ASYMPTOTIC ANALYSIS OF STRUCTURED DETERMINANTS VIA THE RIEMANN-HILBERT APPROACH

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تقدیم به اسما، مجتبی محزون و خانواده عزیزم

برای عشق، راهنمایی و حمایتی که به من ارزانی داشتید

اللَّهُمَّ لَا تَجْعَلْنِي مِنْ الَّذِينَ ضَلَّ سَعْيُهُمْ فِي الْحَيَاةِ الدُّنْيَا
وَهُمْ يَحْسَبُونَ أَنَّهُمْ يُحْسِنُونَ صُنْعًا

رفتم به در مدرسه و گوش کشیدم

حرفی که به انجام برم پی ، نشنیدم

سد اصل سخن رفت و دلیلش همه مدخول

از شک و گمانی به یقینی نرسیدم

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ABBREVIATIONS

RHP	Riemann-Hilbert problem
LUE	Laguerre Unitary Ensemble
GUE	Gaussian Unitary Ensemble
JUE	Jacobi Unitary Ensemble
FH	Fisher-Hartwig
OP	Orthogonal Polynomial
CLT	Central Limit Theorem

ABSTRACT

Gharakhloo, Roozbeh. Ph.D., Purdue University, August 2019. Asymptotic Analysis of Structured Determinants via the Riemann-Hilbert Approach. Major Professor: Alexander Its.

In this work we use and develop Riemann-Hilbert techniques to study the asymptotic behavior of structured determinants. In chapter one we will review the main underlying definitions and ideas which will be extensively used throughout the thesis. Chapter two is devoted to the asymptotic analysis of Hankel determinants with Laguerre-type and Jacobi-type potentials with Fisher-Hartwig singularities. In chapter three we will propose a Riemann-Hilbert problem for Toeplitz+Hankel determinants. We will then analyze this Riemann-Hilbert problem for a certain family of Toeplitz and Hankel symbols. In Chapter four we will study the asymptotics of a certain bordered-Toeplitz determinant which is related to the next-to-diagonal correlations of the anisotropic Ising model. The analysis is based upon relating the bordered-Toeplitz determinant to the solution of the Riemann-Hilbert problem associated to pure Toeplitz determinants. Finally in chapter five we will study the emptiness formation probability in the XXZ-spin 1/2 Heisenberg chain, or equivalently, the asymptotic analysis of the associated Fredholm determinant.

1. INTRODUCTION

1.1 Definitions, notations and preliminaries

The work in this thesis is focused on the asymptotic analysis of *structured determinants* arising in random matrix theory, statistical mechanics, theory of integrable operators and theory of orthogonal polynomials, where we primarily employ the Riemann-Hilbert method. For a given oriented contour Σ in the complex plane (see Figure 1.1) and a function $\mathcal{J} : \Sigma \to GL(k,\mathbb{C})$, the (normalized) Riemann-Hilbert problem $(\Sigma, \mathcal{J})$ consists of determining the unique $k \times k$ matrix function $Y(z)$ satisfying

- $Y(z)$ is analytic in $\mathbb{C} \setminus \Sigma$,
- $Y_+(z) = Y_-(z)\mathcal{J}(z)$, for $z \in \Sigma$, and
- $Y(z) \to I$, as $z \to \infty$,

where $\mathcal{J}(z)$ is called the *jump matrix* of the Riemann-Hilbert problem (RHP), and $Y_\pm(z)$ denote the limit of $Y(\zeta)$ as ζ approaches $z \in \Sigma$ from the $\pm$ side of the oriented contour Σ: As we move along a path in Σ in the direction of the orientation, by convention we say that the + side (respectively the - side) lies to the left (respectively right).

By structured determinants, we mean Toeplitz, bordered Toeplitz, Hankel, Toeplitz+Hankel and integrable Fredholm determinants which arise almost ubiquitously in random matrix theory and statistical mechanics. The $n \times n$ Toeplitz and Hankel matrices associated respectively to the symbols ϕ and w are respectively defined as

$$T_n[\phi] := \{\phi_{j-k}\}, \qquad j,k = 0, \cdots, n-1, \qquad \phi_k = \int_{\mathbb{T}} z^{-k}\phi(z)\frac{dz}{2\pi i z}, \tag{1.1.1}$$

and

$$H_n[w] := \{w_{j+k}\}, \qquad j,k = 0, \cdots, n-1, \qquad w_k = \int_{\mathcal{I}} x^k w(x)dx, \tag{1.1.2}$$

where $\mathcal{I} \subset \mathbb{R}$ and $\mathbb{T}$ denotes the positively oriented unit circle. In the above definition, j is the index of rows and k is the index of columns. The $n \times n$ Toeplitz + Hankel matrix associated to these symbols is naturally defined as $T_n[\phi] + H_n[w]$. A bordered-Toeplitz matrix has the structure of a regular Toeplitz matrix except for its last row or column, i.e. it is of the type

$$\begin{pmatrix} \phi_0 & \phi_{-1} & \cdots & \phi_{-n+2} & b_{-n+1} \\ \phi_1 & \phi_0 & \cdots & \phi_{-n+3} & b_{-n+2} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \phi_{n-1} & \phi_{n-2} & \cdots & \phi_1 & b_0 \end{pmatrix}.$$

The Hankel and Toeplitz+Hankel determinants are of interest both when the Hankel symbol w is supported on an interval and also when it is supported on the unit circle; in the former, w_k is the k-th moment of the weight as defined in (1.1.2) and in the latter w_k is the Fourier coefficient of the weight w:

$$w_k = \int_{\mathbb{T}} z^{-k} w(z) \frac{dz}{2\pi i z}. \tag{1.1.3}$$

For a Toeplitz or Hankel determinant with symbols supported on the unit circle, the so-called *index* or *winding number* of a symbol ϕ is defined as follows: for $z \in \mathbb{T}$ write $\phi(z) = |\phi(z)| \exp[2\pi i b(z)]$ for some choice of b, then the increment of b as the result of a counter-clockwise circuit around $\mathbb{T}$ is an integer solely dependent on ϕ (not on the choice of b), this integer is normally referred to as the winding number or the index of the symbol ϕ and plays an important role in the analysis of the generated determinants.

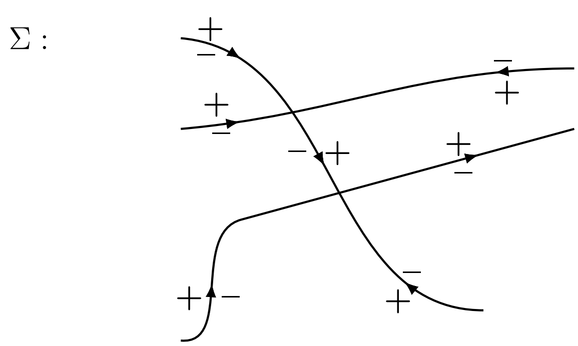

Figure 1.1. The jump contour for the Riemann-Hilbert problem $(\Sigma, \mathcal{J})$.

An important class of Toeplitz or Hankel symbols are those with the so-called *Fisher-Hartwig singularities*. These singularities are named after Fisher and Hartwig, due to their pioneering work in their identification [1]. We say that a symbol ϕ defined on the unit circle possesses Fisher-Hartwig (FH) singularities if it is of the type:

$$\phi(z) = e^{W(z)} z^{\sum_{j=0}^{m} \beta_j} \prod_{j=0}^{m} |z - z_j|^{2\alpha_j} g_j(z) z_j^{-\beta_j}, \qquad z = e^{i\theta}, \qquad \theta \in [0, 2\pi), \tag{1.1.4}$$

for some $m = 0, 1, \cdots$, where $z_j = e^{i\theta_j}, j = 0, \cdots, m, \theta_j \in [0, 2\pi)$,

$$g_j(z) = \begin{cases} e^{i\pi\beta_j}, & 0 \le \arg z < \theta_j, \\ e^{-i\pi\beta_j}, & \theta_j \le \arg z < 2\pi, \end{cases} \tag{1.1.5}$$

where in (1.1.4) and (1.1.5) one assumes that

$$\Re\alpha_j > -\frac{1}{2}, \qquad \beta_j \in \mathbb{C}, \qquad j = 0, \cdots, m. \tag{1.1.6}$$

The term $e^{W(z)}$ in (1.1.4) is sometimes referred to as the smooth part, or the Szegő part of the symbol, while the rest of the terms in (1.1.4) is sometimes referred to as the *pure* Fisher-Hartwig part (e.g. see [2]). Usually, the singularities $|z - z_j|^{2\alpha_j}$ and g_j are respectively called the "root-type" and "jump-type" singularities. One can also consider Hankel weights with FH singularities on the real line(e.g. see [3–5] for particular cases); In chapter 2 we will define such weights as part of a more general class of weights, i.e. weights with Szegő part, FH part and exponentially varying part $e^{-nV(z)}$, for a potential V.

Let Σ be an oriented contour in $\mathbb{C}$, an integral operator acting on $L^2(\Sigma) = L^2(\Sigma, |dz|)$ is *integrable* if it has a kernel of the form

$$K(z, \lambda) = \frac{\sum_{j=1}^{N} f_j(z) h_j(\lambda)}{z - \lambda}, \qquad z, \lambda \in \Sigma, \tag{1.1.7}$$

for some functions f_i, h_j, $1 \le i, j \le N < \infty$. An integrable Fredholm determinant is of the form $\det(\mathbf{1} - K)$, where $\mathbf{1}$ is the identity operator and the determinant is taken in $L^2(\Sigma)$.

For certain choices of symbols, and integrable Fredholm operators, the corresponding structured determinants identify important objects in statistical mechanics and random matrix theory. On these occasions, an asymptotic question in random matrix theory or statistical mechanics can be translated into the question of large-n asymptotics of the corresponding structured determinant. There is an inherent correspondence between these structured determinants and a set of orthogonal polynomials associated to the symbol or integrable operator under consideration. The groundbreaking discovery of Fokas, Its and Kitaev [6], provided the *representation* of the solution to a certain 2×2 Riemann-Hilbert problem in terms of the corresponding orthogonal polynomials and their Cauchy transforms; thus if the Riemann-Hilbert problem could be solved, by independent means, for large values of the parameter n, consequently the large-n asymptotics of associated orthogonal polynomials and structured determinants could be found as well. The celebrated non-linear steepest descent method of Dieft and Zhou [7] was the next paramount breakthrough which provided the needed apparatus for asymptotically solving the Riemann-Hilbert problems with oscillatory jumps in n. In this method one tries to solve an equivalent Riemann-Hilbert problem on an augmented contour, such that the jump matrices on the new set of contours (the so-called *lenses*) converge to the identity matrix away from intersection points of lenses and the old contour for large values of the parameter n, and the jump matrices on the the other parts of the contour are such that they can be factorized to produce the so-called *global parametrix* (away from intersection points of lenses with the old contour) and *local parametrices* (in a neighborhood of intersection points of lenses with the old contour).

1.2 Selected Riemann-Hilbert problems

In this section, we will review the Riemann-Hilbert problems associated to the structured determinants studied in this work.

