \\
\left. \pm \frac{3}{10}(\alpha' - \alpha) \mp \frac{1}{5} \frac{(\alpha' - \alpha)(\beta - \alpha)}{(\beta' - \alpha)} \right] + O(1);
\end{aligned} \tag{4.138}$$

at $z = \beta$, we have

$$\begin{aligned}
(\gamma^2 \pm \gamma^{-2})\psi_\beta^{-3/2}(z) = \frac{(\beta - \alpha)(\beta - \beta')}{2(z - \beta)^2} + \frac{1}{(z - \beta)} \left[\pm \frac{1}{2}(\beta - \alpha') + \frac{3}{10}(\beta - \alpha) \right. \\
\left. + \frac{3}{10}(\beta - \beta') - \frac{1}{5} \frac{(\beta - \alpha)(\beta - \beta')}{(\beta - \alpha')} \right] + O(1);
\end{aligned} \tag{4.139}$$

at $z = \alpha'$, we have

$$\begin{aligned}
(\gamma^2 \pm \gamma^{-2})\psi_{\alpha'}^{-3/2}(z) = \frac{(\alpha' - \alpha)(\beta' - \alpha')}{2(z - \alpha')^2} + \frac{1}{(z - \alpha')} \left[\pm \frac{(\beta - \alpha')}{2} + \frac{3}{10}(\beta' - \alpha') \right. \\
\left. - \frac{3}{10}(\alpha' - \alpha) + \frac{1}{5} \frac{(\alpha' - \alpha)(\beta' - \alpha')}{(\beta - \alpha')} \right] + O(1);
\end{aligned} \tag{4.140}$$

and at $z = \beta'$, we have

$$(\gamma^2 \pm \gamma^{-2})\psi_{\beta'}^{-3/2}(z) = \pm\frac{(\beta'-\alpha')(\beta-\beta')}{2(\beta'-z)^2} + \frac{1}{(\beta'-z)}\left[\frac{(\beta'-\alpha)}{2} \pm \frac{3}{10}(\beta'-\alpha') \mp\frac{3}{10}(\beta-\beta') \pm \frac{1}{5}\frac{(\beta'-\alpha')(\beta-\beta')}{\beta'-\alpha}\right] + O(1). \tag{4.141}$$

Notice also, from the relations (4.129), that the functions $\vartheta_{11}^2 + \vartheta_{12}^2$ and $\vartheta_{11}\vartheta_{12}$ have no jumps in neighborhoods of α or β, and take finite values at $z = \alpha$ and $z = \beta$, thus are analytic in neighborhoods of α and β. Using (4.133) and (4.134), we see that these functions have Taylor expansions about $z = \beta$,

$$\begin{aligned} \vartheta_{11}^2(z) + \vartheta_{12}^2(z) &= 2\frac{\vartheta_3^2(0)}{\vartheta_3^2(\frac{\Omega_n}{2})}\frac{\vartheta_3^2(\tilde{u}_\infty - \frac{\Omega_n}{2})}{\vartheta_3^2(\tilde{u}_\infty)} + \frac{\pi^2}{2A_\beta^2}\frac{d^2}{d\tilde{u}^2}\left(\vartheta_{11}^2 + \vartheta_{12}^2\right)\bigg|_{z=\beta}(z-\beta) + \cdots, \\ \vartheta_{11}(z)\vartheta_{12}(z) &= \frac{\vartheta_3^2(0)}{\vartheta_3^2(\frac{\Omega_n}{2})}\frac{\vartheta_3^2(\tilde{u}_\infty - \frac{\Omega_n}{2})}{\vartheta_3^2(\tilde{u}_\infty)} + \frac{\pi^2}{2A_\beta^2}\frac{d^2}{d\tilde{u}^2}\left(\vartheta_{11}\vartheta_{12}\right)\bigg|_{z=\beta}(z-\beta) + \cdots, \end{aligned} \tag{4.142}$$

and about $z = \alpha$,

$$\begin{aligned} \vartheta_{11}^2(z) + \vartheta_{12}^2(z) &= 2\frac{\vartheta_3^2(0)}{\vartheta_3^2(\frac{\Omega_n}{2})}\frac{\vartheta_4^2(\tilde{u}_\infty - \frac{\Omega_n}{2})}{\vartheta_4^2(\tilde{u}_\infty)} + \frac{\pi^2}{2A_\alpha^2}\frac{d^2}{d\tilde{u}^2}\left(\vartheta_{11}^2 + \vartheta_{12}^2\right)\bigg|_{z=\alpha}(\alpha-z) + \cdots, \\ \vartheta_{11}(z)\vartheta_{12}(z) &= \frac{\vartheta_3^2(0)}{\vartheta_3^2(\frac{\Omega_n}{2})}\frac{\vartheta_4^2(\tilde{u}_\infty - \frac{\Omega_n}{2})}{\vartheta_4^2(\tilde{u}_\infty)} + \frac{\pi^2}{2A_\alpha^2}\frac{d^2}{d\tilde{u}^2}\left(\vartheta_{11}\vartheta_{12}\right)\bigg|_{z=\alpha}(\alpha-z) + \cdots. \end{aligned} \tag{4.143}$$

By a similar argument, $J(z)$ and $\vartheta_{11}\vartheta_{12}$ are also analytic in neighborhoods of α' and β' and using (4.135) and (4.136) we can write their Taylor expansions about $z = \alpha'$,

$$\begin{aligned} J(z) &= 2\frac{\vartheta_3^2(0)}{\vartheta_3^2(\frac{\Omega_n}{2})}\frac{\vartheta_1^2(\tilde{u}_\infty - \frac{\Omega_n}{2})}{\vartheta_1^2(\tilde{u}_\infty)} + \frac{\pi^2}{2A_{\alpha'}^2}\frac{d^2}{d\tilde{u}^2}J(z)\bigg|_{z=\alpha'}(z-\alpha') + \cdots, \\ \vartheta_{11}(z)\vartheta_{12}(z) &= \frac{\vartheta_3^2(0)}{\vartheta_3^2(\frac{\Omega_n}{2})}\frac{\vartheta_1^2(\tilde{u}_\infty - \frac{\Omega_n}{2})}{\vartheta_1^2(\tilde{u}_\infty)} + \frac{\pi^2}{2A_{\alpha'}^2}\frac{d^2}{d\tilde{u}^2}\left(\vartheta_{11}\vartheta_{12}\right)\bigg|_{z=\alpha'}(z-\alpha') + \cdots, \end{aligned} \tag{4.144}$$

and about $z = \beta'$,

$$J(z) = 2\frac{\vartheta_3^2(0)}{\vartheta_3^2(\frac{\Omega_n}{2})}\frac{\vartheta_2^2(\tilde{u}_\infty - \frac{\Omega_n}{2})}{\vartheta_2^2(\tilde{u}_\infty)} + \frac{\pi^2}{2A_{\beta'}^2}\frac{d^2}{d\tilde{u}^2}J(z)\bigg|_{z=\beta'}(\beta' - z) + \cdots ,$$

$$\vartheta_{11}(z)\vartheta_{12}(z) = \frac{\vartheta_3^2(0)}{\vartheta_3^2(\frac{\Omega_n}{2})}\frac{\vartheta_2^2(\tilde{u}_\infty - \frac{\Omega_n}{2})}{\vartheta_2^2(\tilde{u}_\infty)} + \frac{\pi^2}{2A_{\beta'}^2}\frac{d^2}{d\tilde{u}^2}\Big(\vartheta_{11}\vartheta_{12}\Big)\bigg|_{z=\beta'}(\beta' - z) + \cdots . \tag{4.145}$$

Finally, notice that the function $\vartheta_{11}^2 - \vartheta_{12}^2$ is an odd function of $\tilde{u}$, and using (4.133) and (4.134), we can write, for $x \in (\beta, \beta + \varepsilon)$,

$$\vartheta_{11}^2(x) - \vartheta_{12}^2(x) = \sqrt{x - \beta}$$
$$\times \left[\frac{4\pi}{A_\beta}\frac{\vartheta_3^2(0)}{\vartheta_3^2(\frac{\Omega_n}{2})}\frac{\vartheta_3^2(\tilde{u}_\infty - \frac{\Omega_n}{2})}{\vartheta_3^2(\tilde{u}_\infty)}\left(\frac{\vartheta_3'(\tilde{u}_\infty)}{\vartheta_3(\tilde{u}_\infty)} - \frac{\vartheta_3'(\tilde{u}_\infty - \frac{\Omega_n}{2})}{\vartheta_3(\tilde{u}_\infty - \frac{\Omega_n}{2})}\right) + O(x - \beta)\right], \tag{4.146}$$

and for $x \in (\alpha - \varepsilon, \alpha)$,

$$\vartheta_{11}^2(x) - \vartheta_{12}^2(x) = -\sqrt{\alpha - x}$$
$$\times \left[\frac{4\pi}{A_\alpha}\frac{\vartheta_3^2(0)}{\vartheta_3^2(\frac{\Omega_n}{2})}\frac{\vartheta_4^2(\tilde{u}_\infty - \frac{\Omega_n}{2})}{\vartheta_4^2(\tilde{u}_\infty)}\left(\frac{\vartheta_4'(\tilde{u}_\infty)}{\vartheta_4(\tilde{u}_\infty)} - \frac{\vartheta_4'(\tilde{u}_\infty - \frac{\Omega_n}{2})}{\vartheta_4(\tilde{u}_\infty - \frac{\Omega_n}{2})}\right) + O(\alpha - x)\right]. \tag{4.147}$$

Similarly, using (4.135) and (4.136), we can write, for $x \in (\alpha', \alpha' + \varepsilon)$,

$$K(x) = -\sqrt{x - \alpha'}$$
$$\times \left[\frac{4\pi}{A_{\alpha'}}\frac{\vartheta_3^2(0)}{\vartheta_3^2(\frac{\Omega_n}{2})}\frac{\vartheta_1^2(\tilde{u}_\infty - \frac{\Omega_n}{2})}{\vartheta_1^2(\tilde{u}_\infty)}\left(\frac{\vartheta_1'(\tilde{u}_\infty)}{\vartheta_1(\tilde{u}_\infty)} - \frac{\vartheta_1'(\tilde{u}_\infty - \frac{\Omega_n}{2})}{\vartheta_1'(\tilde{u}_\infty - \frac{\Omega_n}{2})}\right) + O(x - \alpha')\right], \tag{4.148}$$

and for $x \in (\beta' - \varepsilon, \beta')$,

$$K(x) = \sqrt{\beta' - x}$$
$$\times \left[\frac{4\pi}{A_{\beta'}}\frac{\vartheta_3^2(0)}{\vartheta_3^2(\frac{\Omega_n}{2})}\frac{\vartheta_2^2(\tilde{u}_\infty - \frac{\Omega_n}{2})}{\vartheta_2^2(\tilde{u}_\infty)}\left(\frac{\vartheta_2'(\tilde{u}_\infty)}{\vartheta_2(\tilde{u}_\infty)} - \frac{\vartheta_2'(\tilde{u}_\infty - \frac{\Omega_n}{2})}{\vartheta_2'(\tilde{u}_\infty - \frac{\Omega_n}{2})}\right) + O(\beta' - x)\right]. \tag{4.149}$$

From equations (4.137), (4.146), (4.147), (4.148), and (4.149), it follows that the functions $\big(\vartheta_{11}^2(z) - \vartheta_{12}^2(z)\big)\psi_\alpha^{-3/2}(z)$ and $\big(\vartheta_{11}^2(z) - \vartheta_{12}^2(z)\big)\psi_\beta^{-3/2}(z)$ are meromorphic in neighborhoods of α and β, respectively, and have simple poles at $z = \alpha$ and $z = \beta$, respectively, and that the functions $K(z)\psi_{\alpha'}^{-3/2}(z)$ and $K(z)\psi_{\beta'}^{-3/2}(z)$ are meromorphic in neighborhoods of α' and β', respectively, and have simple poles at $z=\alpha'$ and $z=\beta'$, respectively.

Let us compute the residues of functions that appear in (4.131). Observe that

$$\frac{\Omega_n}{2}=n\omega+\frac{\pi}{2},\quad \tilde{u}_\infty=-\frac{\omega}{2}+\frac{\pi}{2},\quad \frac{\Omega_n}{2}-\tilde{u}_\infty=n\omega+\frac{\omega}{2}.$$

From (4.138), (4.139), (4.142), and (4.143), we obtain that

$$\begin{aligned}&\operatorname*{Res}_{z=\alpha}3\big(\vartheta_{11}^2(z)+\vartheta_{12}^2(z)\big)\left(\gamma^2(z)+\gamma^{-2}(z)\right)\psi_\alpha^{-3/2}(z)\\&=\frac{\vartheta_3^2(0)\vartheta_4^2(n\omega+\frac{\omega}{2})}{\vartheta_3^2(\frac{\omega}{2})\vartheta_4^2(n\omega)}\left[-3(\beta'-\alpha)-\frac{9}{5}(\beta-\alpha)-\frac{9}{5}(\alpha'-\alpha)+\frac{6}{5}\frac{(\alpha'-\alpha)(\beta-\alpha)}{(\beta'-\alpha)}\right]\\&\quad-\frac{3\pi^2}{4(\beta'-\alpha)}\frac{d^2}{d\tilde{u}^2}\left(\vartheta_{11}^2+\vartheta_{12}^2\right)\Big|_{z=\alpha}\end{aligned}\tag{4.150}$$

and

$$\begin{aligned}&\operatorname*{Res}_{z=\beta}3\big(\vartheta_{11}^2(z)+\vartheta_{12}^2(z)\big)\left(\gamma^2(z)+\gamma^{-2}(z)\right)\psi_\beta^{-3/2}(z)\\&=\frac{\vartheta_3^2(0)\vartheta_3^2(n\omega+\frac{\omega}{2})}{\vartheta_4^2(\frac{\omega}{2})\vartheta_4^2(n\omega)}\left[3(\beta-\alpha')+\frac{9}{5}(\beta-\beta')+\frac{9}{5}(\beta-\alpha)-\frac{6}{5}\frac{(\beta-\alpha)(\beta-\beta')}{(\beta-\alpha')}\right]\\&\quad+\frac{3\pi^2}{4(\beta-\alpha')}\frac{d^2}{d\tilde{u}^2}\left(\vartheta_{11}^2+\vartheta_{12}^2\right)\Big|_{z=\beta}.\end{aligned}$$

Also,

$$\begin{aligned}&\operatorname*{Res}_{z=\alpha}\big(\vartheta_{11}(z)\vartheta_{12}(z)\big)\left(\gamma^2(z)-\gamma^{-2}(z)\right)\psi_\alpha^{-3/2}(z)\\&=\frac{\vartheta_3^2(0)\vartheta_4^2(n\omega+\frac{\omega}{2})}{\vartheta_3^2(\frac{\omega}{2})\vartheta_4^2(n\omega)}\left[-\frac{(\beta'-\alpha)}{2}+\frac{3}{10}(\beta-\alpha)+\frac{3}{10}(\alpha'-\alpha)-\frac{1}{5}\frac{(\alpha'-\alpha)(\beta-\alpha)}{(\beta'-\alpha)}\right]\\&\quad+\frac{\pi^2}{4(\beta'-\alpha)}\frac{d^2}{d\tilde{u}^2}\left(\vartheta_{11}\vartheta_{12}\right)\Big|_{z=\alpha},\end{aligned}\tag{4.151}$$

and

$$\begin{aligned}&\operatorname*{Res}_{z=\beta}\big(\vartheta_{11}(z)\vartheta_{12}(z)\big)\left(\gamma^2(z)-\gamma^{-2}(z)\right)\psi_\beta^{-3/2}(z)\\&=\frac{\vartheta_3^2(0)\vartheta_3^2(n\omega+\frac{\omega}{2})}{\vartheta_4^2(\frac{\omega}{2})\vartheta_4^2(n\omega)}\left[-\frac{(\beta-\alpha')}{2}+\frac{3}{10}(\beta-\beta')+\frac{3}{10}(\beta-\alpha)-\frac{1}{5}\frac{(\beta-\alpha)(\beta-\beta')}{(\beta-\alpha')}\right]\\&\quad+\frac{\pi^2}{4(\beta-\alpha')}\frac{d^2}{d\tilde{u}^2}\left(\vartheta_{11}\vartheta_{12}\right)\Big|_{z=\beta}.\end{aligned}$$

From (4.137), (4.146), and (4.147), we obtain that

$$\operatorname*{Res}_{z=\alpha} 6\big(\vartheta_{11}^2(z)-\vartheta_{12}^2(z)\big)\psi_\alpha^{-3/2}(z) = -12\pi \frac{\vartheta_3^2(0)\vartheta_4^2(n\omega+\frac{\omega}{2})}{\vartheta_3^2(\frac{\omega}{2})\vartheta_4^2(n\omega)}\left[\frac{\vartheta_3'(\frac{\omega}{2})}{\vartheta_3(\frac{\omega}{2})} - \frac{\vartheta_4'(n\omega+\frac{\omega}{2})}{\vartheta_4(n\omega+\frac{\omega}{2})}\right] \tag{4.152}$$

and

$$\operatorname*{Res}_{z=\beta} 6\big(\vartheta_{11}^2(z)-\vartheta_{12}^2(z)\big)\psi_\beta^{-3/2}(z) = -12\pi \frac{\vartheta_3^2(0)\vartheta_3^2(n\omega+\frac{\omega}{2})}{\vartheta_4^2(\frac{\omega}{2})\vartheta_4^2(n\omega)}\left[\frac{\vartheta_4'(\frac{\omega}{2})}{\vartheta_4(\frac{\omega}{2})} - \frac{\vartheta_3'(n\omega+\frac{\omega}{2})}{\vartheta_3'(n\omega+\frac{\omega}{2})}\right].$$

We now turn our attention to the residues at the inner turning points. From (4.140), (4.141), (4.144), and (4.145), we have

$$\begin{aligned}&\operatorname*{Res}_{z=\alpha'} 3\psi_{\alpha'}^{-3/2}(z)J(z)\big(\gamma^2(z)+\gamma^{-2}(z)\big)\\ &= \frac{\vartheta_3^2(0)\vartheta_1^2(n\omega+\frac{\omega}{2})}{\vartheta_2^2(\frac{\omega}{2})\vartheta_4^2(n\omega)}\left[3(\beta-\alpha')+\frac{9}{5}(\beta'-\alpha')-\frac{9}{5}(\alpha'-\alpha)+\frac{6}{5}\frac{(\alpha'-\alpha)(\beta'-\alpha')}{\beta-\alpha'}\right]\\ &\quad+\frac{3\pi^2}{4(\beta-\alpha')}\,\frac{d^2}{d\tilde{u}^2}J(z)\bigg|_{z=\alpha'}\end{aligned}$$

and

$$\begin{aligned}&\operatorname*{Res}_{z=\beta'} 3\psi_{\beta'}^{-3/2}(z)J(z)\big(\gamma^2(z)+\gamma^{-2}(z)\big)\\ &= \frac{\vartheta_3^2(0)\vartheta_2^2(n\omega+\frac{\omega}{2})}{\vartheta_1^2(\frac{\omega}{2})\vartheta_4^2(n\omega)}\left[-3(\beta'-\alpha)-\frac{9}{5}(\beta'-\alpha')+\frac{9}{5}(\beta-\beta')-\frac{6}{5}\frac{(\beta'-\alpha')(\beta-\beta')}{\beta'-\alpha}\right]\\ &\quad-\frac{3\pi^2}{4(\beta'-\alpha)}\,\frac{d^2}{d\tilde{u}^2}J(z)\bigg|_{z=\beta'}.\end{aligned}$$

Also,

$$\begin{aligned}&\operatorname*{Res}_{z=\alpha'} \psi_{\alpha'}^{-3/2}(z)\big(\gamma^2(z)-\gamma^{-2}(z)\big)\vartheta_{11}(z)\vartheta_{12}(z)\\ &= \frac{\vartheta_3^2(0)\vartheta_1^2(n\omega+\frac{\omega}{2})}{\vartheta_2^2(\frac{\omega}{2})\vartheta_4^2(n\omega)}\left[-\frac{(\beta-\alpha')}{2}+\frac{3}{10}(\beta'-\alpha')-\frac{3}{10}(\alpha'-\alpha)+\frac{1}{5}\frac{(\alpha'-\alpha)(\beta'-\alpha')}{\beta-\alpha'}\right]\\ &\quad+\frac{\pi^2}{4(\beta-\alpha')}\,\frac{d^2}{d\tilde{u}^2}\Big(\vartheta_{11}\vartheta_{12}\Big)\bigg|_{z=\alpha'}\end{aligned}$$

and

$$\begin{aligned}&\operatorname*{Res}_{z=\beta'} \psi_{\beta'}^{-3/2}(z)\big(\gamma^2(z)-\gamma^{-2}(z)\big)\vartheta_{11}(z)\vartheta_{12}(z)\\ &= \frac{\vartheta_3^2(0)\vartheta_2^2(n\omega+\frac{\omega}{2})}{\vartheta_1^2(\frac{\omega}{2})\vartheta_4^2(n\omega)}\left[-\frac{(\beta'-\alpha)}{2}+\frac{3}{10}(\beta'-\alpha')-\frac{3}{10}(\beta-\beta')+\frac{1}{5}\frac{(\beta'-\alpha')(\beta-\beta')}{\beta'-\alpha}\right]\\ &\quad+\frac{\pi^2}{4(\beta'-\alpha)}\,\frac{d^2}{d\tilde{u}^2}\Big(\vartheta_{11}\vartheta_{12}\Big)\bigg|_{z=\beta'}.\end{aligned}$$

From (4.137), (4.148), (4.149), we have

$$\operatorname*{Res}_{z=\alpha'} 6\psi_{\alpha'}^{-3/2}(z)K(z) = 12\pi \frac{\vartheta_3^2(0)\vartheta_1^2(n\omega+\frac{\omega}{2})}{\vartheta_2^2(\frac{\omega}{2})\vartheta_4^2(n\omega)}\left[\frac{\vartheta_2'(\frac{\omega}{2})}{\vartheta_2(\frac{\omega}{2})} - \frac{\vartheta_1'(n\omega+\frac{\omega}{2})}{\vartheta_1(n\omega+\frac{\omega}{2})}\right]$$

and

$$\operatorname*{Res}_{z=\beta'} 6\psi_{\beta'}^{-3/2}(z)K(z) = 12\pi \frac{\vartheta_3^2(0)\vartheta_2^2(n\omega+\frac{\omega}{2})}{\vartheta_1^2(\frac{\omega}{2})\vartheta_4^2(n\omega)}\left[\frac{\vartheta_1'(\frac{\omega}{2})}{\vartheta_1(\frac{\omega}{2})} - \frac{\vartheta_2'(n\omega+\frac{\omega}{2})}{\vartheta_2(n\omega+\frac{\omega}{2})}\right].$$

Combining (4.130),(4.131),(4.150),(4.151), and (4.152), we get that

$$\begin{aligned}\operatorname*{Res}_{z=\alpha}[j_X^0(z)] =& \frac{i}{96}\left[\frac{\vartheta_3^2(0)\vartheta_4^2(n\omega+\frac{\omega}{2})}{\vartheta_3^2(\frac{\omega}{2})\vartheta_4^2(n\omega)}\left(-\frac{7}{2}(\beta'-\alpha)-\frac{3}{2}(\beta-\alpha)-\frac{3}{2}(\alpha'-\alpha)\right.\right.\\ &\left.+\frac{(\alpha'-\alpha)(\beta-\alpha)}{\beta'-\alpha} - 12\pi\left(\frac{\vartheta_3'(\frac{\omega}{2})}{\vartheta_3(\frac{\omega}{2})} - \frac{\vartheta_4'(n\omega+\frac{\omega}{2})}{\vartheta_4(n\omega+\frac{\omega}{2})}\right)\right)\\ &\left.-\frac{\pi^2}{4(\beta'-\alpha)}\frac{d^2}{d\tilde{u}^2}\Big[3(\vartheta_{11}^2+\vartheta_{12}^2)-\vartheta_{11}\vartheta_{12}\Big]_{z=\alpha}\right].\end{aligned}$$

Similarly, we have

$$\begin{aligned}\operatorname*{Res}_{z=\alpha'}[j_X^0(z)] = & \frac{i}{96}\left[\frac{\vartheta_3^2(0)\vartheta_1^2(n\omega+\frac{\omega}{2})}{\vartheta_2^2(\frac{\omega}{2})\vartheta_4^2(n\omega)}\left(\frac{7}{2}(\beta-\alpha')+\frac{3}{2}(\beta'-\alpha')-\frac{3}{2}(\alpha'-\alpha)\right.\right.\\ &\left.+\frac{(\alpha'-\alpha)(\beta'-\alpha')}{\beta-\alpha'} + 12\pi\left(\frac{\vartheta_2'(\frac{\omega}{2})}{\vartheta_2(\frac{\omega}{2})} - \frac{\vartheta_1'(n\omega+\frac{\omega}{2})}{\vartheta_1(n\omega+\frac{\omega}{2})}\right)\right)\\ &\left.+\frac{\pi^2}{4(\beta-\alpha')}\frac{d^2}{d\tilde{u}^2}\Big[3J(z)-\vartheta_{11}\vartheta_{12}\Big]_{z=\alpha'}\right],\end{aligned}$$

$$\begin{aligned}\operatorname*{Res}_{z=\beta'}[j_X^0(z)] = & \frac{i}{96}\left[\frac{\vartheta_3^2(0)\vartheta_2^2(n\omega+\frac{\omega}{2})}{\vartheta_1^2(\frac{\omega}{2})\vartheta_4^2(n\omega)}\left(\frac{7}{2}(\beta'-\alpha)+\frac{3}{2}(\beta'-\alpha')-\frac{3}{2}(\beta-\beta')\right.\right.\\ &\left.+\frac{(\beta-\beta')(\beta'-\alpha')}{\beta'-\alpha} - 12\pi\left(\frac{\vartheta_1'(\frac{\omega}{2})}{\vartheta_1(\frac{\omega}{2})} - \frac{\vartheta_2'(n\omega+\frac{\omega}{2})}{\vartheta_2(n\omega+\frac{\omega}{2})}\right)\right)\\ &\left.+\frac{\pi^2}{4(\beta'-\alpha)}\frac{d^2}{d\tilde{u}^2}\Big[3J(z)-\vartheta_{11}\vartheta_{12}\Big]_{z=\beta'}\right],\end{aligned}$$

and

$$\begin{aligned}\operatorname*{Res}_{z=\beta}[j_X^0(z)] =& \frac{i}{96}\left[\frac{\vartheta_3^2(0)\vartheta_3^2(n\omega+\frac{\omega}{2})}{\vartheta_4^2(\frac{\omega}{2})\vartheta_4^2(n\omega)}\left(-\frac{7}{2}(\beta-\alpha')-\frac{3}{2}(\beta-\alpha)-\frac{3}{2}(\beta-\beta')\right.\right.\\ &\left.+\frac{(\beta-\beta')(\beta-\alpha)}{\beta-\alpha'} + 12\pi\left(\frac{\vartheta_4'(\frac{\omega}{2})}{\vartheta_4(\frac{\omega}{2})} - \frac{\vartheta_3'(n\omega+\frac{\omega}{2})}{\vartheta_3(n\omega+\frac{\omega}{2})}\right)\right)\\ &\left.-\frac{\pi^2}{4(\beta-\alpha')}\frac{d^2}{d\tilde{u}^2}\Big[3(\vartheta_{11}^2+\vartheta_{12}^2)-\vartheta_{11}\vartheta_{12}\Big]_{z=\beta}\right].\end{aligned}$$