1.2.1 The RHP for Toeplitz determinants

Given a sufficiently smooth symbol $\phi \in L^1(\mathbb{T})$, one can consider the associated sets of bi-orthonormal polynomials $\{Q_n(z)\}_{n=0}^{\infty}$ and $\{\widehat{Q}_n(z)\}_{n=0}^{\infty}$, where $Q_n(z) = \kappa_n z^n + l_n z^{n-1} + \cdots$, and $\widehat{Q}_n(z) = \widehat{\kappa}_n z^n + \widehat{l}_n z^{n-1} + \cdots$ satisfy the bi-orthonormality conditions

$$\int_{\mathbb{T}} Q_n(z)\widehat{Q}_n(z^{-1})\phi(z)\frac{dz}{2\pi i z} = \delta_{nk}. \tag{1.2.1}$$

The key fact which relates these polynomials to the Toeplitz determinant with symbol ϕ, is that they have determinantal representations given by

$$Q_n(z) = \frac{1}{\sqrt{\det T_n[\phi] \det T_{n+1}[\phi]}} \det \begin{pmatrix} \phi_0 & \phi_{-1} & \cdots & \phi_{-n} \\ \phi_1 & \phi_0 & \cdots & \phi_{-n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n-1} & \phi_{n-2} & \cdots & \phi_{-1} \\ 1 & z & \cdots & z^n \end{pmatrix}, \tag{1.2.2}$$

and

$$\widehat{Q}_n(z) = \frac{1}{\sqrt{\det T_n[\phi] \det T_{n+1}[\phi]}} \det \begin{pmatrix} \phi_0 & \phi_{-1} & \cdots & \phi_{-n+1} & 1 \\ \phi_1 & \phi_0 & \cdots & \phi_{-n+2} & z \\ \vdots & \vdots & \ddots & \vdots & \\ \phi_n & \phi_{n-1} & \cdots & \phi_1 & z^n \end{pmatrix}. \tag{1.2.3}$$

Moreover, from these determinantal representations it is clear that

$$\kappa_n = \widehat{\kappa}_n = \sqrt{\frac{\det T_n[\phi]}{\det T_{n+1}[\phi]}}. \tag{1.2.4}$$

Now let us consider the function

$$X(z) := \begin{pmatrix} \kappa_n^{-1} Q_n(z) & \kappa_n^{-1} \int_{\mathbb{T}} \frac{Q_n(\zeta)}{(\zeta - z)} \frac{\phi(\zeta)d\zeta}{2\pi i \zeta^n} \\ -\kappa_{n-1} z^{n-1}\widehat{Q}_{n-1}(z^{-1}) & -\kappa_{n-1} \int_{\mathbb{T}} \frac{\widehat{Q}_{n-1}(\zeta^{-1})}{(\zeta - z)} \frac{\phi(\zeta)d\zeta}{2\pi i \zeta} \end{pmatrix}. \tag{1.2.5}$$

In [8] it was found by J.Baik, P.Deift and K.Johansson that the function X defined above satisfies the following associated Riemann-Hilbert problem

- **RH-X1** $X : \mathbb{C} \setminus \mathbb{T} \to \mathbb{C}^{2\times 2}$ is analytic,
- **RH-X2** The limits of $X(\zeta)$ as ζ tends to $z \in \mathbb{T}$ from the inside and outside of the unit circle exist, and are denoted $X_{\pm}(z)$ respectively and are related by

$$X_+(z) = X_-(z) \begin{pmatrix} 1 & z^{-n}\phi(z) \\ 0 & 1 \end{pmatrix}, \qquad z \in \mathbb{T}, \tag{1.2.6}$$

- **RH-X3** As $z \to \infty$

$$X(z) = \big(I + \mathcal{O}(z^{-1})\big) z^{n\sigma_3}, \tag{1.2.7}$$

where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the third Pauli matrix. Depending on the analytic features of the particular symbol ϕ, one has to supplement the above Riemann-Hilbert problem with prescribed asymptotic conditions at the singularities of ϕ on the unit circle, if any, to ensure that the X-RHP has a unique solution. In the pioneering work [9] the authors have been able to effectively solve this Riemann-Hilbert problem for a general symbol of the type (1.1.4).

1.2.2 The RHP for Hankel determinants

Although Hankel determinants are mainly studied for weights supported on the real line, in this section we also briefly discuss Hankel determinants whose weight is supported on the unit circle and we will argue, at least on a theoretical level, for why one should be interested in their asymptotics.

Weight supported on the real line

Let $\mathcal{I} \subset \mathbb{R}$ and $w \in L^1(\mathcal{I})$ be a sufficiently smooth function. In this section we will discuss the RHP formulation for determinants of Hankel matrices $H_n[w]$ defined by (1.1.2). One can consider the associated set of monic orthogonal polynomials $\{P_n(z)\}_{n=0}^{\infty}$, $\deg P_n(z) = n$, satisfying the orthogonality conditions

$$\int_{\mathcal{I}} P_n(x) x^k w(x) dx = \gamma_n \delta_{nk}. \tag{1.2.8}$$

The polynomials $P_n(z)$ has the following determinantal representations

$$P_n(z) = \frac{1}{\det H_n[w]} \det \begin{pmatrix} w_0 & w_1 & \cdots & w_{n-1} & w_n \\ w_1 & w_2 & \cdots & w_n & w_{n+1} \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ w_{n-1} & w_n & \cdots & w_{2n-2} & w_{2n-1} \\ 1 & z & \cdots & z^{n-1} & z^n \end{pmatrix}, \qquad \text{and hence} \qquad \gamma_n = \frac{\det H_{n+1}[w]}{\det H_n[w]}. \tag{1.2.9}$$

It is due to Fokas, Its and Kitaev [6] that the following matrix-valued function which is built from the orthogonal polynomials and the Cauchy transforms of the weight w multiplied by the orthogonal polynomials

$$Y(z) = \begin{pmatrix} P_n(z) & \frac{1}{2\pi i}\int_{\mathcal{I}} \frac{P_n(x)w(x)}{x-z}dx \\ -\frac{2\pi i}{h_{n-1}}P_{n-1}(z) & -\frac{1}{h_{n-1}}\int_{\mathcal{I}} \frac{P_{n-1}(x)w(x)}{x-z}dx \end{pmatrix}, \tag{1.2.10}$$

satisfies the following Riemann-Hilbert problem:

- **RH-Y1** $Y : \mathbb{C} \setminus [a,b] \to \mathbb{C}^{2\times 2}$ is analytic.
- **RH-Y2** The limits of $Y(z)$ as z tends to $x \in \mathcal{I}$ from the upper and lower half plane exist, and are denoted $Y_\pm(x)$ respectively and are related by

$$Y_+(x) = Y_-(x)\begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix}, \qquad x \in \mathcal{I}. \tag{1.2.11}$$

- **RH-Y3** As $z \to \infty$,

$$Y(z) = \big(I + \mathcal{O}(z^{-1})\big)z^{n\sigma_3}. \tag{1.2.12}$$

Like what we mentioned about Toeplitz determinants, the analytic properties of w dictates certain asymptotic conditions on the RHP setting, as z approaches singularities of w which belong to $\overline{\mathcal{I}}$. For instance, if w is a modified Jacobi weight $w(x) = h(x)(1-x)^\alpha(1+x)^\beta$, $\mathcal{I} = [-1,1]$, in order to propose a Riemann-Hilbert problem with a unique solution, one has to specify asymptotic conditions as $z \to \pm 1$ as well (see [10]). Furthermore, if w has Fisher-Hartwig singularities at $\{t_j\}_{j=1}^m \subset \overline{\mathcal{I}}$, one has to augment the above Riemann-Hilbert problem with specific asymptotic conditions as $z \to t_j$ (see [3]). The above Reiemann-Hilbert problem can be solved for sufficiently large n via the Deift-Zhou nonlinear steepest descent method, and hence through integration of associated differential identities, the large-n asymptotics of $\det H_n[w]$ can be found (e.g. see [3], [4]).

Weight supported on the unit circle

In this section we will consider the determinants of Hankel matrices $H_n[w] = \{w_{j+k}\}$, whose symbol is supported on the unit circle. We will show that this determinant is, in a natural way, related to a Toeplitz determinant whose symbol contains the large parameter n, making its winding number monotonically decrease as $n \to \infty$. But first let us recall the situation for Toeplitz symbols with *fixed* non-zero winding number. A.Bőttcher and H.Widom considered this problem from an operator-theoretic point of view in [11]. P.Deift, A.Its and I.Krasovsky in [9] prove yet another remarkable result where they relate the Toeplitz determinant with symbol $z^\ell \phi(z)$, where $\ell \in \mathbb{Z}$ is independent of n, to the Toeplitz determinant with symbol ϕ that can be asymptotically analyzed via the RHP method. Here we mention their result:

Lemma 1.2.1 (From [9]) *Let the Toeplitz determinants $D_n(\phi)$ be nonzero for all $n \geq N_0$, for some $N_0 \in \mathbb{N}$. Let $q_k(z) = Q_k(z)/\kappa_k$, $\widehat{q}_k(z) = \widehat{Q}_k/\kappa_k$, $k = N_0, N_0+1, \cdots$ be the system of monic bi-orthogonal polynomials on the unit circle with respect to the weight ϕ. Fix an integer $\ell > 0$, then if*

$$F_k := \det \begin{pmatrix} q_k(0) & q_{k+1}(0) & \cdots & q_{k+\ell-1}(0) \\ \frac{d}{dz} q_k(0) & \frac{d}{dz} q_{k+1}(0) & \cdots & \frac{d}{dz} q_{k+\ell-1}(0) \\ \vdots & \vdots & & \vdots \\ \frac{d^{\ell-1}}{dz^{\ell-1}} q_k(0) & \frac{d^{\ell-1}}{dz^{\ell-1}} q_{k+1}(0) & \cdots & \frac{d^{\ell-1}}{dz^{\ell-1}} q_{k+\ell-1}(0) \end{pmatrix} \neq 0, \tag{1.2.13}$$

for $k = N_0, N_0+1, \cdots, n-1$, we have

$$D_n(z^\ell \phi(z)) = \frac{(-1)^{\ell n} F_n}{\prod_{j=1}^{\ell-1} j!} D_n(\phi(z)), \qquad n \geq N_0. \tag{1.2.14}$$

Furthermore, if

$$\widehat{F}_k := \det \begin{pmatrix} \widehat{q}_k(0) & \widehat{q}_{k+1}(0) & \cdots & \widehat{q}_{k+\ell-1}(0) \\ \frac{d}{dz}\widehat{q}_k(0) & \frac{d}{dz}\widehat{q}_{k+1}(0) & \cdots & \frac{d}{dz}\widehat{q}_{k+\ell-1}(0) \\ \vdots & \vdots & & \vdots \\ \frac{d^{\ell-1}}{dz^{\ell-1}}\widehat{q}_k(0) & \frac{d^{\ell-1}}{dz^{\ell-1}}\widehat{q}_{k+1}(0) & \cdots & \frac{d^{\ell-1}}{dz^{\ell-1}}\widehat{q}_{k+\ell-1}(0) \end{pmatrix} \neq 0, \tag{1.2.15}$$

for $k = N_0, N_0+1, \cdots, n-1$*, we have*

$$D_n(z^{-\ell}\phi(z)) = \frac{(-1)^{\ell n}\widehat{F}_n}{\prod_{j=1}^{\ell-1} j!} D_n(\phi(z)), \qquad n \geq N_0. \tag{1.2.16}$$

However, one could ask: for a Toeplitz matrix $T_n[\phi]$, what if the winding number of the symbol ϕ, itself, depends on n? And whether $\det T_n[\phi]$ can be asymptotically analyzed in particular cases of such symbols? Obviously the results mentioned above do not apply to such Toeplitz determinants. Although at this level it would be difficult to give a decisive answer, at the least, we can point to a concrete example where there are fair prospects of a feasible Rimann-Hilbert approach to such questions. The important point is that the Hankel matrix $H_n[w]$, w supported on $\mathbb{T}$ and w_j is defined by (1.1.3), is related to the Toeplitz matrix $T_n[\psi]$, with

$$\psi(z) = z^{-n+1}w(z). \tag{1.2.17}$$

Indeed, if we denote by A_n the anti-diagonal $n \times n$ matrix whose nonzero elements are all 1, then we notice

$$H_n[w]A_n = \{w_{j-k+n-1}\}_{j,k=0}^{n-1} = \{\psi_{j-k}\}_{j,k=0}^{n-1} = T_n[\psi], \tag{1.2.18}$$

and therefore

$$\det H_n[w] = (-1)^{n-1} \det T_n[\psi]. \tag{1.2.19}$$

Note that the index of ψ is equal to $-n+1+\text{wind}[w]$. So we can categorize ψ as a Toeplitz symbol with *varying index*. By far, there are no asymptotics results for Toeplitz determinants with such symbols.