Using MAPLE for calculations, we get

$$\frac{d^2}{d\tilde{u}^2}[3(\vartheta_{11}^2+\vartheta_{12}^2)-\vartheta_{11}\vartheta_{12}]\Big|_{z=\alpha} = 2\frac{\vartheta_3^2(0)\vartheta_4^2(n\omega+\frac{\omega}{2})}{\vartheta_3^2(\frac{\omega}{2})\vartheta_4^2(n\omega)}\left[5\frac{\vartheta_4''(n\omega+\frac{\omega}{2})}{\vartheta_4(n\omega+\frac{\omega}{2})}-5\frac{\vartheta_3''(\frac{\omega}{2})}{\vartheta_3(\frac{\omega}{2})}\right.$$
$$\left.+7\left(\frac{\vartheta_4'(n\omega+\frac{\omega}{2})}{\vartheta_4(n\omega+\frac{\omega}{2})}\right)^2+17\left(\frac{\vartheta_3'(\frac{\omega}{2})}{\vartheta_3(\frac{\omega}{2})}\right)^2-24\frac{\vartheta_4'(n\omega+\frac{\omega}{2})\vartheta_3'(\frac{\omega}{2})}{\vartheta_4(n\omega+\frac{\omega}{2})\vartheta_3(\frac{\omega}{2})}\right],$$

$$\frac{d^2}{d\tilde{u}^2}[3(\vartheta_{11}^2+\vartheta_{12}^2)-\vartheta_{11}\vartheta_{12}]\Big|_{z=\beta} = 2\frac{\vartheta_3^2(0)\vartheta_3^2(n\omega+\frac{\omega}{2})}{\vartheta_4^2(\frac{\omega}{2})\vartheta_4^2(n\omega)}\left[5\frac{\vartheta_3''(n\omega+\frac{\omega}{2})}{\vartheta_3(n\omega+\frac{\omega}{2})}-5\frac{\vartheta_4''(\frac{\omega}{2})}{\vartheta_4(\frac{\omega}{2})}\right.$$
$$\left.+7\left(\frac{\vartheta_3'(n\omega+\frac{\omega}{2})}{\vartheta_3(n\omega+\frac{\omega}{2})}\right)^2+17\left(\frac{\vartheta_4'(\frac{\omega}{2})}{\vartheta_4(\frac{\omega}{2})}\right)^2-24\frac{\vartheta_3'(n\omega+\frac{\omega}{2})\vartheta_4'(\frac{\omega}{2})}{\vartheta_3(n\omega+\frac{\omega}{2})\vartheta_4(\frac{\omega}{2})}\right],$$

$$\frac{d^2}{d\tilde{u}^2}[3J(z)-\vartheta_{11}\vartheta_{12}]\Big|_{z=\alpha'} = 2\frac{\vartheta_3^2(0)\vartheta_1^2(n\omega+\frac{\omega}{2})}{\vartheta_2^2(\frac{\omega}{2})\vartheta_4^2(n\omega)}\left[5\frac{\vartheta_1''(n\omega+\frac{\omega}{2})}{\vartheta_1(n\omega+\frac{\omega}{2})}-5\frac{\vartheta_2''(\frac{\omega}{2})}{\vartheta_2(\frac{\omega}{2})}\right.$$
$$\left.+7\left(\frac{\vartheta_1'(n\omega+\frac{\omega}{2})}{\vartheta_1(n\omega+\frac{\omega}{2})}\right)^2+17\left(\frac{\vartheta_2'(\frac{\omega}{2})}{\vartheta_2(\frac{\omega}{2})}\right)^2-24\frac{\vartheta_1'(n\omega+\frac{\omega}{2})\vartheta_2'(\frac{\omega}{2})}{\vartheta_1(n\omega+\frac{\omega}{2})\vartheta_2(\frac{\omega}{2})}\right],$$

$$\frac{d^2}{d\tilde{u}^2}[3J(z)-\vartheta_{11}\vartheta_{12}]\Big|_{z=\beta'} = 2\frac{\vartheta_3^2(0)\vartheta_2^2(n\omega+\frac{\omega}{2})}{\vartheta_1^2(\frac{\omega}{2})\vartheta_4^2(n\omega)}\left[5\frac{\vartheta_2''(n\omega+\frac{\omega}{2})}{\vartheta_2(n\omega+\frac{\omega}{2})}-5\frac{\vartheta_1''(\frac{\omega}{2})}{\vartheta_1(\frac{\omega}{2})}\right.$$
$$\left.+7\left(\frac{\vartheta_2'(n\omega+\frac{\omega}{2})}{\vartheta_2(n\omega+\frac{\omega}{2})}\right)^2+17\left(\frac{\vartheta_1'(\frac{\omega}{2})}{\vartheta_1(\frac{\omega}{2})}\right)^2-24\frac{\vartheta_2'(n\omega+\frac{\omega}{2})\vartheta_1'(\frac{\omega}{2})}{\vartheta_2(n\omega+\frac{\omega}{2})\vartheta_1(\frac{\omega}{2})}\right].$$

These formulae, combined with (4.132), prove Lemma 4.17.1.

4.19 Large n asymptotic formula for h_n

We evaluate the large n asymptotic behavior of h_{nn} and then we use formula (4.7). By (4.67), $h_{nn} = [\mathbf{P}_1]_{12}$, and by (4.71),

$$[\mathbf{P}_1]_{12} = [\mathbf{R}_1]_{12}\left(-\frac{n\pi i}{\gamma}\right),$$

hence

$$h_{nn} = [\mathbf{R}_1]_{12}\left(-\frac{n\pi i}{\gamma}\right).$$

Furthermore, from (4.74) we obtain that

$$h_{nn} = e^{nl}[\mathbf{T}_1]_{12}\left(-\frac{n\pi i}{\gamma}\right),$$

and from (4.77), that

$$h_{nn} = e^{nl}[\mathbf{S}_1]_{12}\left(-\frac{n\pi i}{\gamma}\right).$$

It follows from (4.119) that

$$\mathbf{S}_1 = \mathbf{M}_1 + \mathbf{X}_1.$$

By (4.97),

$$[\mathbf{M}_1]_{12} = \frac{iA\vartheta_4\big((n+1)\omega\big)}{\vartheta_4(n\omega)}, \qquad \omega = \frac{\pi(1+\zeta)}{2}, \qquad A = \frac{\pi\vartheta_1'(0)}{2\vartheta_1(\omega)},$$

and by (4.127),

$$[\mathbf{X}_1]_{12} = \frac{c(n)}{n} + O(n^{-2}),$$

where

$$c(n) = X_\alpha + X_{\alpha'} + X_{\beta'} + X_\beta \tag{4.153}$$

is an explicit quasi-periodic function of n. Therefore,

$$h_{nn} = e^{nl}\left[\frac{iA\vartheta_4\big((n+1)\omega\big)}{\vartheta_4(n\omega)} + \frac{c(n)}{n} + O(n^{-2})\right]\left(-\frac{n\pi i}{\gamma}\right).$$

By (4.33),

$$e^{\frac{l}{2}} = \frac{\pi\vartheta_1'(0)}{2e\vartheta_1(\omega)} = \frac{A}{e},$$

hence

$$\begin{aligned} h_{nn} &= \left(\frac{A}{e}\right)^{2n}\left[iA\,\frac{\vartheta_4\big((n+1)\omega\big)}{\vartheta_4(n\omega)} + \frac{c(n)}{n} + O(n^{-2})\right]\left(-\frac{n\pi i}{\gamma}\right) \\ &= \frac{n\pi A^{2n+1}\vartheta_4\big((n+1)\omega\big)}{\gamma e^{2n}\vartheta_4(n\omega)}\left(1 + \frac{c_1(n)}{n} + O(n^{-2})\right), \end{aligned} \tag{4.154}$$

where

$$c_1(n) = \frac{c(n)\vartheta_4(n\omega)}{iA\vartheta_4\big((n+1)\omega\big)}. \tag{4.155}$$

From (4.7) and the Stirling formula we obtain that

$$\frac{h_n}{(n!)^2} = \frac{n^{2n}h_{nn}}{(n!)^2(2\gamma)^{2n}} = \left(\frac{e}{2\gamma}\right)^{2n}\frac{h_{nn}}{2\pi n}\left(1 - \frac{1}{6n} + O(n^{-2})\right),$$

hence by (4.154),

$$\frac{h_n}{(n!)^2} = \left(\frac{e}{2\gamma}\right)^{2n} \frac{1}{2\pi n} \frac{n\pi A^{2n+1}\vartheta_4\big((n+1)\omega\big)}{\gamma e^{2n}\vartheta_4(n\omega)} \left(1 + \frac{c_1(n)}{n} - \frac{1}{6n} + O(n^{-2})\right)$$

$$= G^{2n+1} \frac{\vartheta_4\big((n+1)\omega\big)}{\vartheta_4(n\omega)} \left(1 + \frac{c_2(n)}{n} + O(n^{-2})\right),$$

where

$$G = \frac{A}{2\gamma} = \frac{\pi\vartheta_1'(0)}{4\gamma\vartheta_1(\omega)}, \qquad c_2(n) = c_1(n) - \frac{1}{6}.$$

Observe that $c_1(n)$ has the form,

$$c_1(n) = f(n\omega, \omega),$$

where $f(x, \omega)$ is a real analytic function which is periodic with respect to both x and ω, of periods π and 2π, respectively, so that

$$f(x + \pi, \omega) = f(x, \omega), \qquad f(x, \omega + 2\pi) = f(x, \omega).$$

We can now summarize now the asymptotic formula for $h_n/(n!)^2$.

Proposition 4.19.1 *As $n \to \infty$,*

$$\frac{h_n}{(n!)^2} = G^{2n+1} \frac{\vartheta_4\big((n+1)\omega\big)}{\vartheta_4(n\omega)} \left[1 + \frac{f_0(n\omega, \omega)}{n} + O(n^{-2})\right],$$

where

$$G = \frac{\pi\vartheta_1'(0)}{4\gamma\vartheta_1(\omega)} \tag{4.156}$$

and

$$f_0(x, \omega) = f(x, \omega) - \frac{1}{6}$$

is a real analytic function which satisfies the periodicity conditions

$$f_0(x + \pi, \omega) = f_0(x, \omega), \qquad f_0(x, \omega + 2\pi) = f_0(x, \omega).$$

By (4.155),

$$f(n\omega, \omega) = \frac{X(n\omega, \omega)\vartheta_4(n\omega)}{iA\vartheta_4\big((n+1)\omega\big)}.$$

where

$$X \equiv X_\alpha + X_{\alpha'} + X_{\beta'} + X_\beta$$

and explicit expressions for $X_\alpha, X_{\alpha'}, X_{\beta'}, X_\beta$ are given in Lemma 4.17.1.

In the subsequent sections, we carry out a concrete evaluation of $f(n\omega, \omega)$, obtaining that

$$f_0(x, \omega) \equiv 0, \tag{4.157}$$

thus we can improve Proposition 4.19.1 to the following.

Proposition 4.19.2 *As $n \to \infty$,*

$$\frac{h_n}{(n!)^2} = G^{2n+1} \frac{\vartheta_4\big((n+1)\omega\big)}{\vartheta_4(n\omega)} \left(1 + O(n^{-2})\right), \tag{4.158}$$

where G is defined in (4.156).

To this end, we will first show that $f(n\omega, \omega)$ does not depend on n.

4.20 The function $f(x, \omega)$ is constant in x

Denote

$$z = n\omega + \frac{\omega}{2}, \quad \tilde{f}(x, \omega) \equiv f\left(x - \frac{\omega}{2}, \omega\right),$$

so that

$$\tilde{f}(z, \omega) = f(n\omega, \omega).$$

To prove that $f(x, \omega)$ is constant in x we will prove the following lemmas.

Lemma 4.20.1 *The function $\tilde{f}(z, \omega)$ is doubly periodic in z.*

Lemma 4.20.2 *The function $\tilde{f}(z, \omega)$ is analytic throughout the z-plane.*

From these two lemmas, it follows immediately that $\tilde{f}(z, \omega)$ is constant in z, as it is a doubly periodic entire function of z, and it thus follows that $f(x, \omega)$ is constant in x.

To prove Lemma 4.20.1, we will check that $\tilde{f}(z + \pi\tau, \omega) = \tilde{f}(z, \omega)$. By (4.155), (4.153),

$$c_1(n) \equiv f(n\omega, \omega) = \tilde{f}(z, \omega) = Y_1 + Y_2 + Y_3 + Y_4,$$

where

$$
\begin{aligned}
Y_1 &= \frac{X_\alpha \vartheta_4(n\omega)}{iA\vartheta_4(n\omega+\omega)} \\
&= \frac{\vartheta_1(\omega)\vartheta_3^2(0)\vartheta_4^2(z)}{48\pi\vartheta_1'(0)\vartheta_3^2(\frac{\omega}{2})\vartheta_4(z-\frac{\omega}{2})\vartheta_4(z+\frac{\omega}{2})}\left(C_\alpha + 12\pi\xi_\alpha + \frac{\pi^2\eta_\alpha}{2(\beta'-\alpha)}\right), \\
Y_2 &= \frac{X_{\alpha'} \vartheta_4(n\omega)}{iA\vartheta_4(n\omega+\omega)} \\
&= \frac{\vartheta_1(\omega)\vartheta_3^2(0)\vartheta_1^2(z)}{48\pi\vartheta_1'(0)\vartheta_2^2(\frac{\omega}{2})\vartheta_4(z-\frac{\omega}{2})\vartheta_4(z+\frac{\omega}{2})}\left(C_{\alpha'} + 12\pi\xi_{\alpha'} + \frac{\pi^2\eta_{\alpha'}}{2(\beta-\alpha')}\right), \\
Y_3 &= \frac{X_{\beta'} \vartheta_4(n\omega)}{iA\vartheta_4(n\omega+\omega)} \\
&= \frac{\vartheta_1(\omega)\vartheta_3^2(0)\vartheta_2^2(z)}{48\pi\vartheta_1'(0)\vartheta_1^2(\frac{\omega}{2})\vartheta_4(z-\frac{\omega}{2})\vartheta_4(z+\frac{\omega}{2})}\left(C_{\beta'} + 12\pi\xi_{\beta'} + \frac{\pi^2\eta_{\beta'}}{2(\beta'-\alpha)}\right), \\
Y_4 &= \frac{X_\beta \vartheta_4(n\omega)}{iA\vartheta_4(n\omega+\omega)} \\
&= \frac{\vartheta_1(\omega)\vartheta_3^2(0)\vartheta_3^2(z)}{48\pi\vartheta_1'(0)\vartheta_4^2(\frac{\omega}{2})\vartheta_4(z-\frac{\omega}{2})\vartheta_4(z+\frac{\omega}{2})}\left(C_\beta + 12\pi\xi_\beta + \frac{\pi^2\eta_\beta}{2(\beta-\alpha')}\right).
\end{aligned}
$$

Observe that Y_1, Y_2, Y_3, Y_4 can be written in the form

$$
\begin{aligned}
Y_1 &= Q_{11}h_{11}(z) + Q_{12}h_{12}(z) + Q_{13}h_{13}(z) + Q_{14}h_{14}(z), \\
Y_2 &= Q_{21}h_{21}(z) + Q_{22}h_{22}(z) + Q_{23}h_{23}(z) + Q_{24}h_{24}(z), \\
Y_3 &= Q_{31}h_{31}(z) + Q_{32}h_{32}(z) + Q_{33}h_{33}(z) + Q_{34}h_{34}(z), \\
Y_4 &= Q_{41}h_{41}(z) + Q_{42}h_{42}(z) + Q_{43}h_{43}(z) + Q_{44}h_{44}(z),
\end{aligned}
$$

where

$$h_{11}(z)=\frac{\vartheta_4^2(z)}{\vartheta_4(z-\frac{\omega}{2})\vartheta_4(z+\frac{\omega}{2})},\quad h_{12}(z)=\frac{\vartheta_4'(z)\vartheta_4(z)}{\vartheta_4(z-\frac{\omega}{2})\vartheta_4(z+\frac{\omega}{2})},$$
$$h_{13}(z)=\frac{\vartheta_4'(z)^2}{\vartheta_4(z-\frac{\omega}{2})\vartheta_4(z+\frac{\omega}{2})},\quad h_{14}(z)=\frac{\vartheta_4''(z)\vartheta_4(z)}{\vartheta_4(z-\frac{\omega}{2})\vartheta_4(z+\frac{\omega}{2})},$$
$$h_{21}(z)=\frac{\vartheta_1^2(z)}{\vartheta_4(z-\frac{\omega}{2})\vartheta_4(z+\frac{\omega}{2})},\quad h_{22}(z)=\frac{\vartheta_1'(z)\vartheta_1(z)}{\vartheta_4(z-\frac{\omega}{2})\vartheta_4(z+\frac{\omega}{2})},$$
$$h_{23}(z)=\frac{\vartheta_1'(z)^2}{\vartheta_4(z-\frac{\omega}{2})\vartheta_4(z+\frac{\omega}{2})},\quad h_{24}(z)=\frac{\vartheta_1''(z)\vartheta_1(z)}{\vartheta_4(z-\frac{\omega}{2})\vartheta_4(z+\frac{\omega}{2})},$$
$$h_{31}(z)=\frac{\vartheta_2^2(z)}{\vartheta_4(z-\frac{\omega}{2})\vartheta_4(z+\frac{\omega}{2})},\quad h_{32}(z)=\frac{\vartheta_2'(z)\vartheta_2(z)}{\vartheta_4(z-\frac{\omega}{2})\vartheta_4(z+\frac{\omega}{2})},$$
$$h_{33}(z)=\frac{\vartheta_2'(z)^2}{\vartheta_4(z-\frac{\omega}{2})\vartheta_4(z+\frac{\omega}{2})},\quad h_{34}(z)=\frac{\vartheta_2''(z)\vartheta_2(z)}{\vartheta_4(z-\frac{\omega}{2})\vartheta_4(z+\frac{\omega}{2})},$$
$$h_{41}(z)=\frac{\vartheta_3^2(z)}{\vartheta_4(z-\frac{\omega}{2})\vartheta_4(z+\frac{\omega}{2})},\quad h_{42}(z)=\frac{\vartheta_3'(z)\vartheta_3(z)}{\vartheta_4(z-\frac{\omega}{2})\vartheta_4(z+\frac{\omega}{2})},$$
$$h_{43}(z)=\frac{\vartheta_3'(z)^2}{\vartheta_4(z-\frac{\omega}{2})\vartheta_4(z+\frac{\omega}{2})},\quad h_{44}(z)=\frac{\vartheta_3''(z)\vartheta_3(z)}{\vartheta_4(z-\frac{\omega}{2})\vartheta_4(z+\frac{\omega}{2})},$$

and the numbers Q_{ij} do not depend on z. More specifically,

$$Q_{11}=\frac{\vartheta_1(\omega)\vartheta_3^2(0)}{48\pi\vartheta_1'(0)\vartheta_3^2(\frac{\omega}{2})}\times\left[C_\alpha+12\pi\frac{\vartheta_3'(\frac{\omega}{2})}{\vartheta_3(\frac{\omega}{2})}+\frac{\pi^2}{2(\beta'-\alpha)}\left(-5\frac{\vartheta_3''(\frac{\omega}{2})}{\vartheta_3(\frac{\omega}{2})}+17\left(\frac{\vartheta_3'(\frac{\omega}{2})}{\vartheta_3(\frac{\omega}{2})}\right)^2\right)\right],$$

$$Q_{12}=-\frac{\vartheta_1(\omega)\vartheta_3^2(0)}{4\vartheta_1'(0)\vartheta_3^2(\frac{\omega}{2})}-\frac{\pi\vartheta_1(\omega)\vartheta_3^2(0)\vartheta_3'(\frac{\omega}{2})}{4(\beta'-\alpha)\vartheta_1'(0)\vartheta_3^3(\frac{\omega}{2})},\qquad Q_{13}=\frac{7\pi\vartheta_1(\omega)\vartheta_3^2(0)}{96(\beta'-\alpha)\vartheta_1'(0)\vartheta_3^2(\frac{\omega}{2})},$$

$$Q_{14}=\frac{5\pi\vartheta_1(\omega)\vartheta_3^2(0)}{96(\beta'-\alpha)\vartheta_1'(0)\vartheta_3^2(\frac{\omega}{2})},$$

and similar formulae hold for other Q_{ij}. In particular, notice that

$$Q_{13}+Q_{14}=\frac{\pi\vartheta_1(\omega)\vartheta_3^2(0)}{8(\beta'-\alpha)\vartheta_1'(0)\vartheta_3^2(\frac{\omega}{2})}=\frac{\vartheta_1(\omega)\vartheta_1(\frac{\omega}{2})}{8\vartheta_1'(0)\vartheta_2(\frac{\omega}{2})\vartheta_3(\frac{\omega}{2})\vartheta_4(\frac{\omega}{2})},$$
$$Q_{23}+Q_{24}=-\frac{\pi\vartheta_1(\omega)\vartheta_3^2(0)}{8(\beta-\alpha')\vartheta_1'(0)\vartheta_2^2(\frac{\omega}{2})}=-\frac{\vartheta_1(\omega)\vartheta_4(\frac{\omega}{2})}{8\vartheta_1'(0)\vartheta_1(\frac{\omega}{2})\vartheta_2(\frac{\omega}{2})\vartheta_3(\frac{\omega}{2})},$$
$$Q_{33}+Q_{34}=-\frac{\pi\vartheta_1(\omega)\vartheta_3^2(0)}{8(\beta'-\alpha)\vartheta_1'(0)\vartheta_1^2(\frac{\omega}{2})}=-\frac{\vartheta_1(\omega)\vartheta_3(\frac{\omega}{2})}{8\vartheta_1'(0)\vartheta_1(\frac{\omega}{2})\vartheta_2(\frac{\omega}{2})\vartheta_4(\frac{\omega}{2})},$$
$$Q_{43}+Q_{44}=\frac{\pi\vartheta_1(\omega)\vartheta_3^2(0)}{8(\beta-\alpha')\vartheta_1'(0)\vartheta_4^2(\frac{\omega}{2})}=\frac{\vartheta_1(\omega)\vartheta_2(\frac{\omega}{2})}{8\vartheta_1'(0)\vartheta_1(\frac{\omega}{2})\vartheta_3(\frac{\omega}{2})\vartheta_4(\frac{\omega}{2})}.$$

Observe that all $Q_{ij} = Q_{ij}(\omega)$ are periodic functions of ω of period 2π. It follows from equations (4.9) that the functions

$$h_{j1}(z), \qquad j = 1, 2, 3, 4,$$

are doubly periodic,

$$h_{j1}(z + \pi) = h_{j1}(z)\,, \qquad h_{j1}(z + \pi\tau) = h_{j1}(z),$$

while the functions,

$$h_{j2}(z),\ h_{j3}(z),\ h_{j4}(z), \quad j = 1, 2, 3, 4$$

satisfy the equations

$$h_{jk}(z + \pi) = h_{jk}(z), \qquad k = 2, 3, 4;$$
$$h_{j2}(z + \pi\tau) = h_{j2}(z) - 2ih_{j1}(z), \qquad h_{j3}(z + \pi\tau) = h_{j3}(z) - 4ih_{j2}(z) - 4h_{j1}(z),$$
$$h_4(z + \pi\tau) = h_{j4}(z) - 4ih_{j2}(z) - 4h_{j1}(z).$$

This implies that the functions $Y_j = Y_j(z)$ for $j = 1, 2, 3, 4$, satisfy the equations

$$\begin{aligned} Y_j(z + \pi) &= Y_j(z), \\ Y_j(z + \pi\tau) &= Y_j(z) + (-2iQ_{j2} - 4Q_{j3} - 4Q_{j4})h_{j1}(z) \\ &\quad + (-4iQ_{j3} - 4iQ_{j4})h_{j2}(z). \end{aligned} \tag{4.159}$$

The proof of Lemma 4.20.1 then follows immediately from (4.159) and the following identities:

$$(Q_{13} + Q_{14})h_{11}(z) + (Q_{23} + Q_{24})h_{21}(z) + (Q_{33} + Q_{34})h_{31}(z) + (Q_{43} + Q_{44})h_{41}(z) \equiv 0 \tag{4.160}$$

$$(Q_{13} + Q_{14})h_{12}(z) + (Q_{23} + Q_{24})h_{22}(z) + (Q_{33} + Q_{34})h_{32}(z) + (Q_{43} + Q_{44})h_{42}(z) \equiv 0 \tag{4.161}$$

$$Q_{12}h_{11}(z) + Q_{22}h_{21}(z) + Q_{32}h_{31}(z) + Q_{42}h_{41}(z) \equiv 0, \tag{4.162}$$

which are proven below. Introduce here the notation

$$\vartheta_j \equiv \vartheta_j(\frac{\omega}{2}),\ \vartheta'_j \equiv \vartheta'_j(\frac{\omega}{2}),\ \vartheta''_j \equiv \vartheta''_j(\frac{\omega}{2}) \quad \text{for } j = 1, 2, 3, 4.$$

The sum in (4.160) can be written as

$$\frac{\vartheta_1(\omega)\left[\vartheta_1^2\vartheta_4^2(z)-\vartheta_4^2\vartheta_1^2(z)+\vartheta_2^2\vartheta_3^2(z)-\vartheta_3^2\vartheta_2^2(z)\right]}{8\vartheta_1'(0)\vartheta_1\vartheta_2\vartheta_3\vartheta_4\vartheta_4(z-\frac{\omega}{2})\vartheta_4(z+\frac{\omega}{2})},$$

which is zero by the Jacobi identity (E.9).

The sum in (4.161) can be written as

$$\frac{\vartheta_1(\omega)\left[\vartheta_1^2\vartheta_4'(z)\vartheta_4(z)-\vartheta_4^2\vartheta_1'(z)\vartheta_1(z)+\vartheta_2^2\vartheta_3'(z)\vartheta_3(z)-\vartheta_3^2\vartheta_2'(z)\vartheta_2(z)\right]}{8\vartheta_1'(0)\vartheta_1\vartheta_2\vartheta_3\vartheta_4\vartheta_4(z-\frac{\omega}{2})\vartheta_4(z+\frac{\omega}{2})}. \tag{4.163}$$

Using the identities (E.2), we can write the expression in brackets in the numerator of (4.163) as

$$\begin{aligned}&\frac{\vartheta_1'(z)}{\vartheta_1(z)}\left[\vartheta_1^2\vartheta_4^2(z)-\vartheta_4^2\vartheta_1^2(z)-\vartheta_3^2\vartheta_2^2(z)+\vartheta_2^2\vartheta_3^2(z)\right]\\&\qquad+\frac{\vartheta_2(z)\vartheta_3(z)\vartheta_4(z)}{\vartheta_1(z)}\left[\vartheta_3^2\vartheta_2^2(0)-\vartheta_1^2\vartheta_4^2(0)-\vartheta_2^2\vartheta_3^2(0)\right].\end{aligned} \tag{4.164}$$

The first term in (4.164) vanishes by (E.9) and the second term vanishes by (E.10). Thus (4.161) is proven.

Finally, we can expand the sum in (4.162) and make the substitutions from identities (E.2), to obtain

$$\begin{aligned}&\frac{\vartheta_1(w)}{4\vartheta_1'(0)\vartheta_4(z-\frac{\omega}{2})\vartheta_4(z+\frac{\omega}{2})}\left[\frac{\vartheta_1'\left(\vartheta_4^2\vartheta_1^2(z)-\vartheta_1^2\vartheta_4^2(z)-\vartheta_2^2\vartheta_3^2(z)+\vartheta_3^2\vartheta_2^2(z)\right)}{\vartheta_1^2\vartheta_2\vartheta_3\vartheta_4}\right.\\&\left.+\frac{\vartheta_3^2(z)}{\vartheta_4^2}\left(\vartheta_3^2(0)+\frac{\vartheta_2^2\vartheta_4^2(0)}{\vartheta_1^2}\right)+\frac{\vartheta_1^2(z)}{\vartheta_2^2}\left(\vartheta_3^2(0)-\frac{\vartheta_4^2\vartheta_2^2(0)}{\vartheta_1^2}\right)-\frac{\vartheta_3^2(0)}{\vartheta_1^2}\vartheta_2^2(z)\right].\end{aligned} \tag{4.165}$$

Once again, the first term vanishes by (E.9). If we substitute the identity (E.11) in the second and third terms, (4.165) becomes simply

$$\frac{\vartheta_1(w)}{4\vartheta_1'(0)\vartheta_1\vartheta_4(z-\frac{\omega}{2})\vartheta_4(z+\frac{\omega}{2})}\left[\vartheta_3^2(z)\vartheta_2^2(0)-\vartheta_2^2(z)\vartheta_3^2(0)-\vartheta_1^2(z)\vartheta_4^2(0)\right],$$

which is zero by (E.10). This proves (4.162) and thus Lemma 4.20.1.

We now turn to the proof of Lemma 4.20.2. Notice that in the fundamental rectangle

$$\mathcal{R}=\{z\in\mathbb{C}:\ -\frac{\pi}{2}\le\operatorname{Re}z\le\frac{\pi}{2},\ 0\le\operatorname{Im}z\le\pi\tau\},$$

the function $\tilde{f}(z,\omega)$ has the two possible simple poles,

$$z_{1,2} = \frac{\pi\tau}{2} \pm \frac{\omega}{2}\,.$$

Because of the double-periodicity of $\tilde{f}$, we must have

$$\operatorname*{Res}_{z=z_1} \tilde{f}(z,\omega) = -\operatorname*{Res}_{z=z_2} \tilde{f}(z,\omega).$$

Denote

$$R_{ij}(\omega) = \operatorname*{Res}_{z=z_1} h_{ij}(z,\omega).$$

Then we have that

$$\begin{aligned}
R_{11} &= \frac{\vartheta_1^2}{\vartheta_1'(0)\vartheta_1(\omega)}, & R_{12} &= \frac{(\vartheta_1' - i\vartheta_1)\vartheta_1}{\vartheta_1'(0)\vartheta_1(\omega)}, & R_{13} &= \frac{(\vartheta_1' - i\vartheta_1)^2}{\vartheta_1'(0)\vartheta_1(\omega)},\\
R_{14} &= \frac{(\vartheta_1'' - 2i\vartheta_1' - \vartheta_1)\vartheta_1}{\vartheta_1'(0)\vartheta_1(\omega)}, & R_{21} &= \frac{\vartheta_4^2}{\vartheta_1'(0)\vartheta_1(\omega)}, & R_{22} &= \frac{(\vartheta_4' - i\vartheta_4)\vartheta_4}{\vartheta_1'(0)\vartheta_1(\omega)},\\
R_{23} &= \frac{(\vartheta_4' - i\vartheta_4)^2}{\vartheta_1'(0)\vartheta_1(\omega)}, & R_{24} &= \frac{(\vartheta_4'' - 2i\vartheta_4' - \vartheta_4)\vartheta_4}{\vartheta_1'(0)\vartheta_1(\omega)}, & R_{31} &= \frac{-\vartheta_3^2}{\vartheta_1'(0)\vartheta_1(\omega)},\\
R_{32} &= \frac{-(\vartheta_3' - i\vartheta_3)\vartheta_3}{\vartheta_1'(0)\vartheta_1(\omega)}, & R_{33} &= \frac{-(\vartheta_3' - i\vartheta_3)^2}{\vartheta_1'(0)\vartheta_1(\omega)}, & &\\
R_{41} &= \frac{-\vartheta_2^2}{\vartheta_1'(0)\vartheta_1(\omega)}, & R_{42} &= \frac{-(\vartheta_2' - i\vartheta_2)\vartheta_2}{\vartheta_1'(0)\vartheta_1(\omega)}, & R_{43} &= \frac{-(\vartheta_2' - i\vartheta_2)^2}{\vartheta_1'(0)\vartheta_1(\omega)},\\
R_{44} &= \frac{-(\vartheta_2'' - 2i\vartheta_2' - \vartheta_2)\vartheta_2}{\vartheta_1'(0)\vartheta_1(\omega)}, & R_{34} &= \frac{-(\vartheta_3'' - 2i\vartheta_3' - \vartheta_3)\vartheta_3}{\vartheta_1'(0)\vartheta_1(\omega)}, & &
\end{aligned}$$

and

$$\operatorname*{Res}_{z=z_1} \tilde{f}(z,\omega) = \sum_{j,k=1}^{4} Q_{jk}R_{jk}. \tag{4.166}$$

A priori, the sum in (4.166) is quite complicated, so we evaluate first the imaginary part . Multiplying out (4.166) and again making the substitutions from (E.2), we get

$$\operatorname{Im}\operatorname*{Res}_{z=z_1} \tilde{f}(z,\omega) = \frac{(\vartheta_2^4 - \vartheta_4^4)\big(\vartheta_1^2\vartheta_3^2(0) + \vartheta_2^2\vartheta_4^2(0) - \vartheta_4^2\vartheta_2^2(0)\big)}{4\vartheta_1'(0)^2\vartheta_1^2\vartheta_2^2\vartheta_4^2},$$

which is zero by (E.11).

Substituting the identities (E.5), along with (E.2), into the sum (4.166) gives

$$\operatorname*{Res}_{z=z_1} \tilde{f}(z,\omega) = \frac{24A + 17B + 12C + 10D + 7E + 5F}{96\vartheta_1'(0)^2\vartheta_1^2\vartheta_2^3\vartheta_3\vartheta_4^3},$$

where

$$A = \vartheta_3^2\big(\vartheta_2^4 + \vartheta_4^4\big)\left[\vartheta_2^2\vartheta_4^2\vartheta_2^2(0)\vartheta_4^2(0) - \vartheta_1^2\vartheta_3^2(0)\left(\vartheta_2^2\vartheta_4^2(0) - \vartheta_4^2\vartheta_2^2(0)\right)\right],$$

$$B = -\vartheta_3^2\left(\vartheta_2^8\vartheta_4^4(0) + \vartheta_4^8\vartheta_2^4(0)\right),$$

$$C = \vartheta_1^2\vartheta_2^2\vartheta_4^2\left(\vartheta_2^4 + \vartheta_4^4 - \vartheta_1^4 - \vartheta_3^4\right),$$

$$D = -\vartheta_2^4\vartheta_3^2\vartheta_4^4\vartheta_3^4(0),$$

$$E = -\vartheta_3^2\left[\vartheta_1^4\vartheta_3^4(0)\left(\vartheta_2^4 + \vartheta_4^4\right) + \vartheta_2^4\vartheta_4^4\big(\vartheta_2^4(0) + \vartheta_4^4(0)\big)\right],$$

$$F = \vartheta_2^2\vartheta_4^2\vartheta_3^2(0)\left[\vartheta_2^2\vartheta_4^2(0)\left(\vartheta_3^4 - \vartheta_1^4 - \vartheta_4^4 + \vartheta_2^4\right) + \vartheta_4^2\vartheta_2^2(0)\left(\vartheta_4^4 - \vartheta_2^4 + \vartheta_3^4 - \vartheta_1^4\right)\right].$$

Note that none of these terms involve derivatives of theta functions. We immediately have $C = 0$ by the identity (E.8). We can also use this identity to write

$$F = 2\vartheta_2^2\vartheta_4^2\vartheta_3^2(0)\left[\vartheta_2^2\vartheta_4^2(0)\left(\vartheta_3^4 - \vartheta_4^4\right) + \vartheta_4^2\vartheta_2^2(0)\left(\vartheta_3^4 - \vartheta_2^4\right)\right],$$

and (E.11) to write

$$A = \vartheta_3^2\big(\vartheta_2^4 + \vartheta_4^4\big)\left[\vartheta_2^2\vartheta_4^2\vartheta_2^2(0)\vartheta_4^2(0) + \vartheta_1^4\vartheta_3^2(0)\right].$$

We now combine the terms A, B, and E to obtain

$$\begin{aligned}24A + 17B + 7E =& 24\vartheta_2^2\vartheta_3^2\vartheta_4^2\big(\vartheta_4^2\vartheta_2^2(0) - \vartheta_2^2\vartheta_4^2(0)\big)\big(\vartheta_2^2\vartheta_2^2(0) - \vartheta_4^2\vartheta_4^2(0)\big)\\ &- 21\vartheta_2^4\vartheta_3^2\vartheta_4^4\vartheta_3^4(0) + 7\vartheta_3^6\big(\vartheta_2^4 + \vartheta_4^4\big)\big(\vartheta_2^4(0) + \vartheta_4^4(0)\big)\\ &- 7\vartheta_3^2\big(\vartheta_4^8\vartheta_4^4(0) + \vartheta_2^8\vartheta_2^4(0)\big).\end{aligned}$$

By (E.11) and (E.13) we can write this sum as

$$\begin{aligned}24A + 17B + 7E = & 24\vartheta_1^2\vartheta_2^2\vartheta_3^2\vartheta_4^2\vartheta_3^2(0)\big(\vartheta_2^2\vartheta_2^2(0) - \vartheta_4^2\vartheta_4^2(0)\big) - 21\vartheta_2^4\vartheta_3^2\vartheta_4^4\vartheta_3^4(0)\\ &+ 7\vartheta_3^6\vartheta_3^4(0)\big(\vartheta_2^4 + \vartheta_4^4\big) - 7\vartheta_3^2\big(\vartheta_4^8\vartheta_4^4(0) + \vartheta_2^8\vartheta_2^4(0)\big).\end{aligned}$$

We now combine all terms and use (E.11) and (E.13) to write all terms solely in terms of ϑ_2, ϑ_4, and factors which are constant with respect to ω, yielding

$$\begin{aligned}24A + 17B + 10D + 7E + 5F = & \left(41\frac{\vartheta_2^2\vartheta_4^2\vartheta_4^2(0)}{\vartheta_3^2(0)} + 24\frac{\vartheta_4^4\vartheta_4^4(0)}{\vartheta_2^2(0)\vartheta_3^2(0)} + 17\frac{\vartheta_2^4\vartheta_2^4(0)}{\vartheta_3^2(0)}\right)\\ &\times\left(\vartheta_2^4(0) + \vartheta_4^4(0) - \vartheta_3^4(0)\right),\end{aligned}$$

which is zero by (E.13). Lemma 4.20.2 is thus proven, and it follows that $\tilde{f}(z,\omega)$ is constant in z. To evaluate the constant, we can take $z = 0$.

4.21 Evaluation of $\tilde{f}(0,\omega)$

We will evaluate $\tilde{f}(z,\omega)$ at $z = 0$. Notice that many of the functions $h_{jk}(z)$ vanish at $z = 0$. In fact we have

$$\begin{aligned}\tilde{f}(0,\omega) = Q_{11}h_{11}(0) + Q_{14}h_{14}(0) + Q_{23}h_{23}(0) + Q_{31}h_{31}(0) + Q_{34}h_{34}(0)\\ +Q_{41}h_{41}(0) + Q_{44}h_{44}(0).\end{aligned} \tag{4.167}$$

The identities (E.4) and (E.3) allow us eliminate all derivatives of theta functions from the sum (4.167) except $\vartheta_2', \vartheta_2''$. Making these substitutions and simplifying, (4.167) becomes

$$\begin{aligned}\tilde{f}(0,\omega) =& \frac{\vartheta_1(\omega)\big(\vartheta_1^2\vartheta_4^2(0) + \vartheta_2^2\vartheta_3^2(0) - \vartheta_3^2\vartheta_2^2(0)\big)}{96\vartheta_1'(0)}\Bigg(\frac{5\vartheta_2''(0)}{\vartheta_1\vartheta_2\vartheta_3\vartheta_4^3} + \frac{17(\vartheta_2')^2}{\vartheta_1\vartheta_2^3\vartheta_3\vartheta_4^3} - \frac{5\vartheta_2''}{\vartheta_1\vartheta_2^2\vartheta_3\vartheta_4^3}\\ &+ \frac{24\vartheta_2'\big(\vartheta_3^2\vartheta_2^2(0) + \vartheta_1^2\vartheta_4^2(0)\big)}{\vartheta_1^2\vartheta_2^3\vartheta_3^2\vartheta_4^2}\Bigg) + \frac{\vartheta_1(\omega)(24A + 17B + 7C + 5D + 3E + 2F)}{96\vartheta_1'(0)\vartheta_1^3\vartheta_2^3\vartheta_3^3\vartheta_4^3\vartheta_2^2(0)}\end{aligned} \tag{4.168}$$

where

$$\begin{aligned}
A &= \vartheta_2^2\vartheta_4^2\vartheta_2^2(0)\vartheta_3^2(0)\big(\vartheta_1^4\vartheta_4^4(0) + \vartheta_3^4\vartheta_2^4(0)\big),\\
B &= \vartheta_4^2\vartheta_2^2(0)\big(\vartheta_1^2\vartheta_4^2(0) - \vartheta_3^2\vartheta_2^2(0)\big)\big((\vartheta_3^2\vartheta_2^2(0) + \vartheta_1^2\vartheta_4^2(0))^2 - \vartheta_1^2\vartheta_3^2\vartheta_2^2(0)\vartheta_4^2(0)\big),\\
C &= \vartheta_2^4\vartheta_4^2\vartheta_2^2(0)\vartheta_3^4(0)\big(\vartheta_1^2\vartheta_4^2(0) - \vartheta_3^2\vartheta_2^2(0)\big),\\
D &= \vartheta_1^2\vartheta_3^2\Bigg[\vartheta_4^2\vartheta_2^4(0)\vartheta_4^2(0)\big(\vartheta_3^2\vartheta_2^2(0) - \vartheta_1^2\vartheta_4^2(0)\big) + \vartheta_3^2(0)\Bigg(\vartheta_2^2(0)\big(\vartheta_3^4\vartheta_2^4(0) - \vartheta_1^4\vartheta_4^4(0)\big)\\
&\qquad + \vartheta_2^2\vartheta_1^2(0)\vartheta_4^2(0)\big(\vartheta_1^2(\vartheta_4^2(0) + \vartheta_2^2\vartheta_3^2(0)\big)\Bigg)\Bigg],\\
E &= \vartheta_1^2\vartheta_2^2\vartheta_3^2\vartheta_2^2(0)\vartheta_3^2(0)\Big[\vartheta_4^2(0)\big(\vartheta_1^2\vartheta_3^2(0) - \vartheta_4^2\vartheta_2^2(0)\big) - \vartheta_2^2\vartheta_2^4(0)\Big],\\
F &= \vartheta_1^2\vartheta_2^2\vartheta_3^2\vartheta_2^2(0)\Big[4\vartheta_2^2\vartheta_3^2(0)\vartheta_4^4(0) - \vartheta_2^2(0)\big(\vartheta_1^2\vartheta_2^2(0)\vartheta_4^2(0) + \vartheta_3^2\vartheta_4^4(0) + \vartheta_3^2\vartheta_3^4(0)\big)\Big].
\end{aligned}$$

The identity (E.10) implies that all terms in (4.168) involving derivatives of theta functions in the numerator vanish. Additionally, (E.10)-(E.13) allow us to simplify the numbers B, C, D and E. Namely, we have

$$\begin{aligned}
B &= -\vartheta_2^2\vartheta_4^2\vartheta_2^2(0)\vartheta_3^2(0)\big((\vartheta_3^2\vartheta_2^2(0)+\vartheta_1^2\vartheta_4^2(0))^2-\vartheta_1^2\vartheta_3^2\vartheta_2^2(0)\vartheta_4^2(0)\big),\\
C &= -\vartheta_2^6\vartheta_4^2\vartheta_2^2(0)\vartheta_3^6(0),\\
D &= \vartheta_1^2\vartheta_2^2\vartheta_3^2\vartheta_2^2(0)\vartheta_3^2(0)\Big(\vartheta_1^2\vartheta_3^2(0)\vartheta_4^2(0)+\vartheta_3^2\vartheta_2^2(0)\vartheta_3^2(0)+\vartheta_4^2\vartheta_2^2(0)\vartheta_4^2(0)\Big),\\
E &= -\vartheta_1^2\vartheta_2^4\vartheta_3^2\vartheta_2^2(0)\vartheta_3^6(0).
\end{aligned}$$

Combining these terms gives us

$$\begin{aligned}
&24A+17B+7C+5D+3E+2F=\\
&\vartheta_2^2\vartheta_2^2(0)\Big[-12\vartheta_1^2\vartheta_3^2\vartheta_4^2\vartheta_2^2(0)\vartheta_3^2(0)\vartheta_4^2(0)+7\vartheta_4^2\vartheta_3^2(0)\big(\vartheta_3^4\vartheta_2^4(0)+\vartheta_1^4\vartheta_4^4(0)-\vartheta_2^4\vartheta_3^4(0)\big)\\
&+8\vartheta_1^2\vartheta_2^2\vartheta_3^2\vartheta_3^2(0)\vartheta_4^4(0)+5\vartheta_1^4\vartheta_3^2\vartheta_3^4(0)\vartheta_4^2(0)+3\vartheta_1^2\vartheta_3^2\vartheta_3^4(0)\big(\vartheta_3^2\vartheta_2^2(0)-\vartheta_2^2\vartheta_3^2(0)\big)\\
&-2\vartheta_1^2\vartheta_3^2\vartheta_2^2(0)\vartheta_4^2(0)\big(\vartheta_1^2\vartheta_2^2(0)+\vartheta_3^2\vartheta_4^2(0)\big)\Big].
\end{aligned}$$

Again using (E.10)-(E.13), this expression simplifies to

$$24A+17B+7C+5D+3E+2F=8\vartheta_1^2\vartheta_2^2\vartheta_3^2\vartheta_4^2\vartheta_2^2(0)\vartheta_3^2(0)\vartheta_4^2(0).$$

Inserting this into (4.168), we get

$$\tilde{f}(0,\omega)\equiv\tilde{f}(z,\omega)\equiv f(n\omega,\omega)=\frac{\vartheta_1(\omega)\vartheta_2^2(0)\vartheta_3^2(0)\vartheta_4^2(0)}{12\vartheta_1'(0)\vartheta_1\vartheta_2\vartheta_3\vartheta_4}=\frac{1}{6}$$

by (E.1) and (E.6). It then follows that

$$f_0(n\omega,\omega)=f(n\omega,\omega)-\frac{1}{6}=0.$$

This proves (4.157) and therefore Proposition 4.19.2.

4.22 Large n asymptotics of Z_n

By substituting (4.158) into (4.3) we obtain that

$$\begin{aligned}\frac{\tau_n}{\prod_{k=0}^{n-1}(k!)^2} &= 2^{n^2}\prod_{k=0}^{n-1}\frac{h_k}{(k!)^2}\\ &= 2^{n^2}h_0\prod_{k=1}^{n-1}\left[G^{2k+1}\frac{\vartheta_4((k+1)\omega)}{\vartheta_4(k\omega)}\left(1+O(k^{-2})\right)\right]\\ &= C\vartheta_4(n\omega)(2G)^{n^2}\Big(1+O(n^{-1})\Big),\end{aligned}$$

where $C>0$ does not depend on n. Thus, by (1.5),

$$Z_n=\frac{[\sinh(\gamma-t)\sinh(\gamma+t)]^{n^2}\tau_n}{\left(\prod_{k=0}^{n-1}k!\right)^2}=C\vartheta_4(n\omega)F^{n^2}\Big(1+O(n^{-1})\Big),$$

where

$$F=2G\sinh(\gamma-t)\sinh(\gamma+t)=\frac{\pi\sinh(\gamma-t)\sinh(\gamma+t)\vartheta_1'(0)}{2\gamma\vartheta_1(\omega)}\,.$$

Theorem 4.2.1 is proved.

5. DISCRETE ORTHOGONAL POLYNOMIALS ON AN INFINITE LATTICE

5.1 Introduction

In the preceding chapter, we developed a Riemann-Hilbert approach to asymptotics of a system of discrete orthogonal polynomials on a regular infinite lattice. In fact, this approach can easily be extended to a large class of discrete orthogonal polynomials. In this chapter, originally presented in the paper [10], we obtain asymptotic results for systems of discrete orthogonal polynomials with respect to varying exponential weights on a regular infinite lattice. For a given $N \in \mathbb{N}$, introduce the regular infinite lattice,

$$L_N = \left\{ x_{k,N} = \frac{k}{N}\,,\ k \in \mathbb{Z} \right\}.$$

We consider polynomials orthogonal on L_N with respect to the varying exponential weight

$$w_N(x) = e^{-NV(x)},$$

where $V(x)$ is a real analytic function such that, for some $\varepsilon > 0$, V has analytic extension to the strip

$$|\operatorname{Im} z| < \varepsilon, \tag{5.1}$$

and satisfies the growth condition

$$\frac{\operatorname{Re} V(z)}{\log(|z|^2+1)} \to +\infty \quad \text{as} \quad |z| \to \infty,\ |\operatorname{Im} z| < \varepsilon. \tag{5.2}$$

More specifically, we introduce the system of monic orthogonal polynomials,

$$P_n(x) = x^n + p_{n,n-1}x^{n-1} + \ldots + p_{n0}, \quad n = 0, 1, \ldots,$$

such that

$$\sum_{x \in L_N} P_m(x)P_n(x)w_N(x) = h_n \delta_{mn}\,, \tag{5.3}$$

for some normalizing coefficients h_n. Existence and uniqueness of this system of orthogonal polynomials is guaranteed by condition (5.2). These orthogonal polynomials satisfy the three-term recurrence relation

$$xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n^2 P_{n-1}(x). \tag{5.4}$$

We will explore the asymptotics of the quantities γ_n, β_n, and h_n for $n = N, N-1$, as well as pointwise asymptotics of the polynomials $P_N(x)$ as $N \to \infty$.

The present work has the three predecessors:

1. The work [16] of Deift, Kriecherbauer, McLaughlin, Venakides, and Zhou, in which the large N asymptotics has been obtained for orthogonal polynomials with respect to varying exponential weights on the real line.

2. The work [5] of Baik, Kriecherbauer, McLaughlin, and Miller, in which the large N asymptotics has been obtained for orthogonal polynomials with respect to varying exponential weights on a lattice in a finite interval.

3. The work [8] of Bleher and Liechty, presented in the preceding chapter, in which the large N asymptotics has been obtained for orthogonal polynomials with respect to the varying exponential weight $w_N(x) = e^{-N(|x|-\zeta x)}$ on the infinite lattice L_N.

Also, a very important ingredient comes from the work [32] of Kuijlaars, in which analytic properties of equilibrium measures with constraints are established.

The asymptotic analysis of the polynomials $P_N(x)$ in this work will be based on the Interpolation Problem for discrete orthogonal polynomials, which is introduced in the work [11] of Borodin and Boyarchenko (see also [5], [7], [8]). The asymptotic analysis of $P_N(x)$ will consist of three steps. The first step will be a reduction of the Interpolation Problem to a Riemann-Hilbert problem on a contour on the complex plane, which we accomplish following the general approach introduced in the paper [39] of Miller and in the monograph [26] of Kamvissis, McLaughlin, and Miller. The second step will be an application of the nonlinear steepest descent method of

Deift and Zhou [19] to the Riemann-Hilbert problem under consideration, and the final, third step will be a derivation of the asymptotic formulae both for the orthogonal polynomials $P_N(x)$ and for the recurrence coefficients. To apply the nonlinear steepest descent method to the orthogonal polynomials $P_N(x)$ we need to study the corresponding equilibrium measure.

5.2 Equilibrium measure

The significance of the equilibrium measure is that, as we will see, it gives the limiting distribution of zeros of the polynomial $P_N(x)$. By definition, the equilibrium measure is a solution to a variational problem. Namely, let us consider the following set of probability measures on $\mathbb{R}^1$:

$$\mathcal{M} = \{0 \le \nu \le \sigma, \ \ \nu(\mathbb{R}^1) = 1\},$$

where σ is the Lebesgue measure, and let us introduce the functional

$$H(\nu) = \iint \log \frac{1}{|x-y|} d\nu(x) d\nu(y) + \int V(x) d\nu(x), \qquad \nu \in \mathcal{M}.$$

The equilibrium measure minimizes this functional over some set of measures. In the case of continuous orthogonal polynomials, we minimize over the set of probability measures on the real line. However, in the case of discrete orthogonal polynomials, we must introduce the upper constraint, $\nu \le \sigma$, in order to account for an interlacing property of the zeroes of orthogonal polynomials.

It is a general fact, (see, e.g. [47]) that for any system of polynomials orthogonal on the real line with respect to a real weight, the nth polynomial has n real distinct distinct zeroes. Furthermore, the zeroes of a system of discrete orthogonal polynomials satisfy an interlacing property with regard to the location of the nodes of the lattice L_N, so that no more than one zero may lie between any pair of adjacent nodes. It therefore follows that, if we denote by μ_N the normalized counting measure on the zeroes of the Nth orthogonal polynomial in our system,

$$\mu_N(a,b) \le b - a + \frac{1}{N} \quad \text{for any} \quad -\infty < a < b < \infty,$$

so that $\mu \leq \sigma$, where $\mu = \lim_{N\to\infty} \mu_N$. With this constraint in mind, we define

$$E_0 = \inf_{\nu \in \mathcal{M}} H(\nu).$$

It is possible to prove that there exists a unique minimizer ν_0, so that

$$E_0 = H(\nu_0), \tag{5.5}$$

see, e.g., the works of Saff and Totik [41], Dragnev and Saff [20] and Kuijlaars [32]. The minimizer is called the *equilibrium measure*.

The equilibrium measure ν_0 is uniquely determined by the *Euler-Lagrange variational conditions*: there exists a *Lagrange multiplier* l such that

$$2\int \log|x-y|d\nu_0(y) - V(x) \begin{cases} \geq l & \text{for} \quad x \in \text{supp } \nu_0 \\ \leq l & \text{for} \quad x \in \text{supp } (\sigma - \nu_0), \end{cases}$$

see the works [18] of Deift and McLaughlin and [20]. In particular,

$$2\int \log|x-y|d\nu_0(y) - V(x) = l \quad \text{for} \quad x \in \text{supp } \nu_0 \cap \text{supp } (\sigma - \nu_0).$$

The equilibrium measure ν_0 possesses a number of nice analytical properties, as shown by Kuijlaars in [32]. We will use these analytic properties, so let us discuss the results of [32].

First, observe that the constraint $\nu_0 \leq \sigma$ implies the existence of the density,

$$\rho(x) = \frac{d\nu_0}{dx}.$$

We can partition $\mathbb{R}$ into the three sets

$$\begin{aligned} I^0 &= \left\{ x \in \mathbb{R} : 2\int \log|x-y|d\nu_0(y) - V(x) = l \right\}, \\ I^+ &= \left\{ x \in \mathbb{R} : 2\int \log|x-y|d\nu_0(y) - V(x) > l \right\}, \\ I^- &= \left\{ x \in \mathbb{R} : 2\int \log|x-y|d\nu_0(y) - V(x) < l \right\}. \end{aligned} \tag{5.6}$$

The structure of the equilibrium measure is well described in the following theorem of Kuijlaars, obtained in [32].

Theorem 5.2.1 *(Kuijlaars) For any real analytic potential $V(x)$ satisfying (5.2), the following hold:*

1. *The density $\rho(x)$ of the constrained equilibrium measure ν_0 (defined in (5.5)) is continuous.*

2. *The sets I^+ and I^- are both finite unions of open intervals.*

3. *The density ρ is real analytic on the open set $\{x : 0 < \rho(x) < 1\}$.*

4. *The density ρ has the representation*

$$\rho(x) = \frac{1}{\pi}\sqrt{q_1^+(x)} \quad \textit{for} \quad x \in I^0 \cup I^- ,$$

where q_1^+ is the positive part of a function q_1 defined on $I^0 \cup I^-$ which is real analytic on the interior of $I^0 \cup I^-$. The function q_1 is negative on I^-, so that

$$\rho(x) = 0 \quad \textit{for} \quad x \in I^- ,$$

and it is nonnegative on I^0, so that

$$\rho(x) = \frac{1}{\pi}\sqrt{q_1(x)} \quad \textit{for} \quad x \in I^0. \tag{5.7}$$

5. *The density ρ has the representation*

$$\rho(x) = 1 - \frac{1}{\pi}\sqrt{q_2^+(x)} \quad \textit{for} \quad x \in I^0 \cup I^+ ,$$

where q_2^+ is the positive part of a function q_2 defined on $I^0 \cup I^+$ which is real analytic on the interior of $I^0 \cup I^+$. The function q_2 is negative on I^+, so that

$$\rho(x) = 1 \quad \textit{for} \quad x \in I^+ ,$$

and it is nonnegative on I_0, so that

$$\rho(x) = 1 - \frac{1}{\pi}\sqrt{q_2(x)} \quad \textit{for} \quad x \in I^0. \tag{5.8}$$

Remark: It follows from equations (5.7) and (5.8) that

$$\frac{1}{\pi}\sqrt{q_1(x)} = 1 - \frac{1}{\pi}\sqrt{q_2(x)} \quad \text{for} \quad x \in I^0,$$

hence q_1 and q_2 uniquely determine each other.