However, the equality (1.2.19) implies that the analysis of $\det T_n[\psi]$, ψ being a Toeplitz symbol with varying index, is equivalent to analysis of the Hankel determianant with symbol

w supported on the unit circle. For the former, In the same spirit, we can consider monic orthogonal polynomials $\{\overset{\circ}{P}_n(z)\}_{n=0}^{\infty}$, $\deg \overset{\circ}{P}_n(z) = n$, associated to the weight w satisfying

$$\int_{\mathbb{T}} \overset{\circ}{P}_n(z) z^k w(z^{-1}) \frac{dz}{2\pi i z} = \overset{\circ}{\gamma}_n \delta_{n,k}, \qquad k = 0, 1, \cdots, n. \tag{1.2.20}$$

The polynomials $\overset{\circ}{P}_n(z)$ have the following determinantal representations

$$\overset{\circ}{P}_n(z) = \frac{1}{\det H_n[w]} \det \begin{pmatrix} w_0 & w_1 & \cdots & w_{n-1} & w_n \\ w_1 & w_2 & \cdots & w_n & w_{n+1} \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ w_{n-1} & w_n & \cdots & w_{2n-2} & w_{2n-1} \\ 1 & z & \cdots & z^{n-1} & z^n \end{pmatrix}, \qquad \text{and hence} \qquad \overset{\circ}{\gamma}_n = \frac{\det H_{n+1}[w]}{\det H_n[w]}. \tag{1.2.21}$$

If we consider the following matrix-valued function which is built from the orthogonal polynomials and their Cauchy transforms

$$\overset{\circ}{Y}(z) = \begin{pmatrix} \overset{\circ}{P}_n(z) & \displaystyle\int_{\mathbb{T}} \frac{\overset{\circ}{P}_n(\zeta) w(\zeta^{-1})}{\zeta - z} \frac{d\zeta}{2\pi i \zeta} \\ -\dfrac{1}{\overset{\circ}{\gamma}_{n-1}} \overset{\circ}{P}_{n-1}(z) & -\dfrac{1}{\overset{\circ}{\gamma}_{n-1}} \displaystyle\int_{\mathbb{T}} \frac{\overset{\circ}{P}_{n-1}(\zeta) w(\zeta^{-1})}{\zeta - z} \frac{d\zeta}{2\pi i \zeta} \end{pmatrix}, \tag{1.2.22}$$

then, $\overset{\circ}{Y}(z)$ satisfies the following Riemann-Hilbert problem:

- **RH-$\overset{\circ}{Y}$1** $\quad \overset{\circ}{Y} : \mathbb{C} \setminus \mathbb{T} \to \mathbb{C}^{2\times 2}$ is analytic.
- **RH-$\overset{\circ}{Y}$2** $\quad$ The limits of $\overset{\circ}{Y}(\zeta)$ as ζ tends to $z \in \mathbb{T}$ from the inside and outside of the unit circle exist, these limiting values are denoted by $\overset{\circ}{Y}_{\pm}(z)$ respectively and are related by

$$\overset{\circ}{Y}_+(z) = \overset{\circ}{Y}_-(z) \begin{pmatrix} 1 & z^{-1} w(z^{-1}) \\ 0 & 1 \end{pmatrix}, \qquad z \in \mathbb{T}. \tag{1.2.23}$$

- **RH-$\overset{\circ}{Y}$3** $\quad$ As $z \to \infty$,

$$\overset{\circ}{Y}(z) = \big(I + \mathcal{O}(z^{-1})\big) z^{n\sigma_3}. \tag{1.2.24}$$

Note that the main difference between the Riemann-Hilbert problem for Toeplitz determinants and the Riemann-Hilbert problem for Hankel determinants (with w being supported on $\mathbb{T}$) is in the 12-element of their jump matrices on the unit circle. If an effective analysis of this Riemann-Hilbert problem be feasible, then one can get a hold of $\det T_n[\psi]$, ψ being the symbol with varying index given by (1.2.17), via (1.2.19).

To this end, the work [12] could be a great starting point at least for weights w holomorphic inside of the unit circle. Here, we briefly explain this connection. In fact, the polynomials $\overset{\circ}{P}$ satisfying

$$\int_{\partial D} \overset{\circ}{P}_n(z) z^k f(z) \frac{dz}{2\pi i z} = \overset{\circ}{\gamma}_n \delta_{n,k}, \qquad k = 0, 1, \cdots, n. \tag{1.2.25}$$

are the denominators of the the diagonal Pade' approximants, in D, to a function f holomorphic at infinity. Here, D is assumed to be a connected domain containing the point at infinity in which f is holomorphic and single valued. In particular, when $f \equiv w(z^{-1})$ and $D \equiv \mathbb{C} \setminus \overline{\mathbb{D}}$, $\mathbb{D}$ being the unit disk, then the orthogonality conditions (1.2.20) and (1.2.25) are the same, provided that w be analytic inside of the unit circle, which hence implies that $f \equiv \tilde{w}$ is analytic in D. This means that through the relation (1.2.21), expressing the ratio of Hankel determinants in terms of the the norms $\overset{\circ}{\gamma}_n$ of Pade' approximant denominators, one can obtain the asymptotics of Hankel determinants using the relevant differential identities as usual. This would finally provide us with the asymptotics of Toeplitz determinants $\det T_n[\psi]$ with varying index.

1.2.3 The RHP for integrable integral operators

In this section we will present a Riemann-Hilbert problem for integrable integral operators and the corresponding Fredholm determinants. This Riemann-Hilbert problem was first found by A.Its, A.Izergin, V.Korepin, N.Slavnov in [13]. Let us revisit the kernel $K_n(z,\lambda)$ defined by (1.1.7) and the associated integral operator $\mathbf{K}$:

$$(\mathbf{K}u)(z) := \int_\Sigma K_n(z,\lambda) u(\lambda) d\lambda, \qquad \text{with} \qquad K(z,\lambda) = \frac{f^T(z)h(\lambda)}{z-\lambda}, \tag{1.2.26}$$

where $f, h : \mathbb{C} \to \mathbb{C}^N$, and by f_j and h_j we denote the j-th component of vectors f and g, respectively. One requires $f^T(z)h(z) = 0$ to avoid singularities on the diagonal of the kernel. A key property of operators (1.2.26) is that the Resolvent operator $\mathbf{R} := (\mathbf{1} - \mathbf{K})^{-1} - \mathbf{1}$ also belongs to the class of integrable integral operators, i.e. it can be written as

$$R(z, \lambda) = \frac{F^T(z)H(\lambda)}{z - \lambda}, \tag{1.2.27}$$

where

$$F_j = (\mathbf{1} - \mathbf{K})^{-1} f_j, \qquad H_j = (\mathbf{1} - \mathbf{K}^T)^{-1} h_j. \tag{1.2.28}$$

The vector functions F and H can be computed in terms of a certain matrix Riemann-Hilbert problem. To arrive at this RHP, let us first consider the following $N \times N$ matrix-valued function

$$\mathfrak{Y}(z) = I - \int_\Sigma F(\lambda)h^T(\lambda)\frac{d\lambda}{\lambda - z}. \tag{1.2.29}$$

Thus, by the Plemelj-Sokhotskii formula we have

$$\mathfrak{Y}_+(z) - \mathfrak{Y}_-(z) = -2\pi i F(z)h^T(z), \tag{1.2.30}$$

Since we have assumed that $h^T(z)f(z) = 0$, we get

$$\mathfrak{Y}_+(z)f(z) = \mathfrak{Y}_-(z)f(z). \tag{1.2.31}$$

Also, since $h^T(\lambda)f(z) = f^T(z)h(\lambda)$ is a scalar, we have

$$F(\lambda)h^T(\lambda)f(z) = f^T(z)h(\lambda)F(\lambda). \tag{1.2.32}$$

Using this, (1.2.29) and (1.2.31) we have

$$\begin{aligned} \mathfrak{Y}_\pm(z)f(z) &= f(z) - \int_\Sigma F(\lambda)h^T(\lambda)f(z)\frac{d\lambda}{\lambda - z} = f(z) - \int_\Sigma f^T(z)h(\lambda)F(\lambda)\frac{d\lambda}{\lambda - z} \\ &= f(z) + \int_\Sigma K(z,\lambda)F(\lambda)d\lambda. \end{aligned} \tag{1.2.33}$$

Therefore

$$(\mathbf{K}F)(z) = \mathfrak{Y}_\pm(z)f(z) - f(z). \tag{1.2.34}$$

Note that by definition of F

$$(\mathbf{K}F)(z) = F(z) - f(z),$$

and hence

$$F(z) = \mathfrak{Y}_\pm(z) f(z). \tag{1.2.35}$$

In a similar fashion, we can show that

$$H(z) = (\mathfrak{Y}_\pm^T)^{-1}(z) h(z). \tag{1.2.36}$$

Note that (1.2.30) and (1.2.35) together imply that

$$\mathfrak{Y}_-(z) = \mathfrak{Y}_+(z) \left(I + 2\pi i f(z) h^T(z) \right). \tag{1.2.37}$$

This equation, supplemented by the analytic properties of the Cauchy integral, show that $\mathfrak{Y}(z)$ solves the following $N \times N$ matrix Riemann-Hilbert problem:

- **RH-$\mathfrak{Y}$1** $\mathfrak{Y} : \mathbb{C} \setminus \Sigma \to \mathbb{C}^{N\times N}$ is analytic.
- **RH-$\mathfrak{Y}$2** The limits of $\mathfrak{Y}(\zeta)$ as ζ tends to $z \in \Sigma$ along any non-tangential path exist, and are denoted by $\mathfrak{Y}_\pm(z)$ naturally w.r.t. the orientation of Σ. These limiting values are related by
$$\mathfrak{Y}_-(z) = \mathfrak{Y}_+(z) J_{\mathfrak{Y}}(z), \qquad z \in \Sigma. \tag{1.2.38}$$
- **RH-$\mathfrak{Y}$3** As $z \to \infty$, we have $\mathfrak{Y}(z) = I + \mathcal{O}(z^{-1})$.

In the standard way, one can show that the solution to this Riemann-Hilbert problem, if exists, is unique. Thus, if the solution $\mathfrak{Y}$ of this RHP can be found, then F, H and $\mathbf{R}$ could be found as well via (1.2.35), (1.2.36), and (1.2.27), respectively.

2. ASYMPTOTICS OF HANKEL DETERMINANTS WITH A LAGUERRE-TYPE OR JACOBI-TYPE POTENTIAL AND FISHER-HARTWIG SINGULARITIES

Abstract.

We obtain large n asymptotics of $n \times n$ Hankel determinants whose weight has a one-cut regular potential and Fisher-Hartwig singularities. We restrict our attention to the case where the associated equilibrium measure possesses either one soft edge and one hard edge (Laguerre-type) or two hard edges (Jacobi-type). We also present some applications in the theory of random matrices. In particular, we can deduce from our results asymptotics for partition functions with singularities, central limit theorems, correlations of the characteristic polynomials, and gap probabilities for (piecewise constant) thinned Laguerre and Jacobi-type ensembles. Finally, we mention some links with the topics of rigidity and Gaussian multiplicative chaos. This is a joint work with C. Charlier.

2.1 Introduction

Hankel determinants with Fisher-Hartwig (FH) singularities appear naturally in random matrix theory. Among others, they can express correlations of the characteristic polynomial of a random matrix, or gap probabilities in the point process of the thinned spectrum, see e.g. the introductions of [3, 4, 9] for more details. In these applications, the size n of an $n \times n$ Hankel determinant is equal to the size of the underlying $n \times n$ random matrices. Large n asymptotics for such determinants have already been widely studied, see e.g. [3–5, 14, 15]. Recent developments in the theory of Gaussian multiplicative chaos [15] provide a renewed interest in these asymptotics. For example, such asymptotics provide crucial estimates in the study of rigidity of eigenvalues of a random matrix [16].

In the present work, we restrict our attention on large n asymptotics of Hankel determinants

$$\det\left(\int_{\mathcal{I}} x^{j+k} w(x)dx\right)_{j,k=0,\ldots,n-1}, \tag{2.1.1}$$

whose weight w is supported on an interval $\mathcal{I} \subset \mathbb{R}$, and is of the form

$$w(x) = e^{-nV(x)} e^{W(x)} \omega(x). \tag{2.1.2}$$

The function W is continuous on $\mathcal{I}$ and ω contains the FH singularities (they will be described in more details below). The potential V is real analytic on $\mathcal{I}$ and, in case $\mathcal{I}$ is unbounded, satisfies $\lim_{x\to\pm\infty, x\in\mathcal{I}} V(x)/\log|x| = +\infty$. Furthermore, we assume that V is one-cut and regular. These properties are described in terms of the equilibrium measure μ_V, which is the unique minimizer of the functional

$$\iint \log|x-y|^{-1} d\mu(x)d\mu(y) + \int V(x)d\mu(x) \tag{2.1.3}$$

among all Borel probability measures μ on $\mathcal{I}$. One-cut means that the support of μ_V consists of a single interval. For convenience, and without loss of generality, we will assume that this interval is $[-1,1]$. It is known that μ_V is completely characterized by the Euler-Lagrange variational conditions

$$2\int_{-1}^{1} \log|x-s| d\mu_V(s) = V(x) - \ell, \qquad \text{for } x \in [-1,1], \tag{2.1.4}$$

$$2\int_{-1}^{1} \log|x-s| d\mu_V(s) \leq V(x) - \ell, \qquad \text{for } x \in \mathcal{I}\setminus[-1,1], \tag{2.1.5}$$

where $\ell \in \mathbb{R}$ is a constant. Regular means that the Euler-Lagrange inequality (2.1.5) is strict on $\mathcal{I}\setminus[-1,1]$, and that the density of the equilibrium measure is positive on $(-1,1)$. The three canonical cases are the following:

1. $\mathcal{I} = \mathbb{R}$ and $d\mu_V(x) = \psi(x)\sqrt{1-x^2}dx$,
2. $\mathcal{I} = [-1,\infty)$ and $d\mu_V(x) = \psi(x)\sqrt{\frac{1-x}{1+x}}dx$,
3. $\mathcal{I} = [-1,1]$ and $d\mu_V(x) = \psi(x)\frac{1}{\sqrt{1-x^2}}dx$,

where ψ is real analytic on $\mathcal{I}$, such that $\psi(x) > 0$ for all $x \in [-1,1]$. We will refer to these three cases as Gaussian-type, Laguerre-type and Jacobi-type weights, respectively. Note that (2.1.5) is automatically satisfied for Jacobi-type weights, since $\mathcal{I} = [-1, 1]$. Well-known examples for potentials of such weights are

1. $V(x) = 2x^2$ for Gaussian-type weight, with $\ell = 1 + 2\log 2$ and $\psi(x) = \frac{2}{\pi}$,

2. $V(x) = 2(x+1)$ for Laguerre-type weight, with $\ell = 2 + 2\log 2$ and $\psi(x) = \frac{1}{\pi}$,

3. $V(x) = 0$ for Jacobi-type weight, with $\ell = 2\log 2$ and $\psi(x) = \frac{1}{\pi}$.

In the language of random matrix theory, the interval $(-1, 1)$ is called the bulk, and ± 1 are the edges. An edge is said to be "soft" if there can be eigenvalues beyond it, and "hard" if this is impossible. On the level of the equilibrium measure, a soft edge translates into a square root vanishing of $\frac{d\mu_V}{dx}$, while a hard edge means that $\frac{d\mu_V}{dx}$ blows up like an inverse square root. Thus, there are two soft edges at ± 1 for Gaussian-type weights, one hard edge at -1 and one soft edge at 1 for Laguerre-type weights, and two hard edges for Jacobi-type weights.

The function ω that appears in (2.1.2) is defined by

$$\omega(x) = \prod_{j=1}^{m} \omega_{\alpha_j}(x)\omega_{\beta_j}(x) \times \begin{cases} 1, & \text{for Gaussian-type weights,} \\ (x+1)^{\alpha_0}, & \text{for Laguerre-type weights,} \\ (x+1)^{\alpha_0}(1-x)^{\alpha_{m+1}}, & \text{for Jacobi-type weights,} \end{cases} \tag{2.1.6}$$

where

$$\omega_{\alpha_k}(x) = |x - t_k|^{\alpha_k}, \qquad \omega_{\beta_k}(x) = \begin{cases} e^{i\pi\beta_k}, & \text{if } x < t_k, \\ e^{-i\pi\beta_k}, & \text{if } x > t_k, \end{cases} \tag{2.1.7}$$

with

$$-1 < t_1 < \ldots < t_m < 1. \tag{2.1.8}$$

The functions ω_{α_k} and ω_{β_k} represent the root-type and jump-type singularities at t_k, respectively. These singularities are named after Fisher and Hartwig, due to their pioneering work in their identification [1]. Since $\omega_{\beta_k+1} = -\omega_{\beta_k}$, we can assume without loss of generality that $\Re\beta_k \in (-\frac{1}{2}, \frac{1}{2}]$ for all k. Finally, to ensure integrability of the weight (at least for sufficiently large n), we require that $\Re\alpha_k > -1$ for all k and, in case $\mathcal{I}$ is unbounded, that $W(x) = \mathcal{O}(V(x))$ as $x \to \pm\infty, x \in \mathcal{I}$.

To summarise, the $n \times n$ Hankel determinant given by (2.1.1) depends on n, m, V, W, $\vec{t} = (t_1, \ldots, t_m)$, $\vec{\beta} = (\beta_1, \ldots, \beta_m)$ and $\vec{\alpha}$, where

$$\vec{\alpha} = \begin{cases} (\alpha_1, \ldots, \alpha_m), & \text{for Gaussian-type weight,} \\ (\alpha_0, \alpha_1, \ldots, \alpha_m), & \text{for Laguerre-type weight,} \\ (\alpha_0, \alpha_1, \ldots, \alpha_m, \alpha_{m+1}), & \text{for Jacobi-type weight.} \end{cases}$$

This determinant will be denoted by $G_n(\vec{\alpha}, \vec{\beta}, V, W)$, $L_n(\vec{\alpha}, \vec{\beta}, V, W)$ or $J_n(\vec{\alpha}, \vec{\beta}, V, W)$, depending on whether the weight is of Gaussian, Laguerre or Jacobi-type, respectively.

Many authors have contributed over the years to large n asymptotics for $G_n(\vec{\alpha}, \vec{\beta}, V, W)$ in certain particular cases of the parameters $\vec{\alpha}$, $\vec{\beta}$, V and W (see the introduction of [3] for a global review). The most general result can be found in [3], see also Theorem 2.1.1 below for the precise statement. It is worth to note that these asymptotics are only valid for $\Re\beta_k \in (-\frac{1}{4}, \frac{1}{4})$ and not in the whole strip $\Re\beta_k \in (-\frac{1}{2}, \frac{1}{2}]$. This is due to purely technical reasons, and we comment more on that in Remark 2.1.4 below.

Much less is known about large n asymptotics for $L_n(\vec{\alpha}, \vec{\beta}, V, W)$ and $J_n(\vec{\alpha}, \vec{\beta}, V, W)$, and we briefly discuss this below.

The quantities $L_n(\vec{0}, \vec{0}, V, 0)$ and $J_n(\vec{0}, \vec{0}, V, 0)$ (i.e. no singularities and $W = 0$) represent partition functions of certain random matrix ensembles. In some very special cases of V (like $V(x) = 2(x + 1)$ for Laguerre-type weights and $V(x) = 0$ for Jacobi-type weights), these Hankel determinants reduce to Selberg integrals and are thus computable explicitly. Large n asymptotics for $L_n(\vec{0}, \vec{0}, V, 0)$ and $J_n(\vec{0}, \vec{0}, V, 0)$ for a general V were obtained in [17] (in fact the results of [17] are valid for more general ensembles than we consider). However, we believe our expansions, which are given by Theorem 2.1.2 and Theorem 2.1.3 below with $\vec{\alpha} = \vec{0}$, $\vec{\beta} = \vec{0}$ and $W = 0$, are more explicit (even though less general).

No results are available in the literature for Laguerre-type weight with FH singularities in the bulk (even in the case $V(x) = 2(x + 1)$). There is more known about Jacobi-type weights. Asymptotics for $J_n((\alpha_0, 0, \ldots, 0, \alpha_{m+1}), \vec{0}, 0, W)$ (i.e. root-type singularities only at the edges) were computed in [10], however without the constant term. Major progress were achieved in [9, 14], in which the authors derived large n asymptotics for $J_n(\vec{\alpha}, \vec{\beta}, 0, W)$ including the constant term (under very weak assumption on W, and for general value of $\vec{\beta}$ such that $\Re\beta_k \in (-\frac{1}{2}, \frac{1}{2}])$.

The goal of the present paper is to fill a gap in the literature on large n asymptotics of Hankel determinants with a one-cut potential and FH singularities. In Theorem 2.1.2 and Theorem 2.1.3 below, we find large n asymptotics for $L_n(\vec{\alpha}, \vec{\beta}, V, W)$ and $J_n(\vec{\alpha}, \vec{\beta}, V, W)$ including the constant term. First, we rewrite (in a slightly different way) the result of [3] in Theorem 2.1.1 for the reader's convenience, in order to ease the comparison between the three canonical types of weights.

Theorem 2.1.1 (from [3] for Gaussian-type weight)
Let $m \in \mathbb{N}$, and let t_j, α_j and β_j be such that

$$-1 < t_1 < \ldots < t_m < 1, \qquad \text{and} \qquad \Re\alpha_j > -1, \quad \Re\beta_j \in (-\tfrac{1}{4}, \tfrac{1}{4}) \qquad \text{for } j = 1, \ldots, m.$$

Let V be a one-cut regular potential whose equilibrium measure is supported on $[-1, 1]$ with density $\psi(x)\sqrt{1-x^2}$, and let $W : \mathbb{R} \to \mathbb{R}$ be analytic in a neighbourhood of $[-1, 1]$, locally Hölder-continuous on $\mathbb{R}$ and such that $W(x) = \mathcal{O}(V(x))$, as $|x| \to \infty$. As $n \to \infty$, we have

$$G_n(\vec{\alpha}, \vec{\beta}, V, W) = \exp\left(C_1 n^2 + C_2 n + C_3 \log n + C_4 + \mathcal{O}\Big(\frac{\log n}{n^{1-4\beta_{\max}}}\Big)\right), \tag{2.1.9}$$

with $\beta_{\max} = \max\{|\Re\beta_1|, \ldots, |\Re\beta_m|\}$ and

$$C_1 = -\log 2 - \frac{3}{4} - \frac{1}{2}\int_{-1}^{1}(V(x) - 2x^2)\left(\frac{2}{\pi} + \psi(x)\right)\sqrt{1-x^2}dx, \tag{2.1.10}$$

$$\begin{aligned} C_2 = {} & \log(2\pi) - \mathcal{A}\log 2 - \frac{\mathcal{A}}{2\pi}\int_{-1}^{1}\frac{V(x) - 2x^2}{\sqrt{1-x^2}}dx + \int_{-1}^{1} W(x)\psi(x)\sqrt{1-x^2}dx \\ & + \sum_{j=1}^{m}\frac{\alpha_j}{2}(V(t_j) - 1) + \sum_{j=1}^{m}\pi i\beta_j\left(1 - 2\int_{t_j}^{1}\psi(x)\sqrt{1-x^2}dx\right), \end{aligned} \tag{2.1.11}$$

$$C_3 = -\frac{1}{12} + \sum_{j=1}^{m}\left(\frac{\alpha_j^2}{4} - \beta_j^2\right), \tag{2.1.12}$$