Notice that, according to point (2) of this theorem, the connected components of I^0 are either closed intervals or isolated points. Since ν_0 has compact support, we can write

$$I^0 = \bigsqcup_{j=1}^{q} [\alpha_j, \beta_j]\,,$$

where

$$\begin{aligned} &\alpha_j \le \beta_j && \text{for} \quad j = 1, \ldots, q\,, \\ &\beta_j < \alpha_{j+1} && \text{for} \quad j = 1, \ldots, q-1. \end{aligned}$$

Notice that the intervals $(-\infty, \alpha_1)$ and (β_q, ∞) are components of I^-. The interval (β_j, α_{j+1}) for $1 \le j < q$ is a component of either I^+ or I^-. We therefore adopt the notation

$$\begin{aligned} \mathcal{A}_v &= \Big\{ j \in \{1, \ldots, q-1\} : (\beta_j, \alpha_{j+1}) \subset I^- \Big\} \\ \mathcal{A}_s &= \Big\{ j \in \{1, \ldots, q-1\} : (\beta_j, \alpha_{j+1}) \subset I^+ \Big\}. \end{aligned}$$

We will call an equilibrium measure ν_0 *regular* if the following hold:

1. The functions q_1 and q_2 are non-vanishing on the interior of I^0.

2. I^0 contains no isolated points, so that $\alpha_j < \beta_j$ for all $j = 1, \ldots, q$.

3. If $j \in \mathcal{A}_v$, then $q_1'(\beta_j) \neq 0$ and $q_1'(\alpha_{j+1}) \neq 0$.

4. If $j \in \mathcal{A}_s$, then $q_2'(\beta_j) \neq 0$ and $q_2'(\alpha_{j+1}) \neq 0$.

For the remainder of this paper, we will assume that our equilibrium measure is regular. In this case, the sets I^0, I^+ and I^- are each finite unions of intervals, so that

$$-\infty < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_q < \beta_q < \infty,$$

and we classify these intervals as follows:

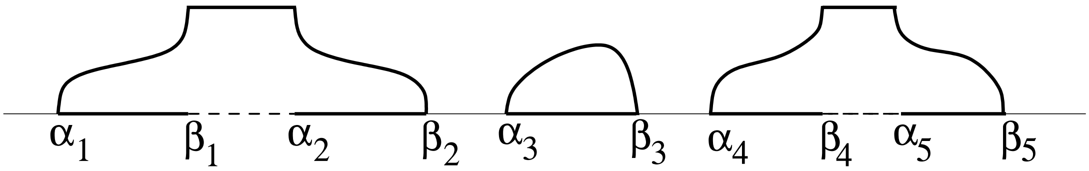

Fig. 5.1. The graph of the density function for a hypothetical equilibrium measure with $q = 5$. Bands are denoted by bold segments, saturated regions by dashed segments, and voids by thin segments.

Definition: A *void* is an open subinterval (β_j, α_{j+1}), $j \in \mathcal{A}_v$, or one of the intervals $(-\infty, \alpha_1)$, (β_q, ∞). The union of all voids is I^-.

Definition: A *saturated region* is an open subinterval (β_j, α_{j+1}), $j \in \mathcal{A}_s$. The union of all saturated regions is I^+.

Definition: A *band* is an open subinterval (α_j, β_j), $j = 1, \dots, q$. The union of all bands is the interior of I^0. Observe that $\rho(x) = 0$ on any void (β_j, α_{j+1}), $\rho(x) = 1$ on any saturated interval (β_j, α_{j+1}), and $0 < \rho(x) < 1$ on any band (α_j, β_j), see Figure 5.1. In addition, at the end-points of any band, $\rho(x)$ has a square-root singularity. Namely, if α_j is a common end-point of a band and a void then as $x \to +0$,

$$\rho(\alpha_j + x) = C\sqrt{x}\,(1 + O(x)), \qquad C = |q_1'(\alpha_j)|^{1/2} > 0.$$

and if α_j is a common end-point of a band and a saturated region then as $x \to +0$,

$$\rho(\alpha_j + x) = 1 - C\sqrt{x}\,(1 + O(x)), \qquad C = |q_2'(\alpha_j)|^{1/2} > 0.$$

Similarly, if β_j is a common end-point of a band and a void then as $x \to +0$,

$$\rho(\beta_j - x) = C\sqrt{x}\,(1 + O(x)), \qquad C = |q_1'(\beta_j)|^{1/2} > 0. \tag{5.9}$$

and if β_j is a common end-point of a band and a saturated region then as $x \to +0$,

$$\rho(\beta_j - x) = 1 - C\sqrt{x}\,(1 + O(x)), \qquad C = |q_2'(\beta_j)|^{1/2} > 0.$$

In the next section we introduce the g-function, which will be our means of exploiting the equilibrium measure.

5.3 The g-function

Define the g-function on $\mathbb{C} \setminus (-\infty, \beta_q]$ as

$$g(z) = \int_{\alpha_1}^{\beta_q} \log(z - x) d\nu_0(x) \tag{5.10}$$

where we take the principal branch for the logarithm. Also, introduce the numbers Ω_j for $j = 1, \ldots, q-1$ as

$$\Omega_j = \begin{cases} 2\pi \displaystyle\int_{\alpha_{j+1}}^{\beta_q} \rho(x)dx & \text{for} \quad j \in \mathcal{A}_v \\ 2\pi \displaystyle\int_{\alpha_{j+1}}^{\beta_q} \rho(x)dx + 2\pi\alpha_{j+1} & \text{for} \quad j \in \mathcal{A}_s. \end{cases}$$

Properties of $g(z)$:

1. $g(z)$ is analytic in $\mathbb{C} \setminus (-\infty, \beta_q]$.

2. For large z,

$$g(z) = \log z - \sum_{j=1}^{\infty} \frac{g_j}{z^j}, \qquad g_j = \int_{\alpha_1}^{\beta_q} \frac{x^j}{j} d\nu_0(x). \tag{5.11}$$

3. The resolvent of the equilibrium measure is given by

$$g'(z) = \int_{\mathbb{R}} \frac{\rho(x)dx}{z - x}$$

4. From (5.6), we have that

$$g_+(x) + g_-(x) \begin{cases} = V(x) + l & \text{for} \quad x \in I^0 \\ > V(x) + l & \text{for} \quad x \in I^+ \\ < V(x) + l & \text{for} \quad x \in I^-, \end{cases} \tag{5.12}$$

   where g_+ and g_- refer to the limiting values from the upper and lower half-planes, respectively.

5. Equation (5.10) implies that the function

$$G(x) \equiv g_+(x) - g_-(x)$$

is pure imaginary for all real x, and

$$G(x) = 2\pi i \int_x^{\beta_q} \rho(s)\, ds. \tag{5.13}$$

Thus

$$G(x) = \begin{cases} i\Omega_j \quad \text{for} \quad \beta_j < x < \alpha_{j+1}, \quad \text{and} \quad j \in \mathcal{A}_v \\ i\Omega_j - 2\pi i x \quad \text{for} \quad \beta_j < x < \alpha_{j+1}, \quad \text{and} \quad j \in \mathcal{A}_s. \end{cases}$$

From (5.12) and (5.13) we obtain that

$$2g_\pm(x) = V(x) + l \pm 2\pi i \int_x^{\beta_q} \rho(s) ds \quad \text{for} \quad x \in I^0. \tag{5.14}$$

6. Also, from (5.13), we get that $G(x)$ is real analytic on the sets I^+, I^-, and on the interior of I^0. We can therefore extend G into a complex neighborhood of any interval of analyticity for ρ, and the Cauchy-Riemann equations imply that

$$\left.\frac{dG(x+iy)}{dy}\right|_{y=0} = 2\pi\rho(x) \geq 0.$$

Observe that from (5.12) we have that

$$G(x) = 2g_+(x) - V(x) - l = -[2g_-(x) - V(x) - l], \quad x \in I^0.$$

5.4 Main results

In this section, we summarize the main results of this chapter. In order to do so, we must first introduce some notations. Introduce the numbers $\Omega_{j,N}$ for $j = 0, \ldots, q$ as

$$\Omega_{j,N} = \begin{cases} N\Omega_j \quad \text{for} \quad j \in \mathcal{A}_v \\ \pi + N\Omega_j \quad \text{for} \quad j \in \mathcal{A}_s \\ 2\pi N \quad \text{for} \quad j = 0 \\ 0 \quad \text{for} \quad j = q. \end{cases}$$

and the vector

$$\Omega_N = (\Omega_{1,N}, \ldots, \Omega_{q-1,N}).$$

Let

$$R(z) \equiv \prod_{j=1}^{q}(z-\alpha_j)(z-\beta_j)$$

and let X be the two-sheeted Riemann surface of genus $g \equiv q-1$ associated with $\sqrt{R(z)}$ with cuts on the intervals (α_j, β_j). We fix the first sheet of X by the condition

$$\sqrt{R(z)} > 0 \quad \text{for} \quad z > \beta_q$$

on the first sheet.

Introduce the following homology basis on X. For any $j \in \{1, \cdots, q-1\}$, let A_j be a cycle enclosing the interval (β_j, α_{j+1}) (passing through the intervals (α_j, β_j) and $(\alpha_{j+1}, \beta_{j+1})$), oriented clockwise, such that the piece of A_j which lies in the upper half-plane also lies on the first sheet of X, while the piece of A_j which lies in the lower half plane also lies on the second sheet of X. Also for any $j \in \{1, \cdots, q-1\}$, let B_j be a cycle enclosing the interval (α_1, β_j) (passing through the intervals $(-\infty, \alpha_1)$ and (β_j, α_{j+1})), oriented clockwise, and lying entirely on the first sheet of X. Then the cycles $(A_1, \ldots, A_{q-1}, B_1, \ldots .B_{q-1})$ form a canonical homology basis for X.

Now consider the the g-dimensional complex linear space Ω of holomorphic one-forms on X,

$$\Omega = \left\{ \omega = \sum_{j=0}^{q-2} \frac{c_j z^j dz}{\sqrt{R(z)}} \right\},$$

and the basis

$$\omega = (\omega_1, \ldots, \omega_{q-1}) \tag{5.15}$$

normalized such that

$$\int_{A_j} \omega_k = \delta_{jk}. \tag{5.16}$$

Notice that the basis ω is real. That is, for the basis elements

$$\omega_j = \sum_{k=1}^{q-1} \frac{c_{jk} z^{k-1} dz}{\sqrt{R(z)}}, \tag{5.17}$$

the coefficients c_{jk} are real.

Now define the associated matrix of B-periods as

$$\tau = (\tau_{jk}), \qquad \tau_{jk} = \int_{B_j} \omega_k, \qquad j,k = 1,\dots,q-1.$$

Since $\sqrt{R(z)}$ is pure imaginary on the intervals (α_j, β_j), the numbers τ_{jk} are pure imaginary. Furthermore, the matrix τ is symmetric and the matrix $-i\tau$ is positive definite (see [22]).

We now define the Riemann theta function associated with τ as

$$\vartheta(s) = \sum_{m\in\mathbb{Z}^g} e^{2\pi i(m,s)+\pi i(m,\tau m)}, \qquad s \in \mathbb{C}^g, \tag{5.18}$$

where $(m,s) = \sum_{j=1}^{q-1} m_j s_j$. Because the quadratic form $i(m,\tau m)$ is negative definite, the sum in (5.18) is absolutely convergent for all $s \in \mathbb{C}^g$, and thus $\vartheta(s)$ is an entire function in $\mathbb{C}^g$. Notice that the theta function is an even function and satisfies the periodicity properties

$$\vartheta(s+e_j) = \vartheta(s), \qquad \vartheta(s+\tau_j) = e^{-2\pi i s_j - \pi i \tau_{jj}}\vartheta(s)$$

where $e_j = (0,\dots,1,\dots,0)$ is the j^{th} canonical basis vector in $\mathbb{C}^g$, and $\tau_j = \tau e_j$.

Introduce now the vector valued function

$$u(z) = \int_{\beta_q}^{z} \omega, \quad \text{for} \quad z \in \mathbb{C}\setminus(\alpha_1,\beta_q),$$

where $\omega = (\omega_1,\dots,\omega_g)$ is defined in (5.15) and (5.16), and the contour of integration lies in $\mathbb{C}\setminus(\alpha_1,\beta_q)$ on the first sheet of X. Notice that $u(z)$ is well defined as a function with values in $\mathbb{C}^g/\mathbb{Z}^g$ except on the interval (α_1,β_q), where it takes limiting values from the upper and lower half-planes.

Introduce also the function

$$\gamma(z) = \prod_{j=1}^{q} \left(\frac{z-\alpha_j}{z-\beta_j}\right)^{1/4}$$

with cuts on I^0, taking the branch such that $\gamma(z) \sim 1$ as $z \to \infty$. It can be seen (see [16]) that, on the first sheet of X, the function $\gamma - \frac{1}{\gamma}$ has exactly one zero in each of the intervals (β_j, α_{j+1}), and is non-zero elsewhere, and that the function $\gamma + \frac{1}{\gamma}$ has no zeroes on the first sheet of X. Define the numbers x_j as

$$x_j \in (\beta_j, \alpha_{j+1}), \quad \gamma(x_j) - \frac{1}{\gamma(x_j)} = 0.$$

Define the vector of Riemann constants

$$K \equiv -\sum_{j=1}^{q-1} u(\beta_j)$$

and the vector

$$d \equiv -K + \sum_{j=1}^{q-1} u(x_j).$$

Then

$$\vartheta(u(x_j) - d) = 0 \quad \text{for} \quad j \in \{1, \ldots, q-1\},$$

and $\{x_j\}_{j=1}^{g}$ are all the zeroes of the function $\vartheta(u(z) - d)$. In addition, the function $\vartheta(u(z) + d)$ has no zeroes on the first sheet of X.

Finally, for $j = 1, \ldots q$, introduce the functions

$$\psi_{\alpha_j}(z) = -\left\{\frac{3\pi}{2}\int_{\alpha_j}^{z} \rho(t)dt\right\}^{2/3}, \quad \psi_{\beta_j}(z) = -\left\{\frac{3\pi}{2}\int_{z}^{\beta_j} \rho(t)dt\right\}^{2/3},$$

and the functions

$$\mathcal{M}_1(z) = \frac{\vartheta(u(\infty) + d)}{\vartheta(u(\infty) + \frac{\Omega_N}{2\pi} + d)} \frac{\gamma(z) + \gamma(z)^{-1}}{2} \frac{\vartheta(u(z) + \frac{\Omega_N}{2\pi} + d)}{\vartheta(u(z) + d)}$$
$$\mathcal{M}_2(z) = \frac{\vartheta(u(\infty) + d)}{\vartheta(u(\infty) + \frac{\Omega_N}{2\pi} + d)} \frac{\gamma(z) - \gamma(z)^{-1}}{2} \frac{\vartheta(u(z) - \frac{\Omega_N}{2\pi} - d)}{\vartheta(u(z) - d)}.$$

Notice that $\mathcal{M}_1$ and $\mathcal{M}_2$ depend quasiperiodically on N, thus are $O(1)$ as $N \to \infty$.

The asymptotics of the normalizing constants in equation (5.3) and of the recurrence coefficients in equation (5.4) are presented in the following theorem.

Theorem 5.4.1 *(Asymptotics of recurrence coefficients) Let $V(x)$ be a real analytic function satisfying (5.2) which yields a regular equilibrium measure (5.5), and let $\{P_n\}_{n=0}^{\infty}$ be the system of orthogonal polynomials defined according to (5.3). Then as*

$N \to \infty$, the normalizing constants in (5.3) and recurrence coefficients in (5.4) admit the following asymptotic expansions.

$$h_N = \frac{N\pi}{2} e^{Nl} \left(\sum_{j=1}^{q} (\beta_j - \alpha_j) \right) \frac{\vartheta(u(\infty)+d)\vartheta(u(\infty) - \frac{\Omega_N}{2\pi} - d)}{\vartheta(u(\infty)-d)\vartheta(u(\infty) + \frac{\Omega_N}{2\pi} + d)} \left[1 + O\left(\frac{1}{N}\right) \right],$$

$$h_{N-1} = 8N\pi e^{Nl} \left(\sum_{j=1}^{q} (\beta_j - \alpha_j) \right)^{-1} \frac{\vartheta(u(\infty)-d)\vartheta(u(\infty) - \frac{\Omega_N}{2\pi} + d)}{\vartheta(u(\infty)+d)\vartheta(u(\infty) + \frac{\Omega_N}{2\pi} - d)} \left[1 + O\left(\frac{1}{N}\right) \right],$$

$$\gamma_N^2 = \left(\frac{\sum_{j=1}^{q} (\beta_j - \alpha_j)}{4} \right)^2 \frac{\vartheta(u(\infty)+d)^2 \vartheta(u(\infty) - \frac{\Omega_N}{2\pi} - d)\vartheta(u(\infty) + \frac{\Omega_N}{2\pi} - d)}{\vartheta(u(\infty)-d)^2 \vartheta(u(\infty) + \frac{\Omega_N}{2\pi} + d)\vartheta(u(\infty) - \frac{\Omega_N}{2\pi} + d)} + O\left(\frac{1}{N}\right),$$

$$\beta_{N-1} = \frac{\sum_{j=1}^{q} (\beta_j^2 - \alpha_j^2)}{2\sum_{j=1}^{q} (\beta_j - \alpha_j)} + \left(\frac{\nabla\vartheta(u(\infty) + \frac{\Omega_N}{2\pi} - d)}{\vartheta(u(\infty) + \frac{\Omega_N}{2\pi} - d)} - \frac{\nabla\vartheta(u(\infty) + \frac{\Omega_N}{2\pi} + d)}{\vartheta(u(\infty) + \frac{\Omega_N}{2\pi} + d)} + \frac{\nabla\vartheta(u(\infty)+b)}{\vartheta(u(\infty)+b)} - \frac{\nabla\vartheta(u(\infty)-d)}{\vartheta(u(\infty)-d)}, u'(\infty) \right) + O\left(\frac{1}{N}\right).$$

where $\nabla\vartheta$ is the gradient of ϑ,

$$u'(\infty) = (c_{1,q-1}, c_{2,q-1}, \ldots, c_{q-1,q-1}), \tag{5.19}$$

and the numbers c_{jk} are defined in (5.17).

Notice that, up to the lattice scaling factor N in the normalizing coefficients, these asymptotics are similar to the results obtained in [16] for continuous orthogonal polynomials.

The remaining theorems in this section present pointwise asymptotics of the polynomials $P_N(z)$ in various regions of the real line and complex plane.

Theorem 5.4.2 *(Asymptotics of $P_N(z)$ in voids) Let $K \subset \mathbb{C}$ be a compact subset of the complex plane such that K does not intersect with the support of the equilibrium measure ν_0. Then for any $z \in K$, we have that*

$$P_N(z) = e^{Ng(z)} \left[\mathcal{M}_1(z) + O(N^{-1}) \right].$$

The error term $O(N^{-1})$ is uniform in K.

The function $e^{Ng(z)}\mathcal{M}_1(z)$ is analytic in a neighborhood of any compact subset of any void, thus this formula gives asymptotics of $P_N(x)$ for x in a void. In particular, notice that this function has no zeroes in the exterior intervals $(-\infty, \alpha_1)$ and (β_q, ∞), and at most one zero in any other void.

Theorem 5.4.3 *(Asymptotics of $P_N(z)$ in bands) Let K be a compact subset of the interior of I^0. Then for any point $x \in K$, we have that*

$$P_N(x) = 2e^{\frac{N}{2}(V(x)+l)} \left[\mathrm{Re}\, \left(e^{iN\pi\phi(x)}\mathcal{M}_{1+}(x)\right) + O(N^{-1})\right],$$

where $\mathcal{M}_{1+}(x)$ refers to the limiting value of the function $\mathcal{M}_1(z)$ from the upper half-plane, and

$$\phi(x) := \int_x^{\beta_q} \rho(t)dt. \tag{5.20}$$

The error term $O(N^{-1})$ is uniform in K.

Theorem 5.4.4 *(Asymptotics of $P_N(z)$ in saturated regions) Let K be a compact subset of I^+. Then the exists $\varepsilon > 0$ such that, for any point $x \in K$, we have that*

$$P_N(x) = e^{NL(x)} \left[2\sin(N\pi x)\left(\mathrm{Im}\, \left(e^{\frac{iN\Omega_j}{2}}\mathcal{M}_{1+}(x)\right) + O(N^{-1})\right) + O(e^{-N\varepsilon})\right],$$

where $\mathcal{M}_{1+}(x)$ refers to the limiting value of the function $\mathcal{M}_1(z)$ from the upper half-plane, and

$$L(x) := \int_{\alpha_1}^{\beta_q} \log|x-t|\rho(t)dt. \tag{5.21}$$

Both of the error terms, $O(N^{-1})$ and $O(e^{-N\varepsilon})$, are uniform in K.

The remaining theorems in this section use the Airy functions Ai and Bi (see, e.g. [40]).

Theorem 5.4.5 *(Asymptotics of $P_N(z)$ at band-void edge points) Let $j \in \mathcal{A}_v \cup \{q\}$, so that the point β_j is the right endpoint of a band and the left endpoint of a void. Then there exists $\varepsilon > 0$ such that, for $|z - \beta_j| < \varepsilon$,*

$$\begin{aligned} P_N(z) &= e^{\frac{N}{2}(V(z)+l)} \\ &\times \left\{ N^{1/6}\psi_{\beta_j}(z)^{1/4}\mathrm{Ai}(N^{2/3}\psi_{\beta_j}(z))\left[e^{\pm\frac{i\Omega_{j,N}}{2}}\mathcal{M}_1(z) + e^{\mp\frac{i\Omega_{j,N}}{2}}\mathcal{M}_2(z) + O(N^{-1})\right]\right. \\ &\left. - N^{-1/6}\psi_{\beta_j}(z)^{-1/4}\mathrm{Ai}'(N^{2/3}\psi_{\beta_j}(z))\left[e^{\pm\frac{i\Omega_{j,N}}{2}}\mathcal{M}_1(z) - e^{\mp\frac{i\Omega_{j,N}}{2}}\mathcal{M}_2(z) + O(N^{-1})\right]\right\} \end{aligned}$$

for $\pm\mathrm{Im}\, z > 0$.

Let $j \in \mathcal{A}_v \cup \{0\}$, *so that the point* α_{j+1} *is the left endpoint of a band and the right endpoint of a void. There exists* $\varepsilon > 0$ *such that, for* $|z - \alpha_{j+1}| < \varepsilon$,

$$P_N(z) = e^{\frac{N}{2}(V(z)+l)}$$
$$\times \Big\{ N^{1/6}\psi_{\alpha_{j+1}j}(z)^{1/4}\mathrm{Ai}(N^{2/3}\psi_{\alpha_{j+1}}(z)) \left[e^{\pm\frac{i\Omega_{j,N}}{2}}\mathcal{M}_1(z) - e^{\mp\frac{i\Omega_{j,N}}{2}}\mathcal{M}_2(z) + O(N^{-1})\right]$$
$$- N^{-1/6}\psi_{\alpha_{j+1}}(z)^{-1/4}\mathrm{Ai}'(N^{2/3}\psi_{\alpha_{j+1}}(z)) \left[e^{\pm\frac{i\Omega_{j,N}}{2}}\mathcal{M}_1(z) + e^{\mp\frac{i\Omega_{j,N}}{2}}\mathcal{M}_2(z) + O(N^{-1})\right] \Big\}$$

for $\pm\mathrm{Im}\, z > 0$.

Theorem 5.4.6 *(Asymptotics of* $P_N(z)$ *at band-saturated region edge points) Let* $j \in \mathcal{A}_s$. *Then the point* β_j *is the right endpoint of a band and the left endpoint of a saturated region. There exists* $\varepsilon > 0$ *such that, for* $|z - \beta_j| < \varepsilon$,

$$P_N(z) = e^{\frac{N}{2}(V(z)+l)}$$
$$\times \Big\{ N^{1/6}\psi_{\beta_j}(z)^{1/4}\mathcal{B}_1(z) \left[-e^{\pm\frac{i\Omega_{j,N}}{2}}\mathcal{M}_1(z) + e^{\mp\frac{i\Omega_{j,N}}{2}}\mathcal{M}_2(z) + O(N^{-1})\right]$$
$$- N^{-1/6}\psi_{\beta_j}(z)^{-1/4}\mathcal{B}_2(z) \left[e^{\pm\frac{i\Omega_{j,N}}{2}}\mathcal{M}_1(z) + e^{\mp\frac{i\Omega_{j,N}}{2}}\mathcal{M}_2(z) + O(N^{-1})\right] \Big\}$$

for $\pm\mathrm{Im}\, z > 0$, *where*

$$\mathcal{B}_1(z) = \cos(N\pi z)\mathrm{Ai}(N^{2/3}\psi_{\beta_j}(z)) + \sin(N\pi z)\mathrm{Bi}(N^{2/3}\psi_{\beta_j}(z)),$$
$$\mathcal{B}_2(z) = \cos(N\pi z)\mathrm{Ai}'(N^{2/3}\psi_{\beta_j}(z)) + \sin(N\pi z)\mathrm{Bi}'(N^{2/3}\psi_{\beta_j}(z)).$$

The point α_{j+1} *is the left endpoint of a band and the right endpoint of a void. There exists* $\varepsilon > 0$ *such that, for* $|z - \alpha_{j+1}| < \varepsilon$,

$$P_N(z) = e^{\frac{N}{2}(V(z)+l)}$$
$$\times \Big\{ N^{1/6}\psi_{\alpha_{j+1}j}(z)^{1/4}\mathcal{B}_3(z) \left[e^{\pm\frac{i\Omega_{j,N}}{2}}\mathcal{M}_1(z) + e^{\mp\frac{i\Omega_{j,N}}{2}}\mathcal{M}_2(z) + O(N^{-1})\right].$$
$$- N^{-1/6}\psi_{\alpha_{j+1}}(z)^{-1/4}\mathcal{B}_4(z) \left[e^{\pm\frac{i\Omega_{j,N}}{2}}\mathcal{M}_1(z) - e^{\mp\frac{i\Omega_{j,N}}{2}}\mathcal{M}_2(z) + O(N^{-1})\right] \Big\}$$

for $\pm\mathrm{Im}\, z > 0$, *where*

$$\begin{aligned}\mathcal{B}_3(z) &= \cos(N\pi z)\mathrm{Ai}(N^{2/3}\psi_{\alpha_{j+1}}(z)) - \sin(N\pi z)\mathrm{Bi}(N^{2/3}\psi_{\alpha_{j+1}}(z)),\\ \mathcal{B}_4(z) &= \cos(N\pi z)\mathrm{Ai}'(N^{2/3}\psi_{\alpha_{j+1}}(z)) - \sin(N\pi z)\mathrm{Bi}'(N^{2/3}\psi_{\alpha_{j+1}}(z)).\end{aligned} \tag{5.22}$$

Remark: Although the above theorems are presented for real analytic potential $V(x)$ these results may be extended to potentials which are continuous and piecewise real analytic satisfying (5.2), assuming that the points of non-analyticity lie strictly within saturated regions and voids, as is the case in the preceding chapter. In this case the preceding results hold, and the asymptotic solution to the associated Riemann-Hilbert Problem does not require local analysis near the points of non-analyticity (see [8]).