$$
\begin{aligned}
C_4 = {} & \zeta'(-1) - \frac{1}{24}\log\Big(\frac{\pi}{2}\psi(-1)\Big) - \frac{1}{24}\log\Big(\frac{\pi}{2}\psi(1)\Big) + \sum_{j=1}^{m}\left(\frac{\alpha_j^2}{4} - \beta_j^2\right)\log\Big(\frac{\pi}{2}\psi(t_j)\Big) \\
& + \sum_{1\le j<k\le m}\left[\log\left(\frac{\big(1 - t_jt_k - \sqrt{(1-t_j^2)(1-t_k^2)}\big)^{2\beta_j\beta_k}}{2^{\frac{\alpha_j\alpha_k}{2}}|t_j - t_k|^{\frac{\alpha_j\alpha_k}{2}+2\beta_j\beta_k}}\right) + \frac{i\pi}{2}(\alpha_k\beta_j - \alpha_j\beta_k)\right] \\
& + \sum_{j=1}^{m}\left(\frac{\alpha_j^2}{4}\log\big(2\sqrt{1-t_j^2}\big) - \beta_j^2\log\Big(8(1-t_j^2)^{3/2}\Big)\right) + \mathcal{A}\sum_{j=1}^{m} i\beta_j \arcsin t_j \\
& + \sum_{j=1}^{m}\log\frac{G(1+\frac{\alpha_j}{2}+\beta_j)G(1+\frac{\alpha_j}{2}-\beta_j)}{G(1+\alpha_j)} \qquad (2.1.13) \\
& + \frac{\mathcal{A}}{2\pi}\int_{-1}^{1}\frac{W(x)}{\sqrt{1-x^2}}dx - \sum_{j=1}^{m}\frac{\alpha_j}{2}W(t_j) + \sum_{j=1}^{m}\frac{i\beta_j}{\pi}\sqrt{1-t_j^2}\,⨍_{-1}^{1}\frac{W(x)}{\sqrt{1-x^2}(t_j - x)}dx \\
& + \frac{1}{4\pi^2}\int_{-1}^{1}\frac{W(x)}{\sqrt{1-x^2}}\left(⨍_{-1}^{1}\frac{W'(y)\sqrt{1-y^2}}{x-y}dy\right)dx,
\end{aligned}
$$

where G is Barnes' G-function, ζ is Riemann's zeta-function, where we use the notations ⨍ for the Cauchy principal value integral, and

$$
\mathcal{A} = \sum_{j=1}^{m}\alpha_j. \qquad (2.1.14)
$$

Furthermore, the error term in (2.1.9) is uniform for all α_k in compact subsets of $\{z \in \mathbb{C} : \Re z > -1\}$, for all β_k in compact subsets of $\{z \in \mathbb{C} : \Re z \in \left(\frac{-1}{4}, \frac{1}{4}\right)\}$, and uniform in $t_1, \ldots, t_m$, as long as there exists $\delta > 0$ independent of n such that

$$
\min_{j\neq k}\{|t_j - t_k|, |t_j - 1|, |t_j + 1|\} \ge \delta. \qquad (2.1.15)
$$

Theorem 2.1.2 (for Laguerre-type weight)

Let $m \in \mathbb{N}$, and let t_j, α_j and β_j be such that

$$
-1 = t_0 < t_1 < \ldots < t_m < 1, \quad \text{and} \quad \Re\alpha_j > -1, \quad \Re\beta_j \in (-\tfrac{1}{4}, \tfrac{1}{4}) \quad \text{for } j = 0, \ldots, m,
$$

with $\beta_0 = 0$. Let V be a one-cut regular potential whose equilibrium measure is supported on $[-1, 1]$ with density $\psi(x)\sqrt{\frac{1-x}{1+x}}$, and let $W : \mathbb{R}^+ \to \mathbb{R}$ be analytic in a neighbourhood of $[-1, 1]$, locally Hölder-continuous on $\mathbb{R}^+$ and such that $W(x) = \mathcal{O}(V(x))$, as $x \to +\infty$. As $n \to \infty$, we have

$$
L_n(\vec{\alpha}, \vec{\beta}, V, W) = \exp\left(C_1n^2 + C_2n + C_3\log n + C_4 + \mathcal{O}\Big(\frac{\log n}{n^{1-4\beta_{\max}}}\Big)\right), \qquad (2.1.16)
$$

with $\beta_{\max} = \max\{|\Re\beta_1|, \dots, |\Re\beta_m|\}$ *and*

$$C_1 = -\log 2 - \frac{3}{2} - \frac{1}{2}\int_{-1}^{1}(V(x) - 2(x+1))\left(\frac{1}{\pi} + \psi(x)\right)\sqrt{\frac{1-x}{1+x}}dx, \tag{2.1.17}$$

$$C_2 = \log(2\pi) - \mathcal{A}\log 2 - \frac{\mathcal{A}}{2\pi}\int_{-1}^{1}\frac{V(x) - 2(x+1)}{\sqrt{1-x^2}}dx + \int_{-1}^{1}W(x)\psi(x)\sqrt{\frac{1-x}{1+x}}dx \tag{2.1.18}$$
$$+ \sum_{j=0}^{m}\frac{\alpha_j}{2}(V(t_j) - 2) + \sum_{j=1}^{m}\pi i\beta_j\left(1 - 2\int_{t_j}^{1}\psi(x)\sqrt{\frac{1-x}{1+x}}dx\right),$$

$$C_3 = -\frac{1}{6} + \frac{\alpha_0^2}{2} + \sum_{j=1}^{m}\left(\frac{\alpha_j^2}{4} - \beta_j^2\right), \tag{2.1.19}$$

$$C_4 = 2\zeta'(-1) - \frac{1-4\alpha_0^2}{8}\log\left(\pi\psi(-1)\right) - \frac{1}{24}\log\left(\pi\psi(1)\right) + \sum_{j=1}^{m}\left(\frac{\alpha_j^2}{4} - \beta_j^2\right)\log\left(\pi\psi(t_j)\right)$$
$$+ \frac{\alpha_0}{2}\log(2\pi) + \sum_{0\le j<k\le m}\left[\log\left(\frac{\left(1 - t_jt_k - \sqrt{(1-t_j^2)(1-t_k^2)}\right)^{2\beta_j\beta_k}}{2^{\frac{\alpha_j\alpha_k}{2}}|t_j - t_k|^{\frac{\alpha_j\alpha_k}{2} + 2\beta_j\beta_k}}\right) + \frac{i\pi}{2}(\alpha_k\beta_j - \alpha_j\beta_k)\right]$$
$$+ \sum_{j=1}^{m}\left(\frac{\alpha_j^2}{4}\log\sqrt{\frac{1-t_j}{1+t_j}} - \beta_j^2\log\left(4(1-t_j)^{3/2}(1+t_j)^{1/2}\right)\right) + \mathcal{A}\sum_{j=1}^{m}i\beta_j\arcsin t_j$$
$$- \log G(1+\alpha_0) + \sum_{j=1}^{m}\log\frac{G(1+\frac{\alpha_j}{2}+\beta_j)G(1+\frac{\alpha_j}{2}-\beta_j)}{G(1+\alpha_j)} \tag{2.1.20}$$
$$+ \frac{\mathcal{A}}{2\pi}\int_{-1}^{1}\frac{W(x)}{\sqrt{1-x^2}}dx - \sum_{j=0}^{m}\frac{\alpha_j}{2}W(t_j) + \sum_{j=1}^{m}\frac{i\beta_j}{\pi}\sqrt{1-t_j^2}\,⨍_{-1}^{1}\frac{W(x)}{\sqrt{1-x^2}(t_j - x)}dx$$
$$+ \frac{1}{4\pi^2}\int_{-1}^{1}\frac{W(x)}{\sqrt{1-x^2}}\left(⨍_{-1}^{1}\frac{W'(y)\sqrt{1-y^2}}{x-y}dy\right)dx,$$

where G *is Barnes' G-function,* ζ *is Riemann's zeta-function, where we use the notations* $⨍$ *for the Cauchy principal value integral, and*

$$\mathcal{A} = \sum_{j=0}^{m}\alpha_j. \tag{2.1.21}$$

Furthermore, the error term in (2.1.16) *is uniform for all* α_k *in compact subsets of* $\{z \in \mathbb{C} : \Re z > -1\}$*, for all* β_k *in compact subsets of* $\{z \in \mathbb{C} : \Re z \in \left(\frac{-1}{4}, \frac{1}{4}\right)\}$*, and uniform in* $t_1, \dots, t_m$*, as long as there exists* $\delta > 0$ *independent of* n *such that*

$$\min_{j\neq k}\{|t_j - t_k|, |t_j - 1|, |t_j + 1|\} \ge \delta. \tag{2.1.22}$$

Theorem 2.1.3 (for Jacobi-type weight)

Let $m \in \mathbb{N}$, and let t_j, α_j and β_j be such that

$$-1 = t_0 < t_1 < \ldots < t_m < t_{m+1} = 1, \quad \text{and} \quad \Re\alpha_j > -1, \quad \Re\beta_j \in (-\tfrac{1}{4}, \tfrac{1}{4}) \quad \text{for } j = 0, \ldots, m+1,$$

with $\beta_0 = 0 = \beta_{m+1}$. Let V be a one-cut regular potential whose equilibrium measure is supported on $[-1,1]$ with density $\frac{\psi(x)}{\sqrt{1-x^2}}$, and let $W : [-1,1] \to \mathbb{R}$ be analytic in a neighbourhood of $[-1,1]$.

As $n \to \infty$, we have

$$J_n(\vec{\alpha}, \vec{\beta}, V, W) = \exp\left(C_1 n^2 + C_2 n + C_3 \log n + C_4 + \mathcal{O}\Big(\frac{\log n}{n^{1-4\beta_{\max}}}\Big)\right), \tag{2.1.23}$$

with $\beta_{\max} = \max\{|\Re\beta_1|, \ldots, |\Re\beta_m|\}$ and

$$C_1 = -\log 2 - \frac{1}{2}\int_{-1}^{1} V(x)\left(\frac{1}{\pi} + \psi(x)\right)\frac{dx}{\sqrt{1-x^2}}, \tag{2.1.24}$$

$$C_2 = \log(2\pi) - \mathcal{A}\log 2 - \frac{\mathcal{A}}{2\pi}\int_{-1}^{1}\frac{V(x)}{\sqrt{1-x^2}}dx + \int_{-1}^{1} W(x)\frac{\psi(x)}{\sqrt{1-x^2}}dx \tag{2.1.25}$$
$$+ \sum_{j=0}^{m+1}\frac{\alpha_j}{2}V(t_j) + \sum_{j=1}^{m}\pi i\beta_j\left(1 - 2\int_{t_j}^{1}\frac{\psi(x)}{\sqrt{1-x^2}}dx\right),$$

$$C_3 = -\frac{1}{4} + \frac{\alpha_0^2 + \alpha_{m+1}^2}{2} + \sum_{j=1}^{m}\left(\frac{\alpha_j^2}{4} - \beta_j^2\right), \tag{2.1.26}$$