Before continuing with the proofs of these theorems, we would also like to remark that the results obtained in this paper match the results obtained in [5] for polynomials orthogonal on a lattice which sits inside a finite interval. Consequently, many corollaries discussed in [5] also apply to infinite lattices. In particular, the authors of [5] discuss the particle statistics of the *discrete orthogonal polynomial ensemble* in different regions of a finite interval of the real line, which are based on asymptotic properties of the associated orthogonal polynomials. The results of this paper imply that their results may be extended to discrete orthogonal polynomial ensembles on an infinite (regular) lattice. Of particular interest may be the *discrete sine kernel* as the scaling limit of the reproducing kernel in the interior of bands, the *Airy kernel* as the scaling limit of the reproducing kernel at band end-points, the *Tracy-Widom distribution* for the location of the left- and right-most particle, and exponential estimates for all correlation functions in voids and saturated regions.

The rest of this chapter is organized as follows. In Section 5.5, we reformulate the orthogonal polynomials (5.2) as the solution to an interpolation problem of complex analysis. In Section 5.6, we reduce the interpolation problem to a Riemann-Hilbert Problem which can be solved by steepest descent analysis, which is done in Sections 5.7-5.12. Finally, in Section 5.13, we give proofs of the preceding theorems.

5.5 Interpolation problem

We will evaluate the asymptotics of the discrete orthogonal polynomials described above via a steepest descent asymptotic analysis of a Riemann-Hilbert problem. To that end, we first introduce the following interpolation problem.

Interpolation Problem. For a given $N = 0, 1, \ldots$, find a 2×2 matrix-valued function $\mathbf{P}_N(z) = (\mathbf{P}_N(z)_{ij})_{1 \le i,j \le 2}$ with the following properties:

1. *Analyticity*: $\mathbf{P}_N(z)$ is an analytic function of z for $z \in \mathbb{C} \setminus L_N$.

2. *Residues at poles*: At each node $x \in L_N$, the elements $\mathbf{P}_N(z)_{11}$ and $\mathbf{P}_N(z)_{21}$ of the matrix $\mathbf{P}_N(z)$ are analytic functions of z, and the elements $\mathbf{P}_N(z)_{12}$ and $\mathbf{P}_N(z)_{22}$ have a simple pole with the residues,
$$\operatorname*{Res}_{z=x} \mathbf{P}_N(z)_{j2} = w_N(x)\mathbf{P}_N(x)_{j1}, \quad j = 1, 2.$$

3. *Asymptotics at infinity*: There exists a function $r(x) > 0$ on L_N such that
$$\lim_{x \to \infty} r(x) = 0,$$
and such that as $z \to \infty$, $\mathbf{P}_N(z)$ admits the asymptotic expansion,
$$\mathbf{P}_N(z) \sim \left(I + \frac{\mathbf{P}_1}{z} + \frac{\mathbf{P}_2}{z^2} + \ldots \right) \begin{pmatrix} z^N & 0 \\ 0 & z^{-N} \end{pmatrix}, \qquad z \in \mathbb{C} \setminus \left[\bigcup_{x \in L_N}^{\infty} D\big(x, r(x)\big) \right], \tag{5.23}$$
where $D(x, r(x))$ denotes a disk of radius $r(x) > 0$ centered at x and I is the identity matrix.

It is not difficult to see (see [11] and [5]) that the IP has a unique solution, which is
$$\mathbf{P}_N(z) = \begin{pmatrix} P_N(z) & C(w_N P_N)(z) \\ (h_{N-1})^{-1} P_{N-1}(z) & (h_{N-1})^{-1} C(w_N P_{N-1})(z) \end{pmatrix}, \tag{5.24}$$
where the Cauchy transformation C is defined by the formula,
$$C(f)(z) = \sum_{x \in L_N} \frac{f(x)}{z - x}.$$

Because of the orthogonality condition, as $z \to \infty$,

$$C(w_N P_n)(z) = \sum_{x \in L_N} \frac{w_N(x) P_n(x)}{z - x} \sim \sum_{x \in L_N} w_N(x) P_n(x) \sum_{j=0}^{\infty} \frac{x^j}{z^{j+1}} = \frac{h_n}{z^{n+1}} + \sum_{j=n+2}^{\infty} \frac{a_j}{z^j},$$

which justifies asymptotic expansion (5.23), and have that

$$h_N = [\mathbf{P}_1]_{12}, \qquad h_{N-1}^{-1} = [\mathbf{P}_1]_{21}. \tag{5.25}$$

Furthermore, the recurrence coefficients in equation (5.4) are given by

$$\gamma_N^2 = [\mathbf{P}_1]_{12}[\mathbf{P}_1]_{21} \quad ; \quad \beta_{N-1} = \frac{[\mathbf{P}_2]_{21}}{[\mathbf{P}_1]_{21}} - [\mathbf{P}_1]_{11}. \tag{5.26}$$

5.6 Reduction of IP to RHP

We would like to reduce the Interpolation Problem to a Riemann-Hilbert Problem (RHP). Introduce the function

$$\Pi(z) = \frac{\sin(N\pi z)}{N\pi}.$$

Notice that

$$\Pi(x_k) = 0, \quad \Pi'(x_k) = \exp(iN\pi x_k) = (-1)^k, \quad \text{for} \quad x_k = \frac{k}{N} \in L_N.$$

Introduce the upper triangular matrices,

$$\mathbf{D}_{\pm}^u(z) = \begin{pmatrix} 1 & -\frac{w_N(z)}{\Pi(z)} e^{\pm iN\pi z} \\ 0 & 1 \end{pmatrix},$$

and the lower triangular matrices,

$$\mathbf{D}_{\pm}^l = \begin{pmatrix} \Pi(z)^{-1} & 0 \\ -\frac{1}{w_N(z)} e^{\pm iN\pi z} & \Pi(z) \end{pmatrix} = \begin{pmatrix} \Pi(z)^{-1} & 0 \\ 0 & \Pi(z) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{\Pi(z) w_N(z)} e^{\pm iN\pi z} & 1 \end{pmatrix}. \tag{5.27}$$

Define the matrix-valued functions,

$$\mathbf{R}_N^u = \mathbf{P}_N(z) \times \begin{cases} \mathbf{D}_+^u(z) & \text{when} \quad \operatorname{Im} z \geq 0 \\ \mathbf{D}_-^u(z) & \text{when} \quad \operatorname{Im} z \leq 0, \end{cases}$$

and

$$\mathbf{R}_N^l = \mathbf{P}_N(z) \times \begin{cases} \mathbf{D}_+^l(z), & \text{when} \quad \operatorname{Im} z \geq 0 \\ \mathbf{D}_-^l(z), & \text{when} \quad \operatorname{Im} z \leq 0. \end{cases}$$

From (5.24) we have that

$$\mathbf{R}_N^u(z) = \begin{pmatrix} P_N(z) & -\frac{w_N(z)P_N(z)}{\Pi(z)} e^{\pm iN\pi z} + C(w_N P_N)(z) \\ h_{N-1}^{-1} P_{N-1}(z) & -\frac{w_N(z)h_{N-1}^{-1}P_{N-1}(z)}{\Pi(z)} e^{\pm iN\pi z} + h_{N-1}^{-1} C(w_N P_{N-1})(z) \end{pmatrix}$$

$$\text{when} \quad \pm \operatorname{Im} z \geq 0,$$

and

$$\mathbf{R}_N^l(z) = \begin{pmatrix} \frac{P_N(z)}{\Pi(z)} - \frac{C(w_N P_N)(z)}{w_N(z)} e^{\pm iN\pi z} & \Pi(z) C(w_N P_N)(z) \\ \frac{h_{N-1}^{-1} P_{N-1}(z)}{\Pi(z)} - \frac{h_{N-1}^{-1} C(w_N P_{N-1})(z)}{w_N(z)} e^{\pm iN\pi z} & \Pi(z) h_{N-1}^{-1} C(w_N P_{N-1})(z) \end{pmatrix}$$

$$\text{when} \quad \pm \operatorname{Im} z \geq 0.$$

Observe that the functions $\mathbf{R}_N^u(z)$, $\mathbf{R}_N^l(z)$ are meromorphic on the closed upper and lower complex planes and they are two-valued on the real axis. Their possible poles are located on the lattice L_N. In fact, due to some cancellations they do not have any poles at all. We have the following proposition.

Proposition 5.6.1 *The matrix-valued functions* $\mathbf{R}_N^u(z)$ *and* $\mathbf{R}_N^l(z)$ *have no poles and on the real line they satisfy the following jump conditions at* $x \in \mathbb{R}$*:*

$$\mathbf{R}_{N+}^u(x) = \mathbf{R}_{N-}^u(x) j_R^u(x), \qquad j_R^u(x) = \begin{pmatrix} 1 & -2N\pi i w_N(x) \\ 0 & 1 \end{pmatrix},$$

and

$$\mathbf{R}_{N+}^l(x) = \mathbf{R}_{N-}^l(x) j_R^l(x), \qquad j_R^l(x) = \begin{pmatrix} 1 & 0 \\ -\frac{2N\pi i}{w_N(x)} & 1 \end{pmatrix}.$$

The proof of this proposition is identical to that of Proposition 4.10.1, and we omit it here.

To reduce the Interpolation Problem to a Riemann-Hilbert Problem, we simply generalize the approach of the previous chapter. Consider the oriented contour Σ on the complex plane depicted in Figure 5.2, in which the horizontal lines are $\operatorname{Im} z =$

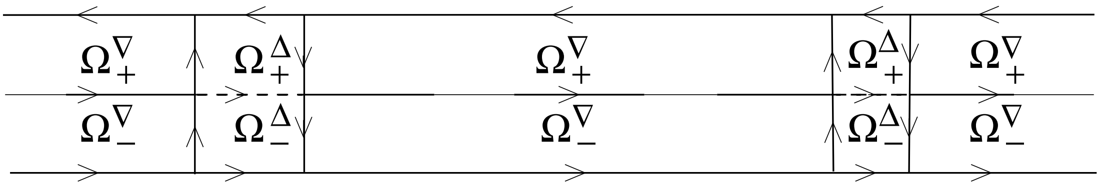

Fig. 5.2. The contour Σ arising from the hypothetical equilibrium measure in Figure 5.1, dividing an ε-neighborhood of the real line into the regions $\Omega_\pm^\Delta$ and $\Omega_\pm^\nabla$.

$\varepsilon, 0, -\varepsilon$, where $\varepsilon > 0$ is a small positive constant which will be determined later, and the vertical segments pass through the endpoints of saturated intervals. Consider the regions

$$\Omega_\pm^\nabla = \{I^0 \cup I^-\} \times (0, \pm i\varepsilon)$$

$$\Omega_\pm^\Delta = I^+ \times (0, \pm i\varepsilon)$$

bounded by the contour Σ.

Define

$$\mathbf{R}_N(z) = \begin{cases} \mathbf{K}_N \mathbf{R}_N^u(z) \mathbf{K}_N^{-1} & \text{for} \quad z \in \Omega_\pm^\nabla \\ \mathbf{K}_N \mathbf{R}_N^l(z) \mathbf{K}_N^{-1} & \text{for} \quad z \in \Omega_\pm^\Delta \\ \mathbf{K}_N \mathbf{P}_N(z) \mathbf{K}_N^{-1} & \text{otherwise,} \end{cases} \tag{5.28}$$

where $\mathbf{K}_N = \begin{pmatrix} 1 & 0 \\ 0 & -2iN\pi \end{pmatrix}$.

Proposition 5.6.2 *The matrix-valued function $\mathbf{R}_N(z)$ has the following jumps on the contour Σ:*

$$\mathbf{R}_{N+}(z) = \mathbf{R}_{N-}(z) j_R(z),$$

where

$$j_R(z) = \begin{cases} \begin{pmatrix} 1 & w_N(z) \\ 0 & 1 \end{pmatrix} & \textit{for} \quad z \in I^- \cup I^0 \\ \begin{pmatrix} 1 & 0 \\ -(2N\pi)^2 w_N(z)^{-1} & 1 \end{pmatrix} & \textit{for} \quad z \in I^+ \\ \mathbf{K}_N \mathbf{D}^u_\pm(z) \mathbf{K}_N^{-1} = \begin{pmatrix} 1 & \frac{1}{2iN\pi} \frac{w_N(z) e^{\pm iN\pi z}}{\Pi(z)} \\ 0 & 1 \end{pmatrix} & \textit{for} \quad z \in \{I^- \cup I^0\} \pm i\varepsilon \\ \mathbf{K}_N \mathbf{D}^l_\pm(z) \mathbf{K}_N^{-1} = \begin{pmatrix} \Pi(z)^{-1} & 0 \\ 2iN\pi \frac{e^{\pm iN\pi z}}{w_N(z)} & \Pi(z) \end{pmatrix} & \textit{for} \quad z \in I^+ \pm i\varepsilon \\ \mathbf{K}_N \mathbf{D}^l_\pm(z)^{-1} \mathbf{D}^u_\pm(z) \mathbf{K}_N^{-1} = \begin{pmatrix} \Pi(z) & \frac{1}{2N\pi i} w_N(z) e^{\pm iN\pi z} \\ -2N\pi i w_N(z)^{-1} e^{\pm iN\pi z} & \mp 2N\pi i e^{\pm iN\pi z} \end{pmatrix} & \\ \qquad \textit{for} \quad z \in (0, \pm i\varepsilon) + \beta_j \quad \textit{or} \quad z \in (0, \pm i\varepsilon) + \alpha_{j+1} \quad \textit{for} \quad j \in \mathcal{A}_s. & \end{cases}$$

5.7 First transformation of the RHP

Define the matrix function $\mathbf{T}_N(z)$ as follows from the equation

$$\mathbf{R}_N(z) = e^{\frac{Nl}{2}\sigma_3} \mathbf{T}_N(z) e^{N(g(z) - \frac{l}{2})\sigma_3}, \tag{5.29}$$

where l is the Lagrange multiplier, the function $g(z)$ is described in Section 5.2, and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the third Pauli matrix. Then $\mathbf{T}_N(z)$ satisfies the following Riemann-Hilbert Problem:

1. $\mathbf{T}_N(z)$ is analytic in $\mathbb{C} \setminus \Sigma$.

2. $\mathbf{T}_{N+}(z) = \mathbf{T}_{N-}(z) j_T(z)$ for $z \in \Sigma$, where

$$j_T(z) = \begin{cases} e^{N(g_-(z) - \frac{l}{2})\sigma_3} j_R(z) e^{-N(g_+(z) - \frac{l}{2})\sigma_3} & \text{for} \quad z \in \mathbb{R} \\ e^{N(g(z) - \frac{l}{2})\sigma_3} j_R(z) e^{-N(g(z) - \frac{l}{2})\sigma_3} & \text{for} \quad z \in \Sigma \setminus \mathbb{R}. \end{cases} \tag{5.30}$$

3. As $\quad z \to \infty$,

$$\mathbf{T}_N(z) \sim I + \frac{\mathbf{T}_1}{z} + \frac{\mathbf{T}_2}{z^2} + \dots.$$

Let's take a closer look at the behavior of the jump matrix j_T described in (5.30) on the horizontal segments of Σ. We have that

$$j_T(z) = \begin{cases} \begin{pmatrix} e^{-NG(z)} & w_N(z)e^{N(g_+(z)+g_-(z)-l)} \\ 0 & e^{NG(z)} \end{pmatrix} & \text{when} \quad z \in I^0 \cup I^- \\ \begin{pmatrix} e^{-NG(z)} & 0 \\ -(2N\pi)^2 e^{-N(g_+(z)+g_-(z)-V(z)-l)} & e^{NG(z)} \end{pmatrix} & \text{when} \quad z \in I^+ \\ \begin{pmatrix} 1 & \pm\frac{e^{N(2g(z)-l-V(z))}}{1-e^{\mp 2iN\pi x}e^{\varepsilon 2N\pi}} \\ 0 & 1 \end{pmatrix} & \text{when} \quad z = x \pm i\varepsilon \in \{I^- \pm i\varepsilon\} \\ \begin{pmatrix} 1 & \pm\frac{e^{\pm NG(z)}}{1-e^{\mp 2iN\pi x}e^{\varepsilon 2N\pi}} \\ 0 & 1 \end{pmatrix} & \text{when} \quad z = x \pm i\varepsilon \in \{I^0 \pm i\varepsilon\} \\ \begin{pmatrix} \Pi(z)^{-1} & 0 \\ 2iN\pi e^{\pm iN\pi x} e^{-N(2g(z)-l-V(z))} & \Pi(z) \end{pmatrix} & \text{when} \quad z = x \pm i\varepsilon \in \{I^+ \pm i\varepsilon\}. \end{cases}$$

5.8 Second transformation of the RHP

We make the second transformation of the RHP in analogue with (4.77), so that

$$\mathbf{S}_N(z) = \begin{cases} \mathbf{T}_N(z) j_+(z)^{-1} & \text{for} \quad z \in I^0 \times (0, i\varepsilon) \\ \mathbf{T}_N(z) j_-(z) & \text{for} \quad z \in I^0 \times (0, -i\varepsilon) \\ \mathbf{T}_N(z)\mathbf{A}_+(z) & \text{for} \quad z \in I^+ \times (0, i\varepsilon) \\ \mathbf{T}_N(z)\mathbf{A}_-(z) & \text{for} \quad z \in I^+ \times (0, -i\varepsilon) \\ \mathbf{T}_N(z) & \text{otherwise,} \end{cases} \tag{5.31}$$

where

$$\mathbf{A}_+(z) = \begin{pmatrix} -\frac{1}{2N\pi i} e^{-iN\pi z} & 0 \\ 0 & -2N\pi i e^{iN\pi z} \end{pmatrix}, \quad \mathbf{A}_-(z) = \begin{pmatrix} \frac{1}{2N\pi i} e^{iN\pi z} & 0 \\ 0 & 2N\pi i e^{-iN\pi z} \end{pmatrix},$$

$$j_+(x) = \begin{pmatrix} 1 & 0 \\ e^{-NG(x)} & 1 \end{pmatrix}, \quad j_-(x) = \begin{pmatrix} 1 & 0 \\ e^{NG(x)} & 1 \end{pmatrix}.$$

This function satisfies a similar RHP to $\mathbf{T}$, but jumps now occur on a new contour, Σ_S, which is obtained from Σ by adding the segments $(\alpha_1 - i\varepsilon, \alpha_1 + i\varepsilon)$, $(\beta_q - i\varepsilon, \beta_q + i\varepsilon)$,$(\alpha_{j+1} - i\varepsilon, \alpha_{j+1} + i\varepsilon)$, $(\beta_j - i\varepsilon, \beta_j + i\varepsilon)$ for $j \in \mathcal{A}_v$, see Figure 5.3.

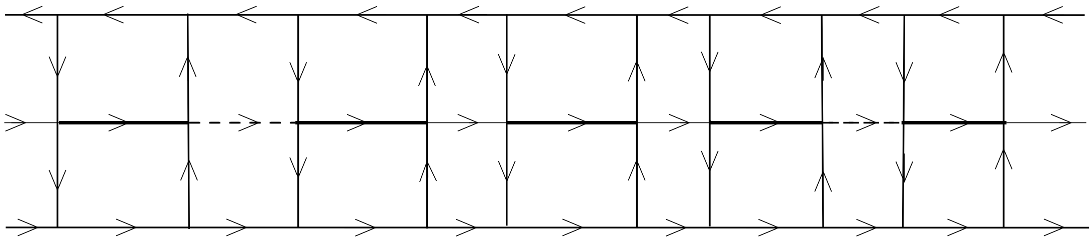

Fig. 5.3. The contour Σ_S arising from the hypothetical equilibrium measure shown in Figure 5.1.

On horizontal segments, we have that

$$j_S(z) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{for } z \in I^0 \\ \begin{pmatrix} 1 + O(e^{-2\varepsilon N\pi}) & O(e^{N(G(z)-2\varepsilon\pi)}) \\ -e^{-NG(z)} & 1 \end{pmatrix} & \text{for } z - i\varepsilon \in I^0 \\ \begin{pmatrix} 1 + O(e^{-2\varepsilon N\pi}) & O(e^{N(-G(z)-2\varepsilon\pi)}) \\ e^{NG(z)} & 1 \end{pmatrix} & \text{for } z + i\varepsilon \in I^0 \\ \begin{pmatrix} 1 + O(e^{-2\varepsilon N\pi}) & 0 \\ 2iN\pi e^{-N(2g(z)-l-V(z))} & 1 + O(e^{-2\varepsilon N\pi}) \end{pmatrix} & \text{for } z \in \{I^+ \pm i\varepsilon\} \\ \begin{pmatrix} -e^{-iN\Omega_j} & 0 \\ -e^{-N(g_+(z)+g_-(z)-l-V(z))} & -e^{iN\Omega_j} \end{pmatrix} & \text{for } z \in (\beta_j, \alpha_{j+1}),\ j \in \mathcal{A}_s \\ \begin{pmatrix} 1 & e^{N(2g(z)-l-V(z))}O(e^{-2\varepsilon N\pi}) \\ 0 & 1 \end{pmatrix} & \text{for } z = x \pm i\varepsilon \in \{I^- \pm i\varepsilon\} \\ \begin{pmatrix} e^{-iN\Omega_j} & e^{N(g_+(z)+g_-(z)-l-V(z))} \\ 0 & e^{iN\Omega_j} \end{pmatrix} & \text{for } z \in (\beta_j, \alpha_{j+1}),\ j \in \mathcal{A}_v. \end{cases}$$

By formula (5.12) for the G-function and the upper constraint on the density ρ, we obtain that, for sufficiently small $\varepsilon > 0$ and $x \in (\alpha_j, \beta_j)$,

$$0 < \mp \operatorname{Re} G(x \pm i\varepsilon) = 2\pi\rho(x) + O(\varepsilon^2) < 2\pi\varepsilon + O(\varepsilon^2).$$

This, combined with property (5.13) of the g-function, implies that all jumps on horizontal segments are exponentially close to the identity matrix, provided that they are bounded away from the segment (α_1, β_q).

5.9 Model RHP

The model RHP appears when we drop in the jump matrix $j_S(z)$ the terms that vanish as $N \to \infty$:

1. $\mathbf{M}(z)$ is analytic in $\mathbb{C} \setminus [\alpha_1, \beta_q]$.
2. $\mathbf{M}_+(z) = \mathbf{M}_-(z) j_M(z)$ for $z \in [\alpha_1, \beta_q]$, where

$$j_M(z) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{for} \quad z \in I^0 \\ e^{-i\Omega_{j,N}\sigma_3} & \text{for} \quad z \in (\beta_j, \alpha_{j+1}), \end{cases} \tag{5.32}$$

3. As $z \to \infty$,

$$\mathbf{M}(z) \sim I + \frac{\mathbf{M}_1}{z} + \frac{\mathbf{M}_2}{z^2} + \dots. \tag{5.33}$$

The solution to this model problem is given as follows (see [16], [15]), using the notation introduced in Section 5.4.

$$\mathbf{M}(z) = \mathbf{F}(\infty)^{-1} \begin{pmatrix} \frac{\gamma(z)+\gamma^{-1}(z)}{2} \frac{\vartheta(u(z)+\frac{\Omega_N}{2\pi}+d)}{\vartheta(u(z)+d)} & \frac{\gamma(z)-\gamma^{-1}(z)}{-2i} \frac{\vartheta(u(z)-\frac{\Omega_N}{2\pi}-d)}{\vartheta(u(z)-d)} \\ \frac{\gamma(z)-\gamma^{-1}(z)}{2i} \frac{\vartheta(u(z)+\frac{\Omega_N}{2\pi}-d)}{\vartheta(u(z)-d)} & \frac{\gamma(z)+\gamma^{-1}(z)}{2} \frac{\vartheta(u(z)-\frac{\Omega_N}{2\pi}+d)}{\vartheta(u(z)+d)} \end{pmatrix}$$

where

$$\mathbf{F}(\infty) = \begin{pmatrix} \frac{\vartheta(u(\infty)+\frac{\Omega_N}{2\pi}+d)}{\vartheta(u(\infty)+d)} & 0 \\ 0 & \frac{\vartheta(u(\infty)-\frac{\Omega_N}{2\pi}+d)}{\vartheta(u(\infty)+d)} \end{pmatrix}.$$

The asymptotics at infinity are given as

$$\mathbf{M}(z) = I + \frac{\mathbf{M}_1}{z} + O(z^{-2}).$$

5.10 Parametrix at band-void edge points

We now consider small disks $D(\alpha_j, \varepsilon)$ for $j-1 \in \mathcal{A}_v \cup \{0\}$, and $D(\beta_j, \varepsilon)$ for $j \in \mathcal{A}_v \cup \{q\}$, centered at the endpoints of bands which are adjacent to a void. Denote

$$D = \left(\bigcup_{j-1 \in \mathcal{A}_v \cup \{0\}} D(\alpha_j, \varepsilon) \right) \bigcup \left(\bigcup_{j \in \mathcal{A}_v \cup \{q\}} D(\beta_j, \varepsilon) \right).$$

We will seek a local parametrix $\mathbf{U}_N(z)$ defined on D such that

1.
$$\mathbf{U}_N(z) \text{ is analytic on } D \setminus \Sigma_S. \tag{5.34}$$

2.
$$\mathbf{U}_{N+}(z) = \mathbf{U}_{N-}(z) j_S(z) \quad \text{for} \quad z \in D \cap \Sigma_S. \tag{5.35}$$

3.
$$\mathbf{U}_N(z) = \mathbf{M}(z)\big(I + O(N^{-1})\big) \quad \text{uniformly for } z \in \partial D. \tag{5.36}$$

We first construct the parametrix near β_j for $j \in \mathcal{A}_v$. The jumps j_S are given by

$$j_S(z) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{for } z \in (\beta_j - \varepsilon, \beta_j) \\ \begin{pmatrix} 1 & 0 \\ -e^{-NG(z)} & 1 \end{pmatrix} & \text{for } z \in (\beta_j, \beta_j + i\varepsilon) \\ \begin{pmatrix} 1 & 0 \\ e^{NG(z)} & 1 \end{pmatrix} & \text{for } z \in (\beta_j, \beta_j - i\varepsilon) \\ \begin{pmatrix} e^{-NG(z)} & e^{N(g_+(z)+g_-(z)-V(z)-l)} \\ 0 & e^{NG(z)} \end{pmatrix} & \text{for } z \in (\beta_j, \beta_j + \varepsilon). \end{cases}$$

If we let

$$\mathbf{U}_N(z) = \mathbf{Q}_N(z) e^{-N(g(z) - \frac{V(z)}{2} - \frac{l}{2})\sigma_3},$$

then the jump conditions on $\mathbf{Q}_N$ become

$$\mathbf{Q}_{N+}(z) = \mathbf{Q}_{N-}(z) j_Q(z)$$

where

$$j_Q(z) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{for } z \in (\beta_j - \varepsilon, \beta_j) \\ \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} & \text{for } z \in (\beta_j, \beta_j + i\varepsilon) \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} & \text{for } z \in (\beta_j, \beta_j - i\varepsilon) \\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \text{for } z \in (\beta_j, \beta_j + \varepsilon). \end{cases} \tag{5.37}$$

where orientation is from left to right on horizontal contours, and down to up on vertical contours, according to Figure 5.3.

$\mathbf{Q}_N$ is constructed using Airy functions. Recall that the Airy function solves the differential equation $y'' = zy$, and has the asymptotics (4.99). If we let

$$y_0(z) = \mathrm{Ai}(z), \quad y_1(z) = \omega \mathrm{Ai}(\omega z), \quad y_2(z) = \omega^2 \mathrm{Ai}(\omega^2 z)$$

where $\omega = e^{\frac{2\pi i}{3}}$, then the functions y_0, y_1, and y_2 satisfy the relation

$$y_0(z) + y_1(z) + y_2(z) = 0.$$

If we take

$$\Phi_{rv}(z) = \begin{cases} \begin{pmatrix} y_0(z) & -y_2(z) \\ y_0'(z) & -y_2'(z) \end{pmatrix} & \text{for} \quad \arg z \in \left(0, \frac{\pi}{2}\right) \\ \begin{pmatrix} -y_1(z) & -y_2(z) \\ -y_1'(z) & -y_2'(z) \end{pmatrix} & \text{for} \quad \arg z \in \left(\frac{\pi}{2}, \pi\right) \\ \begin{pmatrix} -y_2(z) & y_1(z) \\ -y_2'(z) & y_1'(z) \end{pmatrix} & \text{for} \quad \arg z \in \left(-\pi, -\frac{\pi}{2}\right) \\ \begin{pmatrix} y_0(z) & y_1(z) \\ y_0'(z) & y_1'(z) \end{pmatrix} & \text{for} \quad \arg z \in \left(-\frac{\pi}{2}, 0\right), \end{cases}$$

then Φ_{rv} satisfies jump conditions similar to (5.37), but for jumps on rays emanating from the origin rather than from β_j. We thus need to map the disk $D(\beta_j, \varepsilon)$ onto some convex neighborhood of the origin in order to take advantage of the function Φ_{rv}. Our mapping should match the asymptotics of the Airy function in order to have the matching property (5.36).