$$C_4 = 3\zeta'(-1) + \frac{\log 2}{12} - \frac{1-4\alpha_0^2}{8}\log(\pi\psi(-1)) - \frac{1-4\alpha_{m+1}^2}{8}\log(\pi\psi(1)) + \sum_{j=1}^{m}\left(\frac{\alpha_j^2}{4} - \beta_j^2\right)\log(\pi\psi(t_j))$$
$$+ \frac{\alpha_0 + \alpha_{m+1}}{2}\log(2\pi) + \sum_{0 \le j < k \le m+1}\left[\log\left(\frac{\left(1 - t_j t_k - \sqrt{(1-t_j^2)(1-t_k^2)}\right)^{2\beta_j\beta_k}}{2^{\frac{\alpha_j\alpha_k}{2}}|t_j - t_k|^{\frac{\alpha_j\alpha_k}{2} + 2\beta_j\beta_k}}\right) + \frac{i\pi}{2}(\alpha_k\beta_j - \alpha_j\beta_k)\right]$$
$$+ \sum_{j=1}^{m}\left(\frac{\alpha_j^2}{4}\log\frac{1}{\sqrt{1-t_j^2}} - \beta_j^2\log\left(4\sqrt{1-t_j^2}\right)\right) + \mathcal{A}\sum_{j=1}^{m} i\beta_j \arcsin t_j - \frac{\alpha_0^2 + \alpha_{m+1}^2}{2}\log 2$$
$$- \log G(1+\alpha_0) - \log G(1+\alpha_{m+1}) + \sum_{j=1}^{m}\log\frac{G(1 + \frac{\alpha_j}{2} + \beta_j)G(1 + \frac{\alpha_j}{2} - \beta_j)}{G(1+\alpha_j)} \tag{2.1.27}$$
$$+ \frac{\mathcal{A}}{2\pi}\int_{-1}^{1}\frac{W(x)}{\sqrt{1-x^2}}dx - \sum_{j=0}^{m+1}\frac{\alpha_j}{2}W(t_j) + \sum_{j=1}^{m}\frac{i\beta_j}{\pi}\sqrt{1-t_j^2}\,⨍_{-1}^{1}\frac{W(x)}{\sqrt{1-x^2}(t_j - x)}dx$$
$$+ \frac{1}{4\pi^2}\int_{-1}^{1}\frac{W(x)}{\sqrt{1-x^2}}\left(⨍_{-1}^{1}\frac{W'(y)\sqrt{1-y^2}}{x-y}dy\right)dx,$$

where G is Barnes' G-function, ζ is Riemann's zeta-function, where we use the notations $⨍$ for the Cauchy principal value integral, and

$$\mathcal{A} = \sum_{j=0}^{m+1} \alpha_j. \tag{2.1.28}$$

Furthermore, the error term in (2.1.23) is uniform for all α_k in compact subsets of $\{z \in \mathbb{C} : \Re z > -1\}$, for all β_k in compact subsets of $\{z \in \mathbb{C} : \Re z \in \left(\frac{-1}{4}, \frac{1}{4}\right)\}$, and uniform in $t_1, \ldots, t_m$, as long as there exists $\delta > 0$ independent of n such that

$$\min_{j \neq k} \{|t_j - t_k|, |t_j - 1|, |t_j + 1|\} \geq \delta. \tag{2.1.29}$$

Remark 2.1.4 *The assumption $\Re\beta_k \in (-\frac{1}{4}, \frac{1}{4})$ comes from some technicalities in our analysis. Similar difficulties were encountered in [5] for $G_n(0, \vec{\beta}, 2x^2, 0)$ with $m = 1$ (i.e. $\vec{\beta} = \beta_1$), and in [14] for $J_n(\vec{\alpha}, \vec{\beta}, 0, W)$. In [14], the authors overcame these technicalities, and were able to extend their results from $\Re\beta_k \in (-\frac{1}{4}, \frac{1}{4})$ to $\Re\beta_k \in (-\frac{1}{2}, \frac{1}{2})$ by using Vitali's theorem. Their argument relies crucially on w being independent of n (which is true only for Jacobi-type weights with $V = 0$) and can not be adapted straightforwardly to the situation of Theorem 2.1.1, 2.1.2 and 2.1.3. However, the method presented in this paper allows in principle, but with significant extra effort, to obtain asymptotics for the whole region $\Re\beta_k \in (-\frac{1}{2}, \frac{1}{2})$. Finally, extending the result from $\Re\beta_k \in (-\frac{1}{2}, \frac{1}{2})$ to $\Re\beta_k \in (-\frac{1}{2}, \frac{1}{2}]$ would rely on so-called FH representations of the weight, see [9] for more details.*

Remark 2.1.5 *Starting with a function f defined on the unit circle, the associated Toeplitz determinant is given by*

$$\det\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta}) e^{-i(j-k)\theta} d\theta\right)_{j,k=0,\ldots,n-1}. \tag{2.1.30}$$

Asymptotics of large Toeplitz determinants is another topic of high interest, which presents applications similar to those of Hankel determinants, but for point processes defined on the unit circle instead of the real line. In [9], the authors obtained first large n asymptotics for certain Toeplitz determinants (with the zero potential), and deduced from them large n asymptotics for $J_n(\vec{\alpha}, \vec{\beta}, 0, W)$. It is therefore natural to wonder if one can translate the *results of Theorem 2.1.1, 2.1.2 and 2.1.3 into asymptotics for Toeplitz determinants with a one-cut regular potential. We explain here why we believe this is not obvious.*

The main tool used in [9] is a relation of Szegö [18] . If

$$f(e^{i\theta}) = w(\cos\theta)|\sin\theta|, \tag{2.1.31}$$

we can express orthogonal polynomials on the unit circle associated to f in terms of orthogonal polynomials on the real line associated to w. Note that this transformation can only work in all generality from Toeplitz to Hankel, and not the other way around. Indeed, the weight w can be arbitrary, but the function f is of a very particular type (in particular it satisfies $f(e^{i\theta}) = f(e^{-i\theta})$).

We also believe that asymptotics for Toeplitz determinants with a one-cut regular potential and FH singularities would not imply Theorem 2.1.1, 2.1.2 and 2.1.3 (with the exception of $V = 0$ for Jacobi-type weights as done in [9]). The main reason is that, as shown from the change of variables $s = \cos\theta$ in (2.1.4), the potential $\widehat{V}$ on the unit circle is related to the potential V on the interval $[-1,1]$ via the relation $\widehat{V}(e^{i\theta}) = V(\cos\theta)$, which means that at least one potential is not analytic (except if V is a constant as in [9]). Finally, we also point out that regarding e.g. Gaussian-type weights, again the change of variables $s = \cos\theta$ in (2.1.4) shows that the associated equilibrium measure $\mu_{\widehat{V}}$ on the unit circle vanishes as a square at $\theta = 0$ and $\theta = \pi$, which is not a "regular" weight. To avoid this problem, one could by a simple change of variables shrink the support of μ_V into $[-a,a]$ with $0 < a < 1$, but then $\mu_{\widehat{V}}$ would be supported on two disjoint intervals.

Applications

In this section, we provide several applications of Theorem 2.1.1, Theorem 2.1.2 and Theorem 2.1.3 in random matrix theory. For each type of weight, there corresponds a particular type of matrix ensemble. Assume that V is a Gaussian-type potential. The associated

Gaussian-type matrix ensemble consists of the space of $n \times n$ complex Hermitian matrices endowed with the probability measure

$$\frac{1}{\widehat{Z}_n^G} e^{-n\mathrm{Tr}(V(M))} dM, \qquad dM = \prod_{i=1}^{n} dM_{ii} \prod_{1 \le i < j \le n} d\Re M_{ij} d\Im M_{ij}, \tag{2.1.32}$$

with $\widehat{Z}_n^G$ the normalizing constant. Laguerre-type matrix ensembles are usually defined on $n \times n$ complex positive definite Hermitian matrices. Here we instead assume, for Laguerre-type matrix ensembles, that all matrices have eigenvalues geater than -1 (this assumption eases the comparison between the three cases). Such ensembles have a probability measure of the form

$$\frac{1}{\widehat{Z}_n^L} \det(I+M)^{\alpha_0} e^{-n\mathrm{Tr}(V(M))} dM, \qquad \alpha_0 > -1, \tag{2.1.33}$$

where V is of Laguerre-type, and $\widehat{Z}_n^L$ is the normalizing constant. Finally, a Jacobi-type matrix ensemble consists of the space of $n \times n$ Hermitian matrices whose spectrum lies the interval $[-1, 1]$, with a probability measure of the form

$$\frac{1}{\widehat{Z}_n^J} \det(I+M)^{\alpha_0} \det(I-M)^{\alpha_{m+1}} e^{-n\mathrm{Tr}(V(M))} dM, \qquad \alpha_0, \alpha_{m+1} > -1, \tag{2.1.34}$$

with a Jacobi-type potential V and $\widehat{Z}_n^J$ is again the normalizing constant. These three types of matrix ensembles are invariant under unitary conjugation and induce the following probability measures on the eigenvalues $x_1, \ldots, x_n$:

$$\frac{1}{Z_n^G} \prod_{1 \le j < k \le n} (x_k - x_j)^2 \prod_{j=1}^{n} e^{-nV(x_j)} dx_j, \qquad x_1, \ldots, x_n \in \mathbb{R}, \tag{2.1.35}$$

$$\frac{1}{Z_n^L} \prod_{1 \le j < k \le n} (x_k - x_j)^2 \prod_{j=1}^{n} (1+x_j)^{\alpha_0} e^{-nV(x_j)} dx_j, \qquad x_1, \ldots, x_n \in [-1, \infty), \tag{2.1.36}$$

$$\frac{1}{Z_n^J} \prod_{1 \le j < k \le n} (x_k - x_j)^2 \prod_{j=1}^{n} (1+x_j)^{\alpha_0} (1-x_j)^{\alpha_{m+1}} e^{-nV(x_j)} dx_j, \quad x_1, \ldots, x_n \in [-1, 1], \tag{2.1.37}$$

where the first, second and third line read for Gaussian, Laguerre, and Jacobi-type matrix ensembles, respectively, and Z_n^G, Z_n^L and Z_n^J are the normalizing constants, also called the partition functions.

Partition function asymptotics in the one-cut regime. By Heine's formula, the partition functions can be rewritten as Hankel determinants of the form (2.1.1) with $W = 0$, $\vec{\beta} = \vec{0}$ and $\alpha_1 = ... = \alpha_m = 0$ and can thus be deduced from theorems 2.1.1, 2.1.2 and 2.1.3. Large n asymptotics for Z_n^G have been obtained in some particular cases of V in [19, 20] using RH methods. Then large n asymptotics for Z_n^G, Z_n^L and Z_n^J were all obtained in [17] using loop equations, however these asymptotics are valid only without singularities, i.e. only for $\alpha_0 = 0$ for Z_n^L and only for $\alpha_0 = \alpha_{m+1} = 0$ for Z_n^J. Finally, via RH methods, large n asymptotics for Z_n^G have been obtained only recently in [15] for general potential V.