To this end, notice that, by (5.9), for $t \in [\alpha_j, \beta_j]$, as $t \to \beta_j$,

$$\rho(t) = C(\beta_j - t)^{1/2} + O\big((\beta - t)^{3/2}\big), \quad C > 0.$$

It follows that, for $\quad x \in [\alpha_j, \beta_j] \quad$ as $\quad x \to \beta_j$,

$$\int_x^{\beta_j} \rho(t)dt = C(\beta_j - x)^{3/2} + O\big((\beta_j - x)^{5/2}\big) \quad C_0 = \frac{2}{3}C.$$

Thus,

$$\psi_{\beta_j}(z) = -\left\{\frac{3\pi}{2}\int_z^{\beta_j} \rho(t)dt\right\}^{2/3}$$

is analytic at β_j, thus extends to a conformal map from $D(\beta_j, \varepsilon)$ (for small enough ε) onto a convex neighborhood of the origin. Furthermore,

$$\psi_{\beta_j}(\beta_j) = 0 \quad ; \quad \psi'_{\beta_j}(\beta_j) > 0,$$

thus ψ_{β_j} is real negative on $(\beta_j - \varepsilon, \beta_j)$, and real positive on $(\beta_j, \beta_j + \varepsilon)$. Also, we can slightly deform the vertical pieces of the contour Σ_S close to β_j, so that

$$\psi_{\beta_j}\{D(\beta_j, \varepsilon) \cap \Sigma_S\} = (-\varepsilon, \varepsilon) \cup (-i\varepsilon, i\varepsilon).$$

We now set

$$\mathbf{Q}_N(z) = \mathbf{E}_N^{\beta_j}(z)\Phi_{rv}\big(N^{2/3}\psi_{\beta_j}(z)\big)$$

so that

$$\mathbf{U}_N(z) = \mathbf{E}_N^{\beta_j}(z)\Phi_{rv}\big(N^{2/3}\psi_{\beta_j}(z)\big)e^{-N(g(z) - \frac{V(z)}{2} - \frac{l}{2})\sigma_3}, \tag{5.38}$$

where

$$\mathbf{E}_N^{\beta_j}(z) = \mathbf{M}(z)e^{\pm\frac{i\Omega_{j,N}}{2}\sigma_3}\mathbf{L}_N^{\beta_j}(z)^{-1} \quad \text{for} \quad \pm \operatorname{Im} z > 0,$$

$$\mathbf{L}_N^{\beta_j}(z) = \frac{1}{2\sqrt{\pi}}\begin{pmatrix} N^{-1/6}\psi_{\beta_j}^{-1/4}(z) & 0 \\ 0 & N^{1/6}\psi_{\beta_j}^{1/4}(z)\end{pmatrix}\begin{pmatrix} 1 & i \\ -1 & i\end{pmatrix},$$

and we take the principal branch of $\psi_{\beta_j}^{1/4}$, which is positive on $(\beta_j, \beta_j + \varepsilon)$ and has a cut on $(\beta_j - \varepsilon, \beta_j)$. The function $\Phi_{rv}(N^{2/3}\psi_{\beta_j}(z))$ has the jumps j_S, and we claim that the prefactor $\mathbf{E}_N^{\beta_j}$ is analytic in $D(\beta_j,\varepsilon)$, thus does not change these jumps. This can be seen, as

$$\mathbf{M}_+(z)e^{\frac{i\Omega_{j,N}}{2}\sigma_3} = \mathbf{M}_-(z)e^{-\frac{i\Omega_{j,N}}{2}\sigma_3}e^{\frac{i\Omega_{j,N}}{2}\sigma_3}j_M e^{\frac{i\Omega_{j,N}}{2}\sigma_3}\,, \tag{5.39}$$

thus the jump for the function $\mathbf{M}(z)e^{\pm\frac{i\Omega_{j,N}}{2}\sigma_3}$ is

$$e^{\frac{i\Omega_{j,N}}{2}\sigma_3}j_M e^{\frac{i\Omega_{j,N}}{2}\sigma_3} = \begin{cases} e^{\frac{i\Omega_{j,N}}{2}\sigma_3}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} e^{\frac{i\Omega_{j,N}}{2}\sigma_3} & \text{for} \quad z\in(\beta_j-\varepsilon,\beta_j) \\ e^{\frac{i\Omega_{j,N}}{2}\sigma_3}e^{-i\Omega_{j,N}\sigma_3}e^{\frac{i\Omega_{j,N}}{2}\sigma_3} & \text{for} \quad z\in(\beta_j,\beta_j+\varepsilon)\,, \end{cases} \tag{5.40}$$

or equivalently,

$$e^{\frac{i\Omega_{j,N}}{2}\sigma_3}j_M e^{\frac{i\Omega_{j,N}}{2}\sigma_3} = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{for} \quad z\in(\beta_j-\varepsilon,\beta_j) \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{for} \quad z\in(\beta_j,\beta_j+\varepsilon), \end{cases} \tag{5.41}$$

which is exactly the same as the jump conditions for $\mathbf{L}_N^{\beta_j}$. Thus $\mathbf{E}_N^{\beta_j}(z) = \mathbf{M}(z)$ $e^{\pm\frac{i\Omega_{j,N}}{2}\sigma_3}\mathbf{L}_N^{\beta_j}(z)^{-1}$ has no jumps in $D(\beta_j,\varepsilon)$. The only other possible singularity for $\mathbf{E}_N^{\beta_j}$ is at β_j, and this singularity is at most a fourth root singularity, thus removable. Thus, $\mathbf{E}_N^{\beta_j}$ is analytic in $D(\beta_j,\varepsilon)$, and $\mathbf{Q}_N$ has the prescribed jumps. We are left only to prove the matching condition (5.36). Using (4.99), one can check that, for z in each of the sectors of analyticity, $\Phi_{rv}(N^{2/3}\psi_{\beta_j}(z))$ satisfies the following asymptotics as $N\to\infty$:

$$\Phi_{rv}\big(N^{2/3}\psi_{\beta_j}(z)\big) = \frac{1}{2\sqrt{\pi}}N^{-\frac{1}{6}\sigma_3}\psi_{\beta_j}(z)^{-\frac{1}{4}\sigma_3}\left[\begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix} + \frac{\psi_{\beta_j}(z)^{-3/2}}{48N}\begin{pmatrix} -5 & 5i \\ -7 & -7i \end{pmatrix} + O(N^{-2})\right]e^{-\frac{2}{3}N\psi_{\beta_j}(z)^{3/2}\sigma_3}\,, \tag{5.42}$$

where we always take the principal branch of $\psi_{\beta_j}(z)^{3/2}$. As such, $\psi_{\beta_j}(z)^{3/2}$ is two-valued for $z\in(\beta_j-\varepsilon,\beta_j)$, so that

$$\left[\frac{2}{3}\psi_{\beta_j}(x)^{3/2}\right]_\pm = \mp\pi i\int_x^{\beta_j}\rho(t)dt. \tag{5.43}$$

Notice that, by (5.14),

$$2g_\pm(x) - V(x) = l \pm 2\pi i \int_x^{\beta_q} \rho(t)dt = l \pm 2\pi i \int_x^{\beta_j} \rho(t)dt \pm i\Omega_j \tag{5.44}$$

This implies that for $x \in (\beta_j - \varepsilon, \beta_j)$,

$$\begin{aligned} [2g_+(\beta_j) - V(\beta_j)] - [2g_+(x) - V(x)] &= -2\pi i \int_x^{\beta_j} \rho(t)dt\,, \\ [2g_-(\beta_j) - V(\beta_j)] - [2g_-(x) - V(x)] &= 2\pi i \int_x^{\beta_j} \rho(t)dt\,. \end{aligned}$$

Combining these equations with (5.43) gives

$$\left[\frac{2}{3}\psi_{\beta_j}(x)^{3/2}\right]_\pm = \frac{1}{2}\left[\big(2g_\pm(\beta_j) - V(\beta_j)\big) - \big(2g_\pm(x) - V(x)\big)\right]. \tag{5.45}$$

This equation can be extended into the upper and lower planes, respectively, giving

$$\frac{2}{3}\psi_{\beta_j}(z)^{3/2} = \frac{1}{2}\left[\big(2g_\pm(\beta_j) - V(\beta_j)\big) - \big(2g(z) - V(z)\big)\right] \quad \text{for } \pm \operatorname{Im} z > 0.$$

Since, by (5.44), $2g_\pm(\beta_j) - V(\beta_j) = l \pm i\Omega_j$, we get that

$$\frac{2}{3}\psi_{\beta_j}(z)^{3/2} = -g(z) + \frac{V(z)}{2} + \frac{l}{2} \pm \frac{i\Omega_j}{2} \tag{5.46}$$

for $\pm\operatorname{Im} z > 0$. Plugging (5.42) and (5.46) into (5.38), we get, as $N \to \infty$,

$$\begin{aligned} \mathbf{U}_N(z) &= \mathbf{M}(z) e^{\pm\frac{i\Omega_{j,N}}{2}} \mathbf{L}_N^{\beta_j}(z)^{-1} \frac{1}{2\sqrt{\pi}} N^{-\frac{1}{6}\sigma_3} \psi_{\beta_j}(z)^{-\frac{1}{4}\sigma_3} \left[\begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix}\right. \\ &\quad \left. + \frac{\psi_{\beta_j}(z)^{-3/2}}{48N} \begin{pmatrix} -5 & 5i \\ -7 & -7i \end{pmatrix} + O(N^{-2})\right] e^{N(g(z) - \frac{V(z)}{2} - \frac{l}{2} \mp \frac{i\Omega_j}{2})\sigma_3} e^{-N(g(z) - \frac{V(z)}{2} - \frac{l}{2})\sigma_3} \\ &= \mathbf{M}(z) \left[I + \frac{\psi_{\beta_j}(z)^{-3/2}}{48N} \begin{pmatrix} 1 & 6ie^{\pm i\Omega_{j,N}} \\ 6ie^{\mp i\Omega_{j,N}} & -1 \end{pmatrix} + O(N^{-2}) \right] \end{aligned}$$

for $\pm\operatorname{Im} z > 0$. Thus we have that $\mathbf{U}_N$ satisfies conditions (5.34), (5.35), and (5.36).

A similar construction gives the parametrix at the α_j for $j - 1 \in \mathcal{A}_v$. Namely, if we let

$$\psi_{\alpha_j}(z) = -\left\{\frac{3\pi}{2}\int_{\alpha_j}^z \rho(t)dt\right\}^{2/3},$$

then ψ_{α_j} is analytic throughout $D(\alpha_j, \varepsilon)$, real valued on the real line, and has negative derivative at α_j. Close to α_j, the jumps j_Q become

$$j_Q(z) = \begin{cases} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \text{for } z \in (\alpha_j - \varepsilon, \alpha_j) \\ \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} & \text{for } z \in (\alpha_j, \alpha_j + i\varepsilon) \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} & \text{for } z \in (\alpha_j, \alpha_j - i\varepsilon) \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{for } z \in (\alpha_j, \alpha_j + \varepsilon), \end{cases}$$

where orientation is taken left to right on horizontal contours, and up to down on vertical contours according to Figure 5.3. After the change of variables ψ_{α_j} (and a slight deformation of vertical contours), these jumps become the following jumps close to the origin:

$$j_Q\big(\psi_{\alpha_j}(z)\big) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{for } \psi_{\alpha_j}(z) \in (-\varepsilon, 0) \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} & \text{for } \psi_{\alpha_j}(z) \in (0, i\varepsilon) \\ \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} & \text{for } \psi_{\alpha_j}(z) \in (0, -i\varepsilon) \\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \text{for } \psi_{\alpha_j}(z) \in (0, \varepsilon), \end{cases}$$

where orientation is taken right to left on horizontal contours, and down to up on vertical contours. These jump conditions are satisfied by the function

$$\Phi_{lv}(z) = \Phi_{rv}(z) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then we can take

$$\mathbf{U}_N(z) = \mathbf{E}_N^{\alpha_j}(z)\Phi_{lv}\big(N^{2/3}\psi_{\alpha_j}(z)\big)e^{-N(g(z)-\frac{V(z)}{2}-\frac{l}{2})\sigma_3} \tag{5.47}$$

for $z \in D(\alpha_j, \varepsilon)$, where

$$\mathbf{E}_N^{\alpha_j}(z) = \mathbf{M}(z)e^{\pm\frac{i\Omega_{j-1,N}}{2}\sigma_3}\mathbf{L}_N^{\alpha_j}(z)^{-1} \quad \text{for} \quad \pm\operatorname{Im} z > 0,$$

$$\mathbf{L}_N^{\alpha_j}(z) = \frac{1}{2\sqrt{\pi}}\begin{pmatrix} N^{-1/6}\psi_{\alpha_j}^{-1/4}(z) & 0 \\ 0 & N^{1/6}\psi_{\alpha_j}^{1/4}(z)\end{pmatrix}\begin{pmatrix} 1 & -i \\ -1 & -i\end{pmatrix},$$

is an analytic prefactor. Similar to (5.42), we have that in each sector of analyticity, $\Phi_{lv}(N^{2/3}\psi_{\alpha_j}(z))$ satisfies, as $N \to \infty$,

$$\Phi_{lv}\big(N^{2/3}\psi_{\alpha_j}(z)\big) = \frac{1}{2\sqrt{\pi}}N^{-\frac{1}{6}\sigma_3}\psi_{\alpha_j}(z)^{-\frac{1}{4}\sigma_3}\left[\begin{pmatrix} 1 & -i \\ -1 & -i\end{pmatrix} + \frac{\psi_{\alpha_j}(z)^{-3/2}}{48N}\begin{pmatrix} -5 & -5i \\ -7 & 7i\end{pmatrix} \right.$$
$$\left. + O(N^{-2})\right]e^{-\frac{2}{3}N\psi_{\alpha_j}(z)^{3/2}\sigma_3}. \tag{5.48}$$

Once again, we have that, for $x \in (\alpha_j, \alpha_j + \varepsilon)$, $\psi_{\alpha_j}(x)^{3/2}$ takes limiting values from above and below, so that

$$\left[\frac{2}{3}\psi_{\alpha_j}(x)^{3/2}\right]_{\pm} = \pm\pi i\int_{\alpha_j}^{x}\rho(t)dt.$$

In analogue to (5.45), we have

$$\frac{2}{3}\psi_{\alpha_j}(z)^{3/2} = \frac{1}{2}\Big[\big(2g_{\pm}(\alpha_j) - V(\alpha_j)\big) - \big(2g(z) - V(z)\big)\Big] \quad \text{for } \pm\operatorname{Im} z > 0.$$

Since, by (5.44), $2g_{\pm}(\alpha_j) - V(\alpha_j) = l \pm \pi i$, we get that

$$\frac{2}{3}\psi_{\alpha_j}(z)^{3/2} = -g(z) + \frac{V(z)}{2} + \frac{l}{2} \pm \frac{i\Omega_{j-1}}{2} \quad \text{for} \quad \pm\operatorname{Im} z > 0. \tag{5.49}$$

Plugging (5.49) into (5.47) and (5.48) gives, as $N \to \infty$,

$$\begin{aligned}\mathbf{U}_N(z) &= \mathbf{M}(z)\mathbf{L}_N^{\alpha_j}(z)^{-1}\frac{1}{2\sqrt{\pi}}N^{-\frac{1}{6}\sigma_3}\psi_{\alpha_j}(z)^{-\frac{1}{4}\sigma_3}\left[\begin{pmatrix}1 & -i\\ -1 & -i\end{pmatrix}\right.\\ &\qquad\left.+\frac{\psi_{\alpha_j}(z)^{-3/2}}{48N}\begin{pmatrix}-5 & -5i\\ -7 & 7i\end{pmatrix}+O(N^{-2})\right]\\ &\qquad\times e^{N(g(z)-\frac{V(z)}{2}-\frac{l}{2}\mp\frac{i\Omega_{j-1}}{2})\sigma_3}e^{-N(g(z)-\frac{V(z)}{2}-\frac{l}{2})\sigma_3}\\ &= \mathbf{M}(z)\left[I+\frac{\psi_{\alpha_j}(z)^{-3/2}}{48N}\begin{pmatrix}1 & -6ie^{\frac{i\Omega_{j-1,N}}{2}\sigma_3}\\ -6ie^{-\frac{i\Omega_{j-1,N}}{2}\sigma_3} & -1\end{pmatrix}+O(N^{-2})\right].\end{aligned}$$

5.11 Parametrix at the band-saturated region end points

We now consider small disks $D(\alpha_j,\varepsilon)$ for $j-1\in\mathcal{A}_s$, and $D(\beta_j,\varepsilon)$ for $j\in\mathcal{A}_s$, centered at the endpoints of bands which are adjacent to a saturated region. Denote

$$\tilde{D}=\left(\bigcup_{j-1\in\mathcal{A}_s}D(\alpha_j,\varepsilon)\right)\bigcup\left(\bigcup_{j\in\mathcal{A}_s}D(\beta_j,\varepsilon)\right).$$

We will seek a local parametrix $\mathbf{U}_N(z)$ defined on $\tilde{D}$ such that

1. $\mathbf{U}_N(z)$ is analytic on $\tilde{D}\setminus\Sigma_S$.

2. $\mathbf{U}_{N+}(z)=\mathbf{U}_{N-}(z)j_S(z)$ for $z\in\tilde{D}\cap\Sigma_S$.

3. For $z\in\partial\tilde{D}$, we have the uniform estimate,

$$\mathbf{U}_N(z)=\mathbf{M}(z)\big(I+O(N^{-1})\big). \tag{5.50}$$

We first construct the parametrix near β_j for $j\in\mathcal{A}_s$. Let

$$\mathbf{U}_N(z)=\tilde{\mathbf{Q}}_N(z)e^{\mp iN\pi z\sigma_3}e^{-N(g(z)-\frac{V(z)}{2}-\frac{l}{2})\sigma_3}\quad\text{for}\quad\pm\operatorname{Im}z>0.$$

Then the jumps for $\tilde{\mathbf{Q}}_N$ are

$$j_{\tilde{Q}}(z) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{for} \quad z \in (\beta_j - \varepsilon, \beta_j) \\ \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} & \text{for} \quad z \in (\beta_j, \beta_j + \varepsilon) \\ \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} & \text{for} \quad z \in (\beta_j, \beta_j + i\varepsilon) \\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \text{for} \quad z \in (\beta_j, \beta_j - i\varepsilon)\,, \end{cases}$$

where orientation is taken from left to right on horizontal contours, and down to up on vertical contours according to Figure 5.3. We now take

$$\Phi_{rs}(z) = \begin{cases} \begin{pmatrix} y_2(z) & -y_0(z) \\ y_2'(z) & -y_0'(z) \end{pmatrix} & \text{for} \quad \arg z \in \left(0, \frac{\pi}{2}\right) \\ \begin{pmatrix} y_2(z) & y_1(z) \\ y_2'(z) & y_1'(z) \end{pmatrix} & \text{for} \quad \arg z \in \left(\frac{\pi}{2}, \pi\right) \\ \begin{pmatrix} y_1(z) & -y_2(z) \\ y_1'(z) & -y_2'(z) \end{pmatrix} & \text{for} \quad \arg z \in \left(-\pi, -\frac{\pi}{2}\right) \\ \begin{pmatrix} y_1(z) & y_0(z) \\ y_1'(z) & y_0'(z) \end{pmatrix} & \text{for} \quad \arg z \in \left(-\frac{\pi}{2}, 0\right). \end{cases}$$

Then $\Phi_{rs}(z)$ solves a RHP similar to that of $\tilde{\mathbf{Q}}_N$, but for jumps emanating from the origin rather than from β_j.

Once again,

$$\psi_{\beta_j}(z) = -\left\{ \frac{3\pi}{2} \int_z^{\beta_j} (1 - \rho(t))\, dt \right\}^{2/3}$$

extends to a conformal map from $D(\beta_j, \varepsilon)$ onto a convex neighborhood of the origin, with

$$\psi_{\beta_j}(\beta_j) = 0 \quad ; \quad \psi_{\beta_j}'(\beta_j) > 0,$$

Again, we can slightly deform the vertical pieces of the contour Σ_S close to β_j, so that

$$\psi_{\beta_j}\big\{D(\beta_j,\varepsilon)\cap\Sigma_S\big\} = (-\varepsilon,\varepsilon)\cup(-i\varepsilon,i\varepsilon).$$

We now take

$$\tilde{\mathbf{Q}}_N(z) = \mathbf{E}_N^{\beta_j}(z)\Phi_{rs}\big(N^{2/3}\psi_{\beta_j}(z)\big)\,, \tag{5.51}$$

where

$$\mathbf{E}_N^{\beta_j}(z) = \mathbf{M}(z)e^{\pm\frac{i\Omega_{j,N}}{2}\sigma_3}\mathbf{L}_N^{\beta_j}(z)^{-1} \quad \text{for} \quad \pm\operatorname{Im} z\geq 0,$$
$$\mathbf{L}_N^{\beta_j}(z) = \frac{1}{2\sqrt{\pi}}\begin{pmatrix} N^{-1/6}\psi_{\beta_j}^{-1/4}(z) & 0\\ 0 & N^{1/6}\psi_{\beta_j}^{1/4}(z)\end{pmatrix}\begin{pmatrix}1 & i\\ 1 & -i\end{pmatrix},$$

and we take the principal branch of $\psi_{\beta_j}^{1/4}$. The function $\Phi_{rs}(N^{2/3}\psi_{\beta_j}(z))$ has the jumps j_S. Similar to the prefactor $\mathbf{E}_N^{\beta_j}$ at band-void end-points, the prefactor $\mathbf{E}_N^{\beta_j}$ is analytic in $D(\beta_j,\varepsilon)$, thus does not change these jumps.

We now check that $\mathbf{U}_N$ satisfies the matching condition (5.50). The large N asymptotics of $\Phi_{rs}(N^{2/3}\psi_{\beta_j}(z))$ are given in the different regions of analyticity as follows:

$$\Phi_{rs}\big(N^{2/3}\psi_{\beta_j}(z)\big) = \frac{1}{2\sqrt{\pi}}N^{-\frac{1}{6}\sigma_3}\psi_{\beta_j}(z)^{-\frac{1}{4}\sigma_3}\Bigg[\pm\begin{pmatrix}-i & -1\\ -i & 1\end{pmatrix}\pm\frac{\psi_{\beta_j}(z)^{-3/2}}{48N}\begin{pmatrix}-5i & 5\\ 7i & 7\end{pmatrix}$$
$$+O(N^{-2})\Bigg]e^{\frac{2}{3}N\psi_{\beta_j}(z)^{3/2}\sigma_3} \qquad \text{for} \quad \pm\operatorname{Im} z>0, \tag{5.52}$$

where we always take the principal branch of $\psi_{\beta_j}(z)^{3/2}$. As such, $\psi_{\beta_j}(z)^{3/2}$ is two-valued for $x\in(\beta_j-\varepsilon,\beta_j)$, so that

$$\left[\frac{2}{3}\psi_{\beta_j}(x)^{3/2}\right]_{\pm} = \mp\pi i\int_x^{\beta_j}(1-\rho(t))\,dt = \mp\pi i(\beta_j-x)\pm\pi i\int_x^{\beta_j}\rho(t)dt. \tag{5.53}$$

From (5.14) we have that

$$2g_{\pm}(x)-V(x) = l\pm 2\pi i\int_x^{\beta_q}\rho(t)dt = l\pm 2\pi i\int_x^{\beta_j}\rho(t)dt\pm i\Omega_j\mp 2\pi i\beta_j \tag{5.54}$$

for $x\in(\beta_j-\varepsilon,\beta_j)$. These equations imply that

$$\big(2g_{\pm}(x)-V(x)\big)-\big(2g_{\pm}(\beta_j)-V(\beta_j)\big) = \pm 2\pi i\int_x^{\beta_j}\rho(t)dt.$$

We can therefore write (5.53) as

$$\left[\frac{2}{3}\psi_{\beta_j}(x)^{3/2}\right]_{\pm} = \mp\pi i(\beta_j - x) + \frac{1}{2}\left[\big(2g_{\pm}(x) - V(x)\big) - \big(2g_{\pm}(\beta_j) - V(\beta_j)\big)\right].$$

We can extend these equations into the upper and lower half-plane, respectively, obtaining

$$\frac{2}{3}\psi_{\beta_j}(z)^{3/2} = \mp\pi i(\beta_j - z) + \frac{1}{2}\left[\big(2g(z) - V(z)\big) - \big(2g_{\pm}(\beta_j) - V(\beta_j)\big)\right] \quad \text{for} \quad \pm\operatorname{Im} z > 0.$$

Using (5.54) at $x = \beta_j$, we can write

$$\frac{2}{3}\psi_{\beta_j}(z)^{3/2} = g(z) - \frac{V(z)}{2} - \frac{l}{2} \pm \pi i z \mp \frac{i(\Omega_{j,N} - \pi)}{2N} \quad \text{for} \quad \pm\operatorname{Im} z > 0. \tag{5.55}$$

Plugging (5.52) and (5.55) into (5.51) gives, as $N \to \infty$,

$$\begin{aligned}
\mathbf{U}_N(z) &= \mathbf{M}(z) e^{\frac{i\Omega_{j,N}}{2}\sigma_3} \mathbf{L}_N^{\beta_j}(z)^{-1} \frac{1}{2\sqrt{\pi}} N^{-\frac{1}{6}\sigma_3} \psi_{\beta_j}(z)^{-\frac{1}{4}\sigma_3} \\
&\quad \times \left[\pm \begin{pmatrix} -i & -1 \\ -i & 1 \end{pmatrix} \pm \frac{\psi_{\beta_j}(z)^{-3/2}}{48N} \begin{pmatrix} -5i & 5 \\ 7i & 7 \end{pmatrix} + O(N^{-2}) \right] \\
&\quad \times e^{N(g(z) - \frac{l}{2} - \frac{V(z)}{2})\sigma_3} e^{\pm iN\pi z\sigma_3} e^{\mp\frac{i\Omega_{j,N}}{2}\sigma_3} e^{\pm\frac{i\pi}{2}\sigma_3} e^{\mp iN\pi z\sigma_3} e^{-N(g(z) - \frac{V(z)}{2} - \frac{l}{2})\sigma_3} \\
&= \mathbf{M}(z) \left[I + \frac{\psi_{\beta_j}(z)^{-3/2}}{48N} \begin{pmatrix} -1 & -6ie^{\pm i\Omega_{j,N}} \\ -6ie^{\mp i\Omega_{j,N}} & 1 \end{pmatrix} + O(N^{-2}) \right] \\
&\qquad \text{for } \pm\operatorname{Im}(z) > 0.
\end{aligned}$$

We can make a similar construction near α_j for $j - 1 \in \mathcal{A}_s$. Let

$$\psi_{\alpha_j}(z) = -\left\{ \frac{3\pi}{2} \int_{\alpha_j}^{z} (1 - \rho(t))\, dt \right\}^{2/3}. \tag{5.56}$$

This function is analytic in $D(\alpha_j, \varepsilon)$ and has negative derivative at α_j, thus $\operatorname{Im} z$ and $\operatorname{Im} \psi_{\alpha_j}(z)$ have opposite signs for $z \in D(\alpha_j, \varepsilon)$. Then the jumps for $\tilde{\mathbf{Q}}_N$ are

$$j_{\tilde{Q}}(z) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{for} \quad z \in (\alpha_j, \alpha_j + \varepsilon) \\ \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} & \text{for} \quad z \in (\alpha_j - \varepsilon, \alpha_j) \\ \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} & \text{for} \quad z \in (\alpha_j, \alpha_j + i\varepsilon) \\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \text{for} \quad z \in (\alpha_j, \alpha_j - i\varepsilon), \end{cases}$$

where the contour is oriented from left to right on horizontal segments and up to down on vertical segments according to Figure 5.3. After a slight deformation of the vertical contours and the change of variables ψ_{α_j}, these jumps become the following jumps close to the origin:

$$j_{\tilde{Q}}(\psi_{\alpha_j}(z)) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{for} \quad \psi_{\alpha_j}(z) \in (-\varepsilon, 0) \\ \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} & \text{for} \quad \psi_{\alpha_j}(z) \in (0, \varepsilon) \\ \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} & \text{for} \quad \psi_{\alpha_j}(z) \in (-i\varepsilon, 0) \\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \text{for} \quad \psi_{\alpha_j}(z) \in (0, i\varepsilon), \end{cases}$$

where the contour is oriented from right to left on horizontal segments and down to up on vertical segments. These jump conditions are satisfied by the function

$$\Phi_{ls}(z) = \Phi_{rs}(z) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then we can take for $z \in D(\alpha_j, \varepsilon)$,

$$\begin{aligned}\mathbf{U}_N(z) =& \mathbf{M}(z) e^{\frac{\pm i\Omega_{j,N}}{2}\sigma_3} \mathbf{L}_N^{\alpha_j}(z)^{-1} \Phi_{ls}(N^{2/3}\psi_{\alpha_j}(z)) e^{\mp iN\pi z\sigma_3} e^{-N(g(z)-\frac{V(z)}{2}-\frac{l}{2})\sigma_3} \\ & \text{for } \pm \operatorname{Im} z > 0,\end{aligned} \tag{5.57}$$

where

$$\mathbf{L}_N^{\alpha_j}(z) = \frac{1}{2\sqrt{\pi}} \begin{pmatrix} N^{-1/6}\psi_{\alpha_j}^{-1/4}(z) & 0 \\ 0 & N^{1/6}\psi_{\alpha_j}^{1/4}(z) \end{pmatrix} \begin{pmatrix} -1 & i \\ -1 & -i \end{pmatrix}.$$

We once again have, as $N \to \infty$,

$$\begin{aligned}\Phi_{ls}\big(N^{2/3}\psi_{\alpha_j}(z)\big) =& \frac{1}{2\sqrt{\pi}} N^{-\frac{1}{6}\sigma_3} \psi_{\alpha_j}(z)^{-\frac{1}{4}\sigma_3} \Bigg[\pm \begin{pmatrix} -i & 1 \\ -i & -1 \end{pmatrix} \pm \frac{\psi_{\alpha_j}(z)^{-3/2}}{48N} \begin{pmatrix} -5i & -5 \\ 7i & -7 \end{pmatrix} \\ & + O(N^{-2}) \Bigg] e^{\frac{2}{3}N\psi_{\alpha_j}(z)^{3/2}\sigma_3} \quad \text{for } \pm \operatorname{Im} \psi_{\alpha_j}(z) > 0 \text{ (so } \mp \operatorname{Im} z > 0),\end{aligned} \tag{5.58}$$

and for $z \in D(\alpha_j, \varepsilon)$,

$$\frac{2}{3}\psi_{\alpha_j}^{3/2}(z) = \pm i\pi z + g(z) - \frac{V(z)}{2} - \frac{l}{2} \mp \frac{i(\Omega_{j,N} - \pi)}{2N} \quad \text{for} \quad \pm \operatorname{Im} z > 0. \tag{5.59}$$

Combining (5.57), (5.58), and (5.59) gives, as $N \to \infty$,

$$\begin{aligned}\mathbf{U}_N(z) =& \mathbf{M}(z) e^{\frac{\pm i\Omega_{j,N}}{2}\sigma_3} \mathbf{L}_N^{\alpha_j}(z)^{-1} \frac{1}{2\sqrt{\pi}} N^{-\frac{1}{6}\sigma_3} \psi_{\alpha_j}(z)^{-\frac{1}{4}\sigma_3} \\ & \times \left[\pm \begin{pmatrix} i & -1 \\ i & 1 \end{pmatrix} \pm \frac{\psi_{\alpha_j}(z)^{-3/2}}{48N} \begin{pmatrix} 5i & 5 \\ -7i & 7 \end{pmatrix} + O(N^{-2}) \right] \\ & \times e^{\pm iN\pi z\sigma_3} e^{N(g(z)-\frac{V(z)}{2}-\frac{l}{2})\sigma_3} e^{\mp\frac{i\Omega_{j,N}}{2}\sigma_3} e^{\pm\frac{i\pi}{2}\sigma_3} e^{\mp iN\pi z\sigma_3} e^{-N(g(z)-\frac{V(z)}{2}-\frac{l}{2})\sigma_3} \\ =& \mathbf{M}(z) e^{\frac{\pm i\Omega_{j,N}}{2}\sigma_3} \left[I + \frac{\psi_{\alpha_j}(z)^{-3/2}}{48N} \begin{pmatrix} -1 & 6i \\ 6i & 1 \end{pmatrix} + O(N^{-2}) \right] e^{\mp\frac{i\Omega_{j,N}}{2}\sigma_3} \\ =& \mathbf{M}(z) \left[I + \frac{\psi_{\alpha_j}(z)^{-3/2}}{48N} \begin{pmatrix} -1 & 6ie^{\pm i\Omega_{j,N}} \\ 6ie^{\mp i\Omega_{j,N}} & 1 \end{pmatrix} + O(N^{-2}) \right] \\ & \text{for } \pm \operatorname{Im} z > 0.\end{aligned}$$

5.12 The third and final transformation of the RHP

We now consider the contour Σ_X, which consists of the circles $\partial D(\alpha_j, \varepsilon)$, and $\partial D(\beta_j, \varepsilon)$, for $j = 1, \ldots q$, all oriented counterclockwise, together with the parts of

$\Sigma_S \setminus \big(\bigcup_j [\alpha_j, \beta_j]\big)$ which lie outside of the disks $D(\alpha, \varepsilon),\ D(\alpha', \varepsilon),\ D(\beta', \varepsilon)$, and $D(\beta, \varepsilon)$, see Figure 5.4.

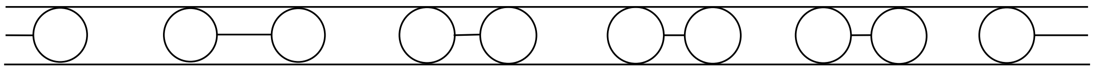

Fig. 5.4. The contour Σ_X arising from the hypothetical equilibrium measure shown in Fig. 5.1.

We let

$$\mathbf{X}_N(z) = \begin{cases} \mathbf{S}_N(z)\mathbf{M}(z)^{-1} & \text{for } z \text{ outside the disks } D(\alpha_j, \varepsilon),\ D(\beta_j, \varepsilon) \\ \mathbf{S}_N(z)\mathbf{U}_N(z)^{-1} & \text{for } z \text{ inside the disks } D(\alpha_j, \varepsilon),\ D(\beta_j, \varepsilon). \end{cases} \tag{5.60}$$

Then $\mathbf{X}_N(z)$ solves the following RHP:

1. $\mathbf{X}_N(z)$ is analytic on $\mathbb{C} \setminus \Sigma_X$.

2. $\mathbf{X}_N(z)$ has the jump properties

$$\mathbf{X}_{N+}(x) = \mathbf{X}_{N-}(z) j_X(z)$$

where

$$j_X(z) = \begin{cases} \mathbf{M}(z)\mathbf{U}_N(z)^{-1} & \text{for } z \text{ on the circles} \\ \mathbf{M}(z) j_S \mathbf{M}(z)^{-1} & \text{otherwise.} \end{cases}$$

3. As $z \to \infty$,

$$\mathbf{X}_N(z) \sim I + \frac{\mathbf{X}_1}{z} + \frac{\mathbf{X}_2}{z^2} + \dots$$

Additionally, we have that $j_X(z)$ is uniformly close to the identity in the following sense:

$$j_X(z) = \begin{cases} I + O(N^{-1}) & \text{uniformly on the circles} \\ I + O(e^{-C(z)N}) & \text{on the rest of } \Sigma_X, \end{cases} \tag{5.61}$$

where $C(z)$ is a positive, continuous function satisfying (5.2). If we set

$$j_X^0(z) = j_X(z) - I,$$

then (5.61) becomes

$$j^0_X(z) = \begin{cases} O(N^{-1}) & \text{uniformly on the circles} \\ O(e^{-C(z)N}) & \text{on the rest of } \Sigma_X. \end{cases}$$

The solution to the RHP for $\mathbf{X}_N$ follows from Lemma 4.16.1 and the arguments following it, which give that

$$\mathbf{X}_N(z) = I + \sum_{k=1}^{\infty} \mathbf{X}_{N,k}(z)$$

where

$$\mathbf{X}_{N,k}(z) = -\frac{1}{2\pi i} \int_{\Sigma_X} \frac{v_{k-1}(u) j^0_X(u)}{z-u} du.$$

In particular, this implies that

$$\mathbf{X}_N \sim I + O\left(\frac{1}{N(|z|+1)}\right) \quad \text{as} \quad N \to \infty \tag{5.62}$$

uniformly for $z \in \mathbb{C} \setminus \Sigma_X$.

5.13 Proof of theorems 5.4.1-5.4.6

The transformations (5.28), (5.29), (5.31), (5.60) give that, for z bounded away from the real line,

$$\mathbf{P}_N(z) = \mathbf{K}_N^{-1} e^{\frac{Nl}{2}\sigma_3} \mathbf{X}_N(z) \mathbf{M}(z) e^{N(g(z)-\frac{l}{2})\sigma_3} \mathbf{K}_N, \tag{5.63}$$

and for z close to the real line but bounded away from the support of the equilibrium measure,

$$\mathbf{P}_N(z) = \mathbf{K}_N^{-1} e^{\frac{Nl}{2}\sigma_3} \mathbf{X}_N(z) \mathbf{M}(z) e^{N(g(z)-\frac{l}{2})\sigma_3} \mathbf{K}_N \mathbf{D}^u_{\pm}(z)^{-1} \quad \text{for} \quad \pm \operatorname{Im} z \geq 0. \tag{5.64}$$

Expanding (5.63) or (5.64), we get that

$$P_N(z) = [\mathbf{P}_N(z)]_{11} = e^{Ng(z)} \left([\mathbf{M}]_{11}[\mathbf{X}]_{11} + [\mathbf{M}]_{21}[\mathbf{X}]_{12}\right)$$

which, along with (5.62), proves Theorem 5.4.2.

The proof of Theorem 5.4.1 requires only the formulae (5.23), (5.25), and (5.26), and a straightforward large z expansion of equation (5.63).

Similar to (5.63), we have that, for any interval J which is contained in and bounded away from the endpoints of a band, in some neighborhood of J, we have

$$\mathbf{P}_N(z) = \begin{cases} \mathbf{K}_N^{-1} e^{\frac{Nl}{2}\sigma_3} \mathbf{X}_N(z)\mathbf{M}(z) j_+(z) e^{N(g(z)-\frac{l}{2})\sigma_3} \mathbf{K}_N \mathbf{D}_+^u(z)^{-1} & \text{for} \quad \operatorname{Im} z \geq 0 \\ \mathbf{K}_N^{-1} e^{\frac{Nl}{2}\sigma_3} \mathbf{X}_N(z)\mathbf{M}(z) j_-^{-1}(z) e^{N(g(z)-\frac{l}{2})\sigma_3} \mathbf{K}_N \mathbf{D}_-^u(z)^{-1} & \text{for} \quad \operatorname{Im} z \leq 0, \end{cases}$$

Expanding the left side of this equation for $\operatorname{Im} z \geq 0$, utilizing (5.14), and taking limits as z approaches the real line, we get that

$$P_N(x) = [\mathbf{P}_N(x)]_{11} = e^{\frac{N}{2}(V(x)+l)} \left(e^{iN\pi\phi(x)} [\mathbf{M}_{11}]_+(x) + e^{-iN\pi\phi(x)} [\mathbf{M}_{12}]_+(x) + O(N^{-1}) \right), \tag{5.65}$$

where $\phi(x)$ is as defined in (5.20), and the + subscript indicates the limiting value from the upper half plane. Notice that $[\mathbf{M}_{12}]_+ = [\mathbf{M}_{11}]_-$ in this region, and that $\mathbf{M}_{11}(\overline{z}) = \overline{\mathbf{M}_{11}(z)}$. This implies that $[\mathbf{M}_{12}]_+(x) = \overline{[\mathbf{M}_{11}]_+(x)}$, and thus we can write (5.65) as

$$P_N(x) = e^{\frac{N}{2}(V(x)+l)} \left(e^{iN\pi\phi(x)} [\mathbf{M}_{11}]_+(x) + \overline{e^{iN\pi\phi(x)} [\mathbf{M}_{11}]_+(x)} + O(N^{-1}) \right),$$

which proves Theorem 5.4.3.

For any interval J which is contained in and bounded away from the endpoints of a saturated region, in some neighborhood of J, we have

$$\mathbf{P}_N(z) = \mathbf{K}_N^{-1} e^{\frac{Nl}{2}\sigma_3} \mathbf{X}_N(z)\mathbf{M}(z)\mathbf{A}_\pm^{-1}(z) e^{N(g(z)-\frac{l}{2})\sigma_3} \mathbf{K}_N \mathbf{D}_\pm^l(z)^{-1} \quad \text{for} \quad \pm \operatorname{Im} z > 0. \tag{5.66}$$

Notice that in this region, we can write

$$g_\pm(x) = L(x) \pm \frac{i\Omega_j}{2} \mp i\pi x \,,$$

where $L(x)$ is defined in (5.21). Notice also that $2g_\pm(x) - V(x) - l$ has positive real part. Expanding (5.66) for $\operatorname{Im} z > 0$ and taking the limit as z approaches the real line gives

$$\begin{aligned}\mathbf{P}_N(x)_{11} &= e^{Ng_+(x)}\big[(1 - e^{2\pi i N x})(\mathbf{M}_{11}\mathbf{X}_{11} + \mathbf{M}_{21}\mathbf{X}_{12}) \\ &\qquad\qquad + e^{-N(2g_+(x)-V(x)-l)}(\mathbf{M}_{12}\mathbf{X}_{11} + \mathbf{M}_{22}\mathbf{X}_{12})\big] \\ &= e^{Ng_+(x)}\big[(1 - e^{2\pi i N x})(\mathbf{M}_{11}\mathbf{X}_{11} + O(N^{-1})) + O(e^{-N\delta})\big] \\ &= e^{NL(x)}\big[-2i\sin(\pi N x)e^{\frac{iN\Omega_j}{2}}[\mathbf{M}_{11}]_+(x)(1 + O(N^{-1})) + O(e^{-N\delta})\big],\end{aligned}$$

which proves Theorem 5.4.4.

Similarly, at the turning points α_j and β_j, explicit formulae can be written for $\mathbf{P}_N$ in terms of explicit transformations in each sector of analyticity of the local parametrix. From these formulae and the properties of the g-function, Theorems 5.4.5 and 5.4.6 are almost immediate, with Theorem 5.4.6 also requiring the identities (see, e.g. [40])

$$\begin{aligned}y_1(z) &= -\frac{1}{2}\big(\mathrm{Ai}(z) - i\mathrm{Bi}(z)\big), \\ y_2(z) &= -\frac{1}{2}\big(\mathrm{Ai}(z) + i\mathrm{Bi}(z)\big).\end{aligned}$$

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APPENDICES

A. PROOF OF LEMMA (1.3.1)

We have

$$\tau_n = \det\left(\phi^{(i+j-2)}(t)\right)_{1\leq i,j\leq n}$$

where

$$\phi^{(k)}(t) = \int_{\mathbb{R}} (-cx)^k e^{-ctx} d\mu(x). \tag{A.1}$$

From equation (A.1) and multi-linearity of the determinant function, we have that τ_n is equal to

$$\int \cdots \int_{\mathbb{R}^n} \det \begin{pmatrix} e^{-ctx_1} & (-cx_1)e^{-ctx_1} & \cdots & (-cx_1)^{n-1}e^{-ctx_1} \\ (-cx_2)e^{-ctx_2} & (-cx_2)^2 e^{-ctx_2} & \cdots & (-cx_2)^n e^{-ctx_2} \\ (-cx_3)^2 e^{-ctx_3} & (-cx_3)^3 e^{-ctx_3} & \cdots & (-cx_3)^{n+1} e^{-ctx_3} \\ \vdots & \vdots & \ddots & \vdots \\ (-cx_n)^{n-1} e^{-ctx_n} & (-cx_n)^n e^{-ctx_n} & \cdots & (-cx_n)^{2n-2} e^{-ctx_n} \end{pmatrix} \prod_{k=1}^{n} d\mu(x_k)$$

$$= \int \cdots \int_{\mathbb{R}^n} \Delta(-cx) \prod_{k=1}^{n} (-cx_k)^{k-1} \prod_{k=1}^{n} e^{-ctx_k} d\mu(x_k)$$

$$= c^{n^2-n} \int \cdots \int_{\mathbb{R}^n} \Delta(x) \prod_{k=1}^{n} (x_k)^{k-1} \prod_{k=1}^{n} e^{-tx_k} d\mu(x_k)$$

where $\Delta(x) = \prod_{i<j}(x_i - x_j)$ is the Vandermonde determinant. Note that, up to sign, this expression for τ_n is invariant with respect to any permutation of x_k *'s*. So, multiplying by their signs and then summing over all permutations, we get

$$n!\tau_n = c^{n^2-n} \int \cdots \int_{\mathbb{R}^n} \Delta(x) \sum_{\pi \in S_n} (-1)^{\pi} \prod_{k=1}^{n} (x_k)^{\pi(k)-1} \prod_{k=1}^{n} e^{-ctx_k} d\mu(x_k),$$

thus

$$\tau_n = \frac{c^{n^2-n}}{n!} \int \cdots \int_{\mathbb{R}^n} \Delta(x)^2 \prod_{k=1}^{n} e^{-ctx_k} d\mu(x_k), \tag{A.2}$$

which is the eigenvalue partition function for an ensemble of random matrices. To express this function in terms of orthogonal polynomials, notice that multilinearity of the determinant function and the form of the Vandermonde matrix allows us to replace $\Delta(x)$ with

$$\det \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ P_1(x_1) & P_1(x_2) & P_1(x_3) & \cdots & P_1(x_n) \\ P_2(x_1) & P_2(x_2) & P_2(x_3) & \cdots & P_2(x_n) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ P_{n-1}(x_1) & P_{n-1}(x_2) & P_{n-1}(x_3) & \cdots & P_{n-1}(x_n) \end{pmatrix},$$

where $\{P_j(x)\}_{j=0}^{\infty}$ is the system of monic polynomials orthogonal with respect to the measure $e^{-ctx}d\mu(x)$. Then (A.2) becomes

$$\tau_n = \frac{c^{n^2-n}}{n!} \int \cdots \int_{\mathbb{R}^n} \left(\sum_{\pi \in S_n} (-1)^{\pi} \prod_{k=1}^{n} P_{\pi(k)-1}(x_k) \right)^2 \prod_{k=1}^{n} e^{-ctx_k} d\mu(x_k).$$

The orthogonality condition ensures that, after integrating, only diagonal terms are non-zero, so we get

$$\tau_n = \frac{c^{n^2-n}}{n!} \int \cdots \int_{\mathbb{R}^n} \left(\sum_{\pi \in S_n} \prod_{k=1}^{n} P^2_{\pi(k)-1}(x_k) \right) \prod_{k=1}^{n} e^{-ctx_k} d\mu(x_k) = c^{n^2-n} \prod_{k=0}^{n-1} h_k.$$

B. PROOF OF FORMULA (3.39)

We use the notations and results from the work [48] of Vanlessen. The essential difference with [48] is that we consider not a fixed but a shrinking neighborhood of the origin,

$$\tilde{U}_{\delta,k} = \left\{ z \in \mathbb{C} : \ |z| \le \frac{\delta}{k} \right\},$$

where $\delta > 0$ is small enough so that the function $V_k(x)$ is analytic in $\tilde{U}_{\delta,k}$, see (3.17). As in [48], we consider a sequence of transformations of the Riemann-Hilbert problem for orthogonal polynomials, and in the end we arrive at the following Riemann-Hilbert problem on a 2×2 matrix-valued function $R(z)$:

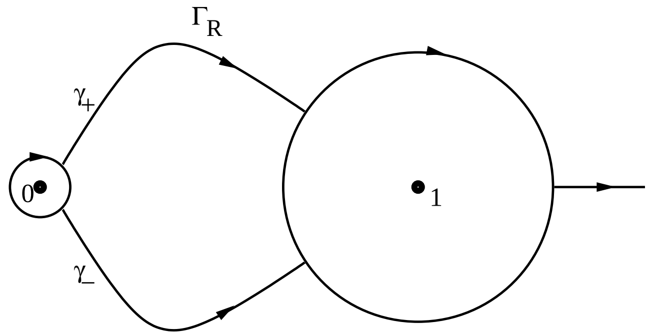

Fig. B.1. The contour Γ_R.

1. $R(z)$ is analytic on $\mathbb{C} \setminus \Gamma_R$, where Γ_R is the contour shown on Fig. 6, and it has limits, $R_+(z)$ and $R_-(z)$, on Γ_R, as z approaches a point on Γ_R from the left and from the right of the contour, with respect to the orientation indicated on Fig. B-1.
2. On Γ_R, $R(z)$ satisfies the jump condition, $R_+(z) = R_-(z) v_R(z)$, where $v_R(z)$ is an explicit matrix-valued function.
3. $R(z) \simeq I + \frac{R_1}{z} + \ldots$ as $z \to \infty$, where I is the identity matrix.

The contour Γ_R consists of the circle $\partial\tilde{U}_{\delta,k}$, the circle ∂U_δ, where

$$U_\delta = \{ z \in \mathbb{C} : \ |z - 1| \le \delta \},$$

the boundaries of the lenses, $\gamma_\pm$, and the semi-infinite interval $[1+\delta,\infty)$. The jump matrix v_R on $\partial\tilde{U}_{\delta,k}$ has the following asymptotics as $k\to\infty$:

$$v_R(z) \simeq I + \sum_{n=1}^{\infty} \tilde{\Delta}_n(z) k^{-n}, \tag{B.1}$$

see formulae (3.105) and (3.98) in [48]. This asymptotics holds under the condition that $k^2 z \to \infty$. Under this condition, for any $N \geq 1$ there exists a constant $C_N > 0$ such that

$$\left| v_R(z) - I - \sum_{n=1}^{N} \tilde{\Delta}_n(z) k^{-n} \right| \leq \frac{C_N}{(k^2|z|)^{\frac{N}{2}} |z|^2}.$$

The condition $k^2 z \to \infty$ is valid for $z \in \partial\tilde{U}_{\delta,k}$, and in this case the last estimate gives that

$$\sup_{z\in\partial\tilde{U}_{\delta,k}} \left| v_R(z) - I - \sum_{n=1}^{N} \tilde{\Delta}_n(z) k^{-n} \right| \leq \frac{\tilde{C}_N}{k^{\frac{N}{2}-2}}, \qquad \tilde{C}_N = \frac{C_N}{\delta^{\left[\frac{N}{2}\right]+2}}. \tag{B.2}$$

The coefficients $\tilde{\Delta}_n(z)$ in (B.1) are given by the following formula:

$$\tilde{\Delta}_n(z) = \frac{1}{\tilde{\phi}_k(z)^{n/2}} P^{(\infty)}(z)(-z)^{\frac{1}{2}\sigma_3} A_n \, (-z)^{-\frac{1}{2}\sigma_3} P^{(\infty)}(z)^{-1}, \tag{B.3}$$

where

$$P^{(\infty)}(z) = 2^{-\sigma_3} \begin{pmatrix} \frac{a(z)+a(z)^{-1}}{2} & \frac{a(z)-a(z)^{-1}}{2i} \\ \frac{a(z)-a(z)^{-1}}{-2i} & \frac{a(z)+a(z)^{-1}}{2} \end{pmatrix} \left(\frac{2z-1+2\sqrt{z(z-1)}}{z} \right)^{\sigma_3/2}, \tag{B.4}$$

$$a(z) = \left(\frac{z-1}{z}\right)^{1/4}, \qquad \tilde{\phi}_k(z) = -\left[\frac{1}{4} \int_0^z \sqrt{\frac{1-s}{s}}\, q_k(s)\, ds \right]^2, \tag{B.5}$$

$$A_n = \frac{\prod_{j=1}^{n}[4-(2j-1)^2]}{16^n n!} \begin{pmatrix} \frac{(-1)^n}{4n}(3+2n) & (n-\frac{1}{2})i \\ (-1)^{n+1}(n-\frac{1}{2})i & \frac{1}{4n}(3+2n) \end{pmatrix}. \tag{B.6}$$

The function $\tilde{\phi}_k(z)$ is analytic in $\tilde{U}_{\delta,k}$. From (3.22), (3.15), and (3.27) we obtain that as $k\to\infty$,

$$\sup_{z\in\tilde{U}_{\delta,k}} |q_k(z) - 4| = O(k^{-1/2}).$$

By (B.5) this implies that

$$\sup_{z\in\tilde{U}_{\delta,k}} \left| \frac{\tilde{\phi}_k(z)}{\phi(z)} - 1 \right| = O(k^{-1/2}), \qquad \phi(z) = -\left(\int_0^z \sqrt{\frac{1-s}{s}}\, ds \right)^2 = -4z + \frac{4z^2}{3} + \dots$$

The function $\tilde{\Delta}_n(z)$ is meromorphic in $\tilde{U}_{\delta,k}$ with the only possible pole at the origin of the order at most $\left[\frac{n+1}{2}\right]$, see [48]. This result, combined with explicit formula (B.3), implies that there exists $c_n > 0$ such that

$$\sup_{z\in\partial\tilde{U}_{\delta,k}} \left|\tilde{\Delta}_n(z)k^{-n}\right| \le c_n k^{-n+\left[\frac{n+1}{2}\right]}.$$

This, in turn, allows us to improve estimate (B.2) as follows: for any $N \ge 1$ there exists $\tilde{c}_N > 0$ such that

$$\sup_{z\in\partial\tilde{U}_{\delta,k}} \left|v_R(z) - I - \sum_{n=1}^{N} \tilde{\Delta}_n(z)k^{-n}\right| \le \tilde{c}_N k^{-N+\left[\frac{N}{2}\right]}.$$

When $N = 1$, this gives that

$$\sup_{z\in\partial\tilde{U}_{\delta,k}} \left|v_R(z) - I - \frac{\tilde{\Delta}_1(z)}{k}\right| = O(k^{-1}). \tag{B.7}$$

The function $\tilde{\Delta}_1(z)$ has a simple pole at 0 and its residue is equal to

$$B_k = \frac{3}{16q_k(0)}\, 2^{-\sigma_3} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} 2^{\sigma_3},$$

see equation (4.11) in [48]. The function $\tilde{\Delta}_1(z) - \frac{B_k}{z}$ is regular at $z = 0$ and from explicit formula (B.3) we obtain that as $k \to \infty$,

$$\sup_{z\in\partial\tilde{U}_{\delta,k}} \left|\tilde{\Delta}_1(z) - \frac{B_k}{z}\right| = O(1),$$

hence from (B.7) we obtain that

$$\sup_{z\in\partial\tilde{U}_{\delta,k}} \left|v_R(z) - I - \frac{B_k}{kz}\right| = O(k^{-1}). \tag{B.8}$$

The problem here is that $v_R(z)$ is not close to I on $\partial\tilde{U}_{\delta,k}$, but we will overcome this obstacle by a transformation of the Riemann-Hilbert problem for $R(z)$.