Corollary 2.1.5.1 *As $n \to +\infty$, we have*

$$\begin{aligned} Z_n^G = \quad & \exp\Bigg(- \bigg(\log 2 + \frac{3}{4} + \frac{1}{2} \int_{-1}^{1} (V(x) - 2x^2) \Big(\frac{2}{\pi} + \psi(x) \Big) \sqrt{1-x^2} dx \bigg) n^2 \qquad (2.1.38) \\ & + \log(2\pi) n - \frac{1}{12} \log n + \zeta'(-1) - \frac{1}{24} \log \Big(\frac{\pi}{2} \psi(-1) \Big) - \frac{1}{24} \log \Big(\frac{\pi}{2} \psi(1) \Big) + \mathcal{O}\Big(\frac{\log n}{n} \Big) \Bigg), \\ Z_n^L = \quad & \exp\Bigg(- \bigg(\log 2 + \frac{3}{2} + \frac{1}{2} \int_{-1}^{1} (V(x) - 2(x+1)) \Big(\frac{1}{\pi} + \psi(x) \Big) \sqrt{\frac{1-x}{1+x}} dx \bigg) n^2 \qquad (2.1.39) \\ & + \bigg(\log(2\pi) - \alpha_0 \log 2 - \frac{\alpha_0}{2\pi} \int_{-1}^{1} \frac{V(x) - 2(x+1)}{\sqrt{1-x^2}} dx + \frac{\alpha_0}{2} (V(-1) - 2) \bigg) n + \Big(\frac{\alpha_0^2}{2} - \frac{1}{6} \Big) \log n \\ & + 2\zeta'(-1) - \frac{1-4\alpha_0^2}{8} \log \big(\pi \psi(-1) \big) - \frac{1}{24} \log(\pi \psi(1) + \frac{\alpha_0}{2} \log(2\pi) - \log G(1+\alpha_0) + \mathcal{O}\Big(\frac{\log n}{n} \Big) \Bigg), \end{aligned}$$

$$\begin{aligned} Z_n^J = \quad & \exp\Bigg(- \bigg(\log 2 + \frac{1}{2} \int_{-1}^{1} V(x) \Big(\frac{1}{\pi} + \psi(x) \Big) \frac{dx}{\sqrt{1-x^2}} \bigg) n^2 \qquad (2.1.40) \\ & + \bigg(\log(2\pi) - (\alpha_0 + \alpha_{m+1}) \log 2 - \frac{\alpha_0 + \alpha_{m+1}}{2\pi} \int_{-1}^{1} \frac{V(x)}{\sqrt{1-x^2}} dx + \frac{\alpha_0}{2} V(-1) + \frac{\alpha_{m+1}}{2} V(1) \bigg) n \\ & + \bigg(-\frac{1}{4} + \frac{\alpha_0^2 + \alpha_{m+1}^2}{2} \bigg) \log n + 3\zeta'(-1) + \frac{\log 2}{12} - \frac{1-4\alpha_0^2}{8} \log \big(\pi \psi(-1) \big) - \frac{1-4\alpha_{m+1}^2}{8} \log \big(\pi \psi(1) \big) \\ & + \frac{\alpha_0 + \alpha_{m+1}}{2} \log(2\pi) - \frac{(\alpha_0 + \alpha_{m+1})^2}{2} \log 2 - \log \big(G(1+\alpha_0) G(1+\alpha_{m+1}) \big) + \mathcal{O}\Big(\frac{\log n}{n} \Big) \Bigg). \end{aligned}$$

Central limit theorems (CLTs). The function W allows to obtain information about the global fluctuation properties of the spectrum around the equilibrium measure. In [21], Johansson obtained a CLT for Gaussian-type ensembles (and is reproduced in (2.1.41) below for convenience). Until now, there were no CLTs in the literature for Laguerre and Jacobi-type ensembles. These CLTs are obtained in Corollary 2.1.5.2 below, as a rather straightforward consequence of Theorem 2.1.2 and Theorem 2.1.3.

Corollary 2.1.5.2 *(a) Let $x_1, \ldots, x_n$ be distributed according to* (2.1.35) *and V and W be as in Theorem 2.1.1. As $n \to +\infty$, we have*

$$\sum_{i=1}^{n} W(x_i) - n\int_{-1}^{1} W(x)\psi(x)\sqrt{1-x^2}dx \quad \xrightarrow{d} \quad \mathcal{N}(0,\sigma^2), \tag{2.1.41}$$

where $\xrightarrow{d}$ means convergence in distribution, and $\mathcal{N}(0,\sigma^2)$ is a zero-mean normal random variable with variance given by

$$\sigma^2 = \frac{1}{2\pi^2}\int_{-1}^{1} \frac{W(x)}{\sqrt{1-x^2}}\left(⨍_{-1}^{1} \frac{W'(y)\sqrt{1-y^2}}{x-y}dy\right)dx. \tag{2.1.42}$$

(b) Let $x_1, \ldots, x_n$ be distributed according to (2.1.36) *and V and W be as in Theorem 2.1.2. As $n \to +\infty$, we have*

$$\sum_{i=1}^{n} W(x_i) - n\int_{-1}^{1} W(x)\psi(x)\sqrt{\frac{1-x}{1+x}}dx \quad \xrightarrow{d} \quad \mathcal{N}(\mu_L,\sigma^2), \tag{2.1.43}$$

where σ^2 is given by (2.1.42) *and the mean μ_L is given by*

$$\mu_L = \frac{\alpha_0}{2\pi}\int_{-1}^{1}\frac{W(x)}{\sqrt{1-x^2}}dx - \frac{\alpha_0}{2}W(-1). \tag{2.1.44}$$

(c) Let $x_1, \ldots, x_n$ be distributed according to (2.1.37) *and V and W be as in Theorem 2.1.3. As $n \to +\infty$, we have*

$$\sum_{i=1}^{n} W(x_i) - n\int_{-1}^{1} W(x)\frac{\psi(x)}{\sqrt{1-x^2}}dx \quad \xrightarrow{d} \quad \mathcal{N}(\mu_J,\sigma^2), \tag{2.1.45}$$

where σ^2 is given by (2.1.42) *and the mean μ_J is given by*

$$\mu_J = \frac{\alpha_0+\alpha_{m+1}}{2\pi}\int_{-1}^{1}\frac{W(x)}{\sqrt{1-x^2}}dx - \frac{\alpha_0}{2}W(-1) - \frac{\alpha_{m+1}}{2}W(1). \tag{2.1.46}$$

Proof We only prove the result for Jacobi-type ensembles. The proofs for the other cases are similar. From Heine's formula, we have

$$\mathbb{E}_J\left[e^{t\sum_{j=1}^{n}W(x_j)}\right] = \frac{J_n\big((\alpha_0,0,...,0,\alpha_{m+1}),\vec{0},V,tW\big)}{J_n\big((\alpha_0,0,...,0,\alpha_{m+1}),\vec{0},V,0\big)}, \qquad t\in\mathbb{R}, \tag{2.1.47}$$

where $\mathbb{E}_J$ means that the expectation is taken with respect to (2.1.37). Let X_n be the random variable defined by

$$X_n = \sum_{j=1}^{n} W(x_j) - n \int_{-1}^{1} W(x) \frac{\psi(x)}{\sqrt{1-x^2}} dx. \tag{2.1.48}$$

Theorem 2.1.3 then implies

$$\mathbb{E}_J \left[e^{tX_n} \right] = \exp\left(t\mu_J + \frac{t^2}{2}\sigma^2 + \mathcal{O}\Big(\frac{\log n}{n}\Big) \right), \qquad \text{as } n \to +\infty. \tag{2.1.49}$$

Thus, for each $t \in \mathbb{R}$, (X_n) is a sequence of random variables whose moment generating functions converge to $e^{t\mu_J + \frac{t^2}{2}\sigma^2}$ as $n \to +\infty$ (the convergence is pointwise in $t \in \mathbb{R}$). Convergence in distribution follows from well-known convergence theorems (see e.g. [22]).

■

Correlations of the characteristic polynomials. Let $p_n(t) = \prod_{j=1}^{n}(t - x_j)$ be the characteristic polynomial associated to a matrix from a Gaussian-type, Laguerre-type or Jacobi-type ensemble. Supported by numerical evidence, numerous conjectures in the literature have been formulated about links between $p_n(t)$ and the behavior of the Riemann ζ-functions along the critical line (see e.g. [23]). For Gaussian-type ensembles, correlations with root-type singularities were studied in [4] for $V(x) = 2x^2$ and in [15] for general V. Large n asymptotics for more general correlations with both root-type and jump-type singularities were obtained in [3]. However, the cases of Laguerre or Jacobi-type ensembles were still open. In the same way as noticed in [3, equation (1.16)], we can express these correlations in terms of Hankel determinants with FH singularities as follows[1]:

$$\mathbb{E}_D \left[\prod_{k=1}^{m} |p_n(t_k)|^{\alpha_k} e^{2i\beta_k \arg p_n(t_k)} \right] = \frac{D_n(\vec{\alpha}, \vec{\beta}, V, 0)}{Z_n^D} \prod_{k=1}^{m} e^{-in\pi\beta_k}, \qquad D = G, L, J, \tag{2.1.50}$$

where $\mathbb{E}_G$, $\mathbb{E}_L$ and $\mathbb{E}_J$ are the expectations taken with respect to (2.1.35), (2.1.36) and (2.1.37), respectively, and where

$$\arg p_n(t) = \sum_{j=1}^{n} \arg(t - x_j), \qquad \text{with} \qquad \arg(t - x_j) = \begin{cases} 0, & \text{if } x_j < t, \\ -\pi, & \text{if } x_j > t. \end{cases} \tag{2.1.51}$$

[1]There is a n missing in [3, equations (1.16) and (1.22)]: $e^{-i\pi\beta_k}$ should instead be $e^{-in\pi\beta_k}$ and $s_k^{1/2}$ should instead be $s_k^{n/2}$. The correct expressions are given by (2.1.50) and (2.1.52) of the present work.

Therefore, as an immediate corollary of Theorem 2.1.2 and Theorem 2.1.3, we obtain large n asymptotics for the correlations given in (2.1.50) for Laguerre and Jacobi-type ensembles.

Gap probabilities in piecewise constant thinned point processes. Given a point process, a constant thinning consists of removing each point independently with a certain probability $s \in [0,1]$. The remaining points, denoted by $y_1, \ldots, y_N$, form a thinned point process, and can be interpreted in certain applications as observed points [24, 25]. Probabilities of observing a large gap in the thinned sine point process, as well as for thinned eigenvalues of Haar distributed unitary matrices, have been studied in [26] and [27], respectively. A more general operation consists of applying a piecewise constant thinning, and was first considered in [3] for Gaussian-type ensembles. Large gap asymptotics for the piecewise constant thinned Airy and Bessel point processes were obtained recently in [28] and [29], respectively. From Theorem 2.1.2 and Theorem 2.1.3, we can deduce large gap asymptotics for (piecewise constant) thinned Laguerre and Jacobi-type ensembles. Following [3], we consider $\mathcal{K} \subseteq \{1, ..., m+1\}$. For each $k \in \mathcal{K}$, we remove each point on (t_{k-1}, t_k) with a probability $s_k \in (0,1]$. In the same way as shown in [3, equations (1.20)–(1.22)], we can express gap probabilities in the piecewise thinned spectrum of Gaussian, Laguerre and Jacobi-type ensembles as follows:

$$\mathbb{P}_D\Big(\sharp\{y_j \in \bigcup_{k\in\mathcal{K}}(t_{k-1}, t_k)\} = 0\Big) = \frac{D_n(\vec{\alpha}, \vec{\beta}, V, 0)}{Z_n^D} \prod_{k\in\mathcal{K}} s_k^{n/2}, \qquad D = G, L, J, \tag{2.1.52}$$

with $\alpha_1 = ... = \alpha_m = 0$ and $\vec{\beta} = (\beta_1, ..., \beta_m)$ given by

$$2i\pi\beta_j = \log\left(\frac{\widetilde{s}_j}{\widetilde{s}_{j+1}}\right), \qquad \widetilde{s}_j = \begin{cases} s_j, & \text{if } j \in \mathcal{K}, \\ 1, & \text{if } j \notin \mathcal{K}, \end{cases} \tag{2.1.53}$$

and where again $\mathbb{P}_G$, $\mathbb{P}_L$ and $\mathbb{P}_J$ are probabilities taken with respect to (2.1.35), (2.1.36) and (2.1.37), respectively.

Rigidity and Gaussian multiplicative chaos. Let us consider a sequence of matrices M_n taken from either Gaussian, Laguerre, or Jacobi-type ensembles. As $n \to +\infty$, the logarithm of the characteristic polynomial of M_n behaves like a log-correlated field. A fundamental tool in describing some properties of the limiting field is a class of random measures, known as

Gaussian multiplicative chaos measures. Roughly speaking, these measures are exponential of the field, however a precise definition is rather subtle. This subject was introduced by Kahane in [30], and we refer to [31] for a recent review. For Gaussian-type ensembles, it is known (from [15]) that a sufficiently small power of the absolute value of the characteristic polynomial converges weakly in distribution to a Gaussian multiplicative chaos measure. Large n asymptotics for Hankel determinants with root-type singularities provide crucial estimates in the proof. Theorem 2.1.2 and Theorem 2.1.3 provide similar estimates for Laguerre and Jacobi-type ensembles, which could probably be used to prove analogous results for the Laguerre and Jacobi cases. Another related topic is the study of rigidity, which attempts to answer the question: "How much can the eigenvalues of a random matrix fluctuate?". For Gaussian-type ensembles, this question has been answered in [16]. This time, it is large n asymptotics for Hankel determinants with jump-type singularities that are crucial in the analysis. In particular, the proof of [16] relies heavily on Theorem 2.1.1 (with $\vec{\alpha} = \vec{0}$). Theorem 2.1.2 and Theorem 2.1.3 provide similar estimates for Laguerre and Jacobi-type ensembles, which we believe are relevant to prove similar rigidity results for these ensembles.