Observe that

$$\mathrm{Tr} B_k = 0, \qquad \det B_k = 0,$$

hence

$$\det\left(I + \frac{B_k}{kz}\right) = 1,$$

hence the matrix $I + \frac{B_k}{kz}$ is invertible for any $z \neq 0$. Let us make the substitution,

$$R(z) = \begin{cases} \tilde{R}(z), & z \in \tilde{U}_{\delta,k}, \\ \tilde{R}(z)\left(I + \dfrac{B_k}{kz}\right), & z \notin \tilde{U}_{\delta,k}. \end{cases} \tag{B.9}$$

Then $\tilde{R}(z)$ solves the Riemann-Hilbert problem similar to the one for $R(z)$, with the jump matrix $\tilde{v}_R(z)$ such that

$$\tilde{v}_R(z) = v_R(z)\left(I + \frac{B_k}{kz}\right)^{-1}, \qquad z \in \partial\tilde{U}_{\delta,k}$$

and

$$\tilde{v}_R(z) = \left(I + \frac{B_k}{kz}\right) v_R(z) \left(I + \frac{B_k}{kz}\right)^{-1}, \qquad z \in \Gamma_R \setminus \partial\tilde{U}_{\delta,k}.$$

From (B.8) we obtain that

$$\sup_{z \in \partial\tilde{U}_{\delta,k}} |\tilde{v}_R(z) - I| = O(k^{-1}).$$

Also, since the equilibrium density function diverges as $z^{-1/2}$ at the origin, we obtain that $v_R(z)$ is sub-exponentially small on the boundary of lenses,

$$\sup_{z \in \gamma_+ \cup \gamma_-} |v_R(z) - I| = O(e^{-c\sqrt{k}}), \qquad c > 0.$$

This implies that $\tilde{v}_R(z)$ satisfies a similar estimate,

$$\sup_{z \in \gamma_+ \cup \gamma_-} |\tilde{v}_R(z) - I| = O(e^{-c\sqrt{k}}), \qquad c > 0.$$

In addition,

$$\sup_{z \in \partial U_\delta} |\tilde{v}_R(z) - I| = O(k^{-1}),$$

and

$$|\tilde{v}_R(z) - I| = O(e^{-ckz}), \quad z \geq 1; \qquad c > 0.$$

These estimates of smallness of $(\tilde{v}_R(z) - I)$ on Γ_R enable us to solve the Riemann-Hilbert problem for $\tilde{R}(z)$ by a series of perturbation theory. The fact that the radius of $\tilde{U}_{\delta,k}$, $r = \frac{\delta}{k}$, is tending to zero does not cause a problem, see the appendix to the work [6] of Bleher and Kuijlaars.

The rest of the proof of formula (3.39) goes along the lines of [48]. Namely, by formula (4.17) in [48],

$$h_{k,\alpha} = \frac{\pi}{8}\,\beta_k^{2k+2}e^{kl_k}\left[1-16i(R_1)_{12}+O(k^{-2})\right]. \tag{B.10}$$

By (B.9),

$$(R_1)_{12} = (\tilde{R}_1)_{12} + \frac{(B_k)_{12}}{k} + O(k^{-2}) = (\tilde{R}_1)_{12} + \frac{3i}{64q_k(0)k} + O(k^{-2})$$

By applying formula (4.11) in [48] to $\tilde{R}_1$, we obtain that

$$(\tilde{R}_1)_{12} = -\frac{q_k'(1)i}{64q_k(1)^2k} + \frac{47i}{192q_k(1)k} + O(k^{-2}).$$

Observe that the first term in formula (4.11) in [48] is missing in this case, because the function $\left[\tilde{\Delta}_1(z) - \frac{B_k}{z}\right]$ is regular at $z=0$. From the last two formulae we obtain that

$$-16i(R_1)_{12} = \left[\frac{3}{4q_k(0)} - \frac{q_k'(1)}{4q_k(1)^2} + \frac{47}{12q_k(1)}\right]\frac{1}{k} + O(k^{-2}).$$

By substituting this into (B.10) we obtain (3.39).

C. PROOF OF FORMULA (4.108)

From, (4.107), (4.76), (4.77), and (4.30), we have that the jump $j_{\tilde{Q}}$ on $(\alpha'-\varepsilon,\alpha')$ is given by

$$\begin{aligned} j_{\tilde{Q}} &= e^{\frac{in\pi z}{2\gamma}\sigma_3} e^{-n(g_-(z)-\frac{V(z)}{2}-\frac{l}{2})\sigma_3} j_S e^{n(g_+(z)-\frac{V(z)}{2}-\frac{l}{2})\sigma_3} e^{\frac{in\pi z}{2\gamma}\sigma_3} \\ &= e^{\frac{in\pi z}{2\gamma}\sigma_3} e^{-n(g_-(z)-\frac{V(z)}{2}-\frac{l}{2})\sigma_3} j_-^{-1} j_T j_+^{-1} e^{n(g_+(z)-\frac{V(z)}{2}-\frac{l}{2})\sigma_3} e^{\frac{in\pi z}{2\gamma}\sigma_3} \\ &= e^{\frac{in\pi z}{2\gamma}\sigma_3} e^{-n(g_-(z)-\frac{V(z)}{2}-\frac{l}{2})\sigma_3} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} e^{n(g_+(z)-\frac{V(z)}{2}-\frac{l}{2})\sigma_3} e^{\frac{in\pi z}{2\gamma}\sigma_3} \\ &= e^{\frac{in\pi z}{2\gamma}\sigma_3} \begin{pmatrix} 0 & e^{-n(g_+(z)+g_-(z)-V(z)-l)} \\ -e^{n(g_+(z)+g_-(z)-V(z)-l)} & 0 \end{pmatrix} e^{\frac{in\pi z}{2\gamma}\sigma_3} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \end{aligned}$$

From, (4.107), (4.77), (4.73), and (4.72), we have that the jump $j_{\tilde{Q}}$ on $(\alpha',\alpha'+\varepsilon)$ is given by

$$\begin{aligned} j_{\tilde{Q}} &= e^{\frac{in\pi z}{2\gamma}\sigma_3} e^{-n(g_-(z)-\frac{V(z)}{2}-\frac{l}{2})\sigma_3} j_S e^{n(g_+(z)-\frac{V(z)}{2}-\frac{l}{2})\sigma_3} e^{\frac{in\pi z}{2\gamma}\sigma_3} \\ &= e^{\frac{in\pi z}{2\gamma}\sigma_3} e^{-n(g_-(z)-\frac{V(z)}{2}-\frac{l}{2})\sigma_3} \begin{pmatrix} \frac{\gamma}{n\pi i} e^{\frac{in\pi z}{2\gamma}} & 0 \\ 0 & \frac{n\pi i}{\gamma} e^{-\frac{in\pi z}{2\gamma}} \end{pmatrix}^{-1} j_T \\ &\quad \times \begin{pmatrix} -\frac{\gamma}{n\pi i} e^{-\frac{in\pi z}{2\gamma}} & 0 \\ 0 & -\frac{n\pi i}{\gamma} e^{\frac{in\pi z}{2\gamma}} \end{pmatrix} e^{n(g_+(z)-\frac{V(z)}{2}-\frac{l}{2})\sigma_3} e^{\frac{in\pi z}{2\gamma}\sigma_3} \\ &= e^{-n(g_-(z)-\frac{V(z)}{2}-\frac{l}{2})\sigma_3} \begin{pmatrix} \frac{n\pi i}{\gamma} & 0 \\ 0 & \frac{\gamma}{n\pi i} \end{pmatrix} e^{n(g_-(z)-\frac{l}{2})\sigma_3} j_R e^{-n(g_+(z)-\frac{l}{2})\sigma_3} \begin{pmatrix} -\frac{\gamma}{n\pi i} & 0 \\ 0 & -\frac{n\pi i}{\gamma} \end{pmatrix} \\ &\quad \times e^{n(g_+(z)-\frac{V(z)}{2}-\frac{l}{2})\sigma_3} \\ &= e^{n\frac{V(z)}{2}\sigma_3} \begin{pmatrix} \frac{n\pi i}{\gamma} & 0 \\ 0 & \frac{\gamma}{n\pi i} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ (\frac{n\pi i}{\gamma})^2 e^{nV(z)} & 1 \end{pmatrix} \begin{pmatrix} -\frac{\gamma}{n\pi i} & 0 \\ 0 & -\frac{n\pi i}{\gamma} \end{pmatrix} e^{-n\frac{V(z)}{2}\sigma_3} \\ &= e^{n\frac{V(z)}{2}\sigma_3} \begin{pmatrix} -1 & 0 \\ -e^{nV(z)} & -1 \end{pmatrix} e^{-n\frac{V(z)}{2}\sigma_3} = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}. \end{aligned}$$

From (4.107), (4.77), (4.73), (4.72), and analytic continuation of (4.30) into a neighborhood of $[\alpha, \alpha']$, we have that the jump $j_{\tilde{Q}}$ on $(\alpha', \alpha' + i\varepsilon)$ is given by

$$
\begin{aligned}
j_{\tilde{Q}} &= e^{-n(g_+(z)-\frac{V(z)}{2}-\frac{l}{2})\sigma_3} \begin{pmatrix} -\frac{n\pi i}{\gamma} & 0 \\ 0 & -\frac{\gamma}{n\pi i} \end{pmatrix} e^{n(g_+(z)-\frac{l}{2})\sigma_3} j_R e^{-n(g_+(z)-\frac{l}{2})\sigma_3} j_+(z)^{-1} \\
&\quad \times e^{n(g_+(z)-\frac{V(z)}{2}-\frac{l}{2})\sigma_3} e^{\frac{in\pi z}{2\gamma}\sigma_3} \\
&= e^{n\frac{V(z)}{2}\sigma_3} \begin{pmatrix} -\frac{n\pi i}{\gamma} & 0 \\ 0 & -\frac{\gamma}{n\pi i} \end{pmatrix} j_R e^{-n(g_+(z)-\frac{l}{2})\sigma_3} j_+(z)^{-1} e^{n(g_+(z)-\frac{V(z)}{2}-\frac{l}{2})\sigma_3} e^{\frac{in\pi z}{2\gamma}\sigma_3} \\
&= e^{n\frac{V(z)}{2}\sigma_3} \begin{pmatrix} -\frac{n\pi i}{\gamma} & 0 \\ 0 & -\frac{\gamma}{n\pi i} \end{pmatrix} \begin{pmatrix} \Pi(z) & \frac{\gamma}{n\pi i} e^{-nV(z)} e^{\frac{in\pi z}{2\gamma}} \\ -\frac{n\pi i}{\gamma} e^{nV(z)} e^{\frac{in\pi z}{2\gamma}} & -\frac{n\pi i}{\gamma} e^{\frac{in\pi z}{2\gamma}} \end{pmatrix} \\
&\quad \times e^{-n(g_+(z)-\frac{l}{2})\sigma_3} \begin{pmatrix} 1 & 0 \\ -e^{-n(g_+(z)-g_-(z))} & 1 \end{pmatrix} e^{n(g_+(z)-\frac{V(z)}{2}-\frac{l}{2})\sigma_3} e^{\frac{in\pi z}{2\gamma}\sigma_3} \\
&= e^{n\frac{V(z)}{2}\sigma_3} \begin{pmatrix} -2i\sin(\frac{n\pi z}{2\gamma}) & -e^{-nV(z)} e^{\frac{in\pi z}{2\gamma}} \\ e^{nV(z)} e^{\frac{in\pi z}{2\gamma}} & e^{\frac{in\pi z}{2\gamma}} \end{pmatrix} \\
&\quad \times \begin{pmatrix} e^{-n\frac{V(z)}{2}} & 0 \\ -e^{n(g_+(z)+g_-(z)-l-\frac{V(z)}{2})} & e^{n\frac{V(z)}{2}} \end{pmatrix} e^{\frac{in\pi z}{2\gamma}\sigma_3} \\
&= \begin{pmatrix} -2i\sin(\frac{n\pi z}{2\gamma}) e^{n\frac{V(z)}{2}} & -e^{-n\frac{V(z)}{2}} e^{\frac{in\pi z}{2\gamma}} \\ e^{n\frac{V(z)}{2}} e^{\frac{in\pi z}{2\gamma}} & e^{-n\frac{V(z)}{2}} e^{\frac{in\pi z}{2\gamma}} \end{pmatrix} \\
&\quad \times \begin{pmatrix} e^{-n\frac{V(z)}{2}} & 0 \\ -e^{n(g_+(z)+g_-(z)-l-\frac{V(z)}{2})} & e^{n\frac{V(z)}{2}} \end{pmatrix} e^{\frac{in\pi z}{2\gamma}\sigma_3} \\
&= \begin{pmatrix} -2i\sin(\frac{n\pi z}{2\gamma}) + e^{\frac{in\pi z}{2\gamma}} e^{n(g_+(z)+g_-(z)-l-V(z))} & -e^{\frac{in\pi z}{2\gamma}} \\ e^{\frac{in\pi z}{2\gamma}} - e^{\frac{in\pi z}{2\gamma}} e^{n(g_+(z)+g_-(z)-l-V(z))} & e^{\frac{in\pi z}{2\gamma}} \end{pmatrix} e^{\frac{in\pi z}{2\gamma}\sigma_3} \\
&= \begin{pmatrix} e^{-\frac{in\pi z}{2\gamma}} & -e^{\frac{in\pi z}{2\gamma}} \\ 0 & e^{\frac{in\pi z}{2\gamma}} \end{pmatrix} e^{\frac{in\pi z}{2\gamma}\sigma_3} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.
\end{aligned}
$$

Similarly, we have that the jump $j_{\tilde{Q}}$ on $(\alpha' - i\varepsilon, \alpha')$ is given by

$$\begin{aligned}
j_{\tilde{Q}} &= e^{\frac{in\pi z}{2\gamma}\sigma_3} e^{-n(g_-(z)-\frac{V(z)}{2}-\frac{l}{2})\sigma_3} j_S e^{n(g_-(z)-\frac{V(z)}{2}-\frac{l}{2})\sigma_3} e^{-\frac{in\pi z}{2\gamma}\sigma_3} \\
&= e^{\frac{in\pi z}{2\gamma}\sigma_3} e^{-n(g_-(z)-\frac{V(z)}{2}-\frac{l}{2})\sigma_3} \begin{pmatrix} \frac{\gamma}{n\pi i} e^{\frac{in\pi z}{2\gamma}} & 0 \\ 0 & \frac{n\pi i}{\gamma} e^{-\frac{in\pi z}{2\gamma}} \end{pmatrix}^{-1} j_T j_-(z) \\
&\quad \times e^{n(g_-(z)-\frac{V(z)}{2}-\frac{l}{2})\sigma_3} e^{-\frac{in\pi z}{2\gamma}\sigma_3} \\
&= e^{n\frac{V(z)}{2}\sigma_3} \begin{pmatrix} \frac{n\pi i}{\gamma} & 0 \\ 0 & \frac{\gamma}{n\pi i} \end{pmatrix} j_R e^{-n(g_-(z)-\frac{l}{2})\sigma_3} j_-(z) e^{n(g_-(z)-\frac{V(z)}{2}-\frac{l}{2})\sigma_3} e^{-\frac{in\pi z}{2\gamma}\sigma_3}
\end{aligned}$$

$$\begin{aligned}
&= e^{n\frac{V(z)}{2}\sigma_3} \begin{pmatrix} \frac{n\pi i}{\gamma} & 0 \\ 0 & \frac{\gamma}{n\pi i} \end{pmatrix} \begin{pmatrix} \Pi(z) & \frac{\gamma}{n\pi i} e^{-nV(z)} e^{-\frac{in\pi z}{2\gamma}} \\ -\frac{n\pi i}{\gamma} e^{nV(z)} e^{-\frac{in\pi z}{2\gamma}} & \frac{n\pi i}{\gamma} e^{-\frac{in\pi z}{2\gamma}} \end{pmatrix} \\
&\quad \times \begin{pmatrix} e^{-n\frac{V(z)}{2}} & 0 \\ e^{n(g_+(z)+g_-(z)-l-\frac{V(z)}{2})} & e^{n\frac{V(z)}{2}} \end{pmatrix} e^{-\frac{in\pi z}{2\gamma}\sigma_3} \\
&= \begin{pmatrix} 2i\sin(\frac{n\pi z}{2\gamma}) e^{n\frac{V(z)}{2}} & e^{-n\frac{V(z)}{2}} e^{-\frac{in\pi z}{2\gamma}} \\ -e^{n\frac{V(z)}{2}} e^{-\frac{in\pi z}{2\gamma}} & e^{-n\frac{V(z)}{2}} e^{-\frac{in\pi z}{2\gamma}} \end{pmatrix} \begin{pmatrix} e^{-n\frac{V(z)}{2}} & 0 \\ e^{n(g_+(z)+g_-(z)-l-\frac{V(z)}{2})} & e^{n\frac{V(z)}{2}} \end{pmatrix} e^{-\frac{in\pi z}{2\gamma}\sigma_3} \\
&= \begin{pmatrix} 2i\sin(\frac{n\pi z}{2\gamma}) + e^{-\frac{in\pi z}{2\gamma}} e^{n(g_+(z)+g_-(z)-l-V(z))} & e^{-\frac{in\pi z}{2\gamma}} \\ -e^{-\frac{in\pi z}{2\gamma}} + e^{-\frac{in\pi z}{2\gamma}} e^{n(g_+(z)+g_-(z)-l-V(z))} & e^{-\frac{in\pi z}{2\gamma}} \end{pmatrix} e^{-\frac{in\pi z}{2\gamma}\sigma_3} \\
&= \begin{pmatrix} e^{\frac{in\pi z}{2\gamma}} & e^{-\frac{in\pi z}{2\gamma}} \\ 0 & e^{-\frac{in\pi z}{2\gamma}} \end{pmatrix} e^{-\frac{in\pi z}{2\gamma}\sigma_3} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.
\end{aligned}$$

D. PROOF OF PROPOSITION 4.13.1

From (4.43) and (4.84), we have

$$\tilde{u}_\infty = \frac{\pi}{4}(1-\zeta)$$

This, combined with formula (4.78) for Ω_n immediately gives

$$\frac{\vartheta_3(\tilde{u}_\infty + d)\vartheta_3(-\tilde{u}_\infty + d + \frac{\Omega_n}{2})}{\vartheta_3(-\tilde{u}_\infty + d)\vartheta_3(\tilde{u}_\infty + d + \frac{\Omega_n}{2})} = \frac{\vartheta_3(0)\vartheta_4((n+1)\omega)}{\vartheta_3(\omega)\vartheta_4(n\omega)}. \tag{D.1}$$

Formulae (4.21) give that

$$\frac{(\beta-\beta') + (\alpha'-\alpha)}{4i} = \frac{\pi\vartheta_4^2(0)}{4i}\left[\frac{\vartheta_1^2(\frac{\omega}{2})\vartheta_4^2(\frac{\omega}{2}) + \vartheta_2^2(\frac{\omega}{2})\vartheta_3^2(\frac{\omega}{2})}{\vartheta_1(\frac{\omega}{2})\vartheta_2(\frac{\omega}{2})\vartheta_3(\frac{\omega}{2})\vartheta_4(\frac{\omega}{2})}\right]. \tag{D.2}$$

Plugging the duplication formulae (E.6) and (E.7) into (D.2) yields

$$\frac{(\beta-\beta') + (\alpha'-\alpha)}{4i} = \frac{\pi}{2i}\frac{\vartheta_2(0)\vartheta_4(0)\vartheta_3(\omega)}{\vartheta_1(\omega)}. \tag{D.3}$$

Combining (D.1) and (D.3), we can write the [12] entry of (4.96) as

$$\begin{aligned}[\mathbf{M}_1]_{12} &= \frac{i\pi}{2}\frac{\vartheta_4\big((n+1)\omega\big)}{\vartheta_4(n\omega)}\frac{\vartheta_3(0)}{\vartheta_3(\omega)}\frac{\vartheta_2(0)\vartheta_4(0)\vartheta_3(\omega)}{\vartheta_1(\omega)} \\ &= \frac{i\pi}{2}\frac{\vartheta_4\big((n+1)\omega\big)}{\vartheta_4(n\omega)}\frac{\vartheta_1'(0)}{\vartheta_2(\frac{\pi\zeta}{2})} = \frac{iA\vartheta_4\big((n+1)\omega\big)}{\vartheta_4(n\omega)}.\end{aligned}$$

Similarly, we can write the [21] entry of (4.96) as

$$[\mathbf{M}_1]_{21} = \frac{A\vartheta_4(n\omega)}{i\vartheta_4\big((n-1)\omega\big)}.$$

E. THETA FUNCTION IDENTITIES

The following identities (see [50]) are used in chapter 4 of this dissertation. There are the identities involving derivatives of theta functions:

$$\vartheta_1'(0) = \vartheta_2(0)\vartheta_3(0)\vartheta_4(0), \tag{E.1}$$

$$\vartheta_4'(z) = \frac{\vartheta_1'(z)\vartheta_4(z) - \vartheta_4(0)^2\vartheta_2(z)\vartheta_3(z)}{\vartheta_1(z)}, \quad \vartheta_2'(z) = \frac{\vartheta_1'(z)\vartheta_2(z) - \vartheta_2(0)^3\vartheta_3(z)\vartheta_4(z)}{\vartheta_1(z)},$$
$$\vartheta_3'(z) = \frac{\vartheta_1'(z)\vartheta_3(z) - \vartheta_3(0)^2\vartheta_2(z)\vartheta_4(z)}{\vartheta_1(z)}, \tag{E.2}$$

$$\vartheta_4'(z) = \frac{\vartheta_2'(z)\vartheta_4(z) + \vartheta_1(z)\vartheta_3(z)\vartheta_3^2(0)}{\vartheta_2(z)}, \quad \vartheta_1'(z) = \frac{\vartheta_2'(z)\vartheta_1(z) + \vartheta_3(z)\vartheta_4(z)\vartheta_2^2(0)}{\vartheta_2(z)},$$
$$\vartheta_3'(z) = \frac{\vartheta_2'(z)\vartheta_3(z) + \vartheta_1(z)\vartheta_4(z)\vartheta_4^2(0)}{\vartheta_2(z)}, \tag{E.3}$$

$$\begin{aligned}
\vartheta_4''(z) &= \frac{\vartheta_2''(z)\vartheta_4(z)}{\vartheta_2(z)} + \frac{2\vartheta_2'(z)\vartheta_1(z)\vartheta_3(z)\vartheta_3^2(0)}{\vartheta_2^2(z)} \\
&\quad + \frac{\vartheta_4(z)\vartheta_3^2(0)}{\vartheta_2^2(z)}\Big(\vartheta_3^2(z)\vartheta_2^2(0) + \vartheta_1^2(z)\vartheta_4^2(0)\Big), \\
\vartheta_1''(z) &= \frac{\vartheta_2''(z)\vartheta_1(z)}{\vartheta_2(z)} + \frac{2\vartheta_2'(z)\vartheta_3(z)\vartheta_4(z)\vartheta_2^2(0)}{\vartheta_2^2(z)} \\
&\quad + \frac{\vartheta_1(z)\vartheta_2^2(0)}{\vartheta_2^2(z)}\Big(\vartheta_3^2(z)\vartheta_3^2(0) + \vartheta_4^2(z)\vartheta_4^2(0)\Big), \\
\vartheta_3''(z) &= \frac{\vartheta_2''(z)\vartheta_3(z)}{\vartheta_2(z)} + \frac{2\vartheta_2'(z)\vartheta_1(z)\vartheta_4(z)\vartheta_4^2(0)}{\vartheta_2^2(z)} \\
&\quad + \frac{\vartheta_3(z)\vartheta_3^2(0)}{\vartheta_2^2(z)}\Big(\vartheta_4^2(z)\vartheta_2^2(0) + \vartheta_1^2(z)\vartheta_4^2(0)\Big),
\end{aligned} \tag{E.4}$$

$$\begin{aligned}
\vartheta_4''(z) &= \frac{\vartheta_1''(z)\vartheta_4(z)}{\vartheta_1(z)} - \frac{2\vartheta_1'(z)\vartheta_2(z)\vartheta_3(z)\vartheta_4^2(0)}{\vartheta_1^2(z)} \\
&\quad + \frac{\vartheta_4^2(0)\vartheta_4^2(z)}{\vartheta_1^2(z)}\Big(\vartheta_3^2(z)\vartheta_2^2(0) + \vartheta_2^2(z)\vartheta_3^2(0)\Big), \\
\vartheta_2''(z) &= \frac{\vartheta_1''(z)\vartheta_2(z)}{\vartheta_1(z)} - \frac{2\vartheta_1'(z)\vartheta_3(z)\vartheta_4(z)\vartheta_2^2(0)}{\vartheta_1^2(z)} \\
&\quad + \frac{\vartheta_2^2(0)\vartheta_2^2(z)}{\vartheta_1^2(z)}\Big(\vartheta_4^2(z)\vartheta_3^2(0) + \vartheta_3^2(z)\vartheta_4^2(0)\Big), \\
\vartheta_3''(z) &= \frac{\vartheta_1''(z)\vartheta_3(z)}{\vartheta_1(z)} - \frac{2\vartheta_1'(z)\vartheta_2(z)\vartheta_4(z)\vartheta_3^2(0)}{\vartheta_1^2(z)} \\
&\quad + \frac{\vartheta_3^2(0)\vartheta_3^2(z)}{\vartheta_1^2(z)}\Big(\vartheta_4^2(z)\vartheta_2^2(0) + \vartheta_2^2(z)\vartheta_4^2(0)\Big),
\end{aligned} \tag{E.5}$$

the duplication formulae

$$\vartheta_1(2z) = \frac{2\vartheta_1(z)\vartheta_2(z)\vartheta_3(z)\vartheta_4(z)}{\vartheta_1'(0)}, \tag{E.6}$$

$$\vartheta_3(2z)\vartheta_3(0)\vartheta_2^2(0) = \vartheta_1(z)^2\vartheta_4(z)^2 + \vartheta_2(z)^2\vartheta_3(z)^2, \tag{E.7}$$

$$\vartheta_4(2z)\vartheta_4^3(0) = \vartheta_3^4(z) - \vartheta_2^4(z) = \vartheta_4^4(z) - \vartheta_1^4(z), \tag{E.8}$$

the addition formula

$$\begin{aligned}
\vartheta_3(y+z)\vartheta_3(y-z)\vartheta_2^2(0) &= \vartheta_3^2(y)\vartheta_2^2(z) + \vartheta_4^2(y)\vartheta_1^2(z) \\
&= \vartheta_1^2(y)\vartheta_4^2(z) + \vartheta_2^2(y)\vartheta_3^2(z),
\end{aligned} \tag{E.9}$$

and the identities relating squares of theta functions

$$\vartheta_1^2(z)\vartheta_4^2(0) = \vartheta_3^2(z)\vartheta_2^2(0) - \vartheta_2^2(z)\vartheta_3^2(0), \tag{E.10}$$

$$\vartheta_2^2(z)\vartheta_4^2(0) = \vartheta_4^2(z)\vartheta_2^2(0) - \vartheta_1^2(z)\vartheta_3^2(0), \tag{E.11}$$

$$\vartheta_3^2(z)\vartheta_4^2(0) = \vartheta_4^2(z)\vartheta_3^2(0) - \vartheta_1^2(z)\vartheta_2^2(0), \tag{E.12}$$

$$\vartheta_4^2(z)\vartheta_4^2(0) = \vartheta_3^2(z)\vartheta_3^2(0) - \vartheta_2^2(z)\vartheta_2^2(0). \tag{E.13}$$

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Karl Edmund Liechty was born May 27, 1981 in Indianapolis, IN. He enjoyed a charmed midwestern childhood, graduating from North Central High School in Indianapolis in 1999. After spending several years as a musician (receiving a Bachelor of Music degree from the Indiana University School of Music in 2003), Karl began studying mathematics in earnest, matriculating at Indiana University-Purdue University Indianapolis in 2005 and obtaining a Ph.D. in mathematics under the guidance of Professor Pavel Bleher in 2010.