Outline

The general strategy of our proof is close to the one done in [3], and can be schematized as

$$\begin{array}{ccccccc} L_n(\vec{0},\vec{0},2(x+1),0) & \mapsto & L_n(\vec{\alpha},\vec{\beta},2(x+1),0) & \mapsto & L_n(\vec{\alpha},\vec{\beta},V,0) & \mapsto & L_n(\vec{\alpha},\vec{\beta},V,W), \\ & & J_n(\vec{\alpha},\vec{\beta},0,0) & \mapsto & J_n(\vec{\alpha},\vec{\beta},V,0) & \mapsto & J_n(\vec{\alpha},\vec{\beta},V,W). \end{array} \tag{2.1.54}$$

In Section 2.2, we recall a well-known correspondence between Hankel determinants and orthogonal polynomials (OPs), and the characterization of these OPs in terms of a Riemann-Hilbert (RH) problem found by Fokas, Its and Kitaev [6], and whose solution is denoted by Y. In Section 2.3, we derive suitable differential identities, which express the quantities

$$\begin{array}{lll} \partial_\nu \log L_n(\vec{\alpha},\vec{\beta},2(x+1),0), & \partial_s \log L_n(\vec{\alpha},\vec{\beta},V_s,0), & \partial_t \log L_n(\vec{\alpha},\vec{\beta},V,W_t), \\ & \partial_s \log J_n(\vec{\alpha},\vec{\beta},V_s,0), & \partial_t \log J_n(\vec{\alpha},\vec{\beta},V,W_t), \end{array} \tag{2.1.55}$$

in terms of Y, where $\nu \in \{\alpha_0, \dots, \alpha_m, \beta_1, \dots, \beta_m\}$, and $s \in [0,1]$ and $t \in [0,1]$ are smooth deformation parameters (more details on these deformations are given in Section 2.7 and Section 2.8). In Section 2.4, we perform a Deift/Zhou steepest descent analysis of the RH problem to obtain large n asymptotics for Y. We deduce from them asymptotics for the log derivatives given in (2.1.55), and we also proceed with their successive integrations (represented schematically by an arrow in (2.1.54)). These computations are rather long, and we organise them in several sections: Section 2.6 is devoted to integration in $\vec{\alpha}$ and $\vec{\beta}$, Section 2.7 to integration in s and Section 2.8 to integration in t. Each integration only gives us asymptotics for a ratio of Hankel determinants. Therefore, it is important to chose carefully the starting point of integration in the set of parameters $(\vec{\alpha}, \vec{\beta}, V, W)$. For Laguerre-type weights, we chose this point to be $(\vec{0}, \vec{0}, 2(x+1), 0)$ and for Jacobi-type weights, we use the result of [9] and chose $(\vec{\alpha}, \vec{\beta}, 0, 0)$. We recall large n asymptotics for $L_n(\vec{0}, \vec{0}, 2(x+1), 0)$ and for $J_n(\vec{\alpha}, \vec{\beta}, 0, 0)$ in Section 2.5.

Notations. We will use repetitively through the paper the convention $t_0 = -1$, $t_{m+1} = 1$, $\beta_0 = 0$ and $\beta_{m+1} = 0$. Furthermore, for Laguerre-type weights, we define $\alpha_{m+1} = 0$ and for Gaussian-type weights, we define $\alpha_0 = 0$ and $\alpha_{m+1} = 0$. This allows us for example to rewrite ω given in (2.1.6) as

$$\omega(x) = \prod_{j=0}^{m+1} \omega_{\alpha_j}(x)\omega_{\beta_j}(x). \tag{2.1.56}$$

2.2 A Riemann-Hilbert problem for orthogonal polynomials

We consider the family of OPs associated to the weight w given in (2.1.2). The degree k polynomial p_k is characterized by the relations

$$\int_{\mathcal{I}} p_k(x) x^j w(x) dx = \kappa_k^{-1} \delta_{jk}, \qquad j = 0, 1, 2, \dots, k, \tag{2.2.1}$$

where $\kappa_k \neq 0$ is the leading order coefficient of p_k. If $\beta_j \in i\mathbb{R}$ and $\Re\alpha_j > -1$, $j = 0, \dots, m+1$, then $w(x) > 0$ for almost all $x \in \mathcal{I}$. In this case, we can rewrite (2.2.1) as an inner product and it is a simple consequence of Gram-Schmidt that the OPs exist. However, for general values of α_j and β_j, the weight w is complex-valued and existence is no more guaranteed.

This fact introduces some technicalities in the analysis that are briefly discussed in Section 2.6, Section 2.7 and Section 2.8.

We associate to these OPs a RH problem for a 2×2 matrix-valued function Y, due to [6]. As mentioned in the outline, it will play a crucial role in our proof.

RH problem for Y

(a) $Y : \mathbb{C} \setminus \mathcal{I} \to \mathbb{C}^{2\times 2}$ is analytic.

(b) The limits of $Y(z)$ as z tends to $x \in \mathcal{I} \setminus \{-1, t_1, \ldots, t_m, 1\}$ from the upper and lower half plane exist, and are denoted $Y_\pm(x)$ respectively. Furthermore, the functions $x \mapsto Y_\pm(x)$ are continuous on $\mathcal{I} \setminus \{-1, t_1, \ldots, t_m, 1\}$ and are related by

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix}, \qquad x \in \mathcal{I} \setminus \{-1, t_1, \ldots, t_m, 1\}. \tag{2.2.2}$$

(c) As $z \to \infty$,

$$Y(z) = \big(I + \mathcal{O}(z^{-1})\big) z^{n\sigma_3}, \quad \text{where} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.2.3}$$

(d) As $z \to t_j$, for $j = 0, 1, \ldots, m+1$ (with $t_0 := -1$ and $t_{m+1} := 1$), we have

$$Y(z) = \begin{cases} \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) + \mathcal{O}((z - t_j)^{\alpha_j}) \\ \mathcal{O}(1) & \mathcal{O}(1) + \mathcal{O}((z - t_j)^{\alpha_j}) \end{pmatrix}, & \text{if } \Re\alpha_j \neq 0, \\ \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\log(z - t_j)) \\ \mathcal{O}(1) & \mathcal{O}(\log(z - t_j)) \end{pmatrix}, & \text{if } \Re\alpha_j = 0. \end{cases} \tag{2.2.4}$$

The solution of the RH problem for Y is always unique, exists if and only if p_n and p_{n-1} exist, and is explicitly given by

$$Y(z) = \begin{pmatrix} \kappa_n^{-1} p_n(z) & \dfrac{\kappa_n^{-1}}{2\pi i} \displaystyle\int_{\mathcal{I}} \frac{p_n(x) w(x)}{x - z} dx \\ -2\pi i \kappa_{n-1} p_{n-1}(z) & -\kappa_{n-1} \displaystyle\int_{\mathcal{I}} \frac{p_{n-1}(x) w(x)}{x - z} dx \end{pmatrix}. \tag{2.2.5}$$

The fact that Y given by (2.2.5) satisfies the condition (b) of the RH problem for Y follows from the Sokhotski formula and relies on the assumption that W is locally Hölder continuous on $\mathcal{I}$ (see e.g. [32]).

2.3 Differential identities

In this section, we express the logarithmic derivatives given in (2.1.55) in terms of Y.

2.3.1 Identity for $\partial_\nu \log L_n(\vec{\alpha}, \vec{\beta}, 2(x+1), 0)$ with $\nu \in \{\alpha_0, \dots, \alpha_m, \beta_1, \dots, \beta_m\}$

In this subsection, we specialize to the Laguerre-type weight $w(x) = \omega(x)e^{-2n(x+1)}$.

Note that the second column of Y blows up as $z \to t_k$, $k = 0, 1, \dots, m$ as shown in (2.2.4). The terms of order 1 in these asymptotics will contribute in our identity for $\partial_\nu \log L_n(\vec{\alpha}, \vec{\beta}, 2(x+1), 0)$. To prepare ourselves for that matter, following [3, eq (3.6)], for each $k \in \{1, \dots, m\}$ we define a regularized integral by

$$\mathrm{Reg}_k(f) = \lim_{\varepsilon \to 0_+} \left[\alpha_k \int_{\mathcal{I} \setminus [t_k - \epsilon, t_k + \epsilon]} \frac{f(x)\omega(x)}{x - t_k} dx - f(t_k)\omega_{t_k}(t_k)(e^{\pi i \beta_k} - e^{-\pi i \beta_k})\varepsilon^{\alpha_k} \right], \tag{2.3.1}$$

where f is a smooth function on $\mathcal{I} = [-1, +\infty)$, and

$$\omega_{t_k}(x) = \prod_{\substack{0 \le j \le m \\ j \ne k}} \omega_{\alpha_j}(x)\omega_{\beta_j}(x). \tag{2.3.2}$$

For $k = 0$, we define the regularized integral as above, with $e^{\pi i \beta_k}$ replaced by 0 and $e^{-\pi i \beta_k}$ replaced by 1 (we also recall that $t_0 = -1$), i.e. we have

$$\mathrm{Reg}_0(f) := \lim_{\varepsilon \to 0_+} \left[\alpha_0 \int_{\mathcal{I} \setminus [t_0, t_0 + \epsilon]} \frac{f(x)\omega(x)}{x - t_0} dx + f(t_0)\omega_{-1}(t_0)\varepsilon^{\alpha_0} \right]. \tag{2.3.3}$$

Proposition 2.3.1 *The regularized integrals (2.3.1) and (2.3.3) satisfy*

$$Reg_k(f) = \lim_{z \to t_k} \alpha_k \int_{\mathcal{I}} \frac{f(x)\omega(x)}{x - z} dx - \mathcal{J}_k(z), \tag{2.3.4}$$

where the limit is taken along a path in the upper-half plane which is non-tangential to the real line. For $k = 1, \dots, m$, $\mathcal{J}_k(z)$ is given by

$$\mathcal{J}_k(z) = \begin{cases} \frac{\pi\alpha_k}{\sin(\pi\alpha_k)} f(t_k)\omega_{t_k}(t_k)(e^{\pi i\beta_k} - e^{-\pi i\alpha_k}e^{-\pi i\beta_k})(z-t_k)^{\alpha_k}, & \text{if } \Re\alpha_k \le 0,\ \alpha_k \neq 0, \\ f(t_k)\omega_{t_k}(t_k)(e^{\pi i\beta_k} - e^{-\pi i\beta_k}), & \text{if } \alpha_k = 0, \\ 0, & \text{if } \Re\alpha_k > 0. \end{cases} \tag{2.3.5}$$

For $k = 0$, we have

$$\mathcal{J}_0(z) = \begin{cases} -\frac{\pi\alpha_0 e^{-\pi i\alpha_0}}{\sin(\pi\alpha_0)} f(t_0)\omega_{-1}(t_0)(z-t_0)^{\alpha_0}, & \text{if } \Re\alpha_0 \le 0,\ \alpha_0 \neq 0, \\ -f(t_0)\omega_{-1}(t_0), & \text{if } \alpha_0 = 0, \\ 0, & \text{if } \Re\alpha_0 > 0. \end{cases} \tag{2.3.6}$$

Proof The proof for $k = 1, \dots, m$ can be found in [3, Proposition 3.1] (which is itself based on [4]). The proof for $k = 0$ can be proved similarly by a straightforward adaptation. It suffices to replace $e^{\pi i\beta_k}$ by 0 and $e^{-\pi i\beta_k}$ by 1 in the proof of [3, Proposition 3.1]. ■

Since the second column of $Y(z)$ blows up as $z \to t_j$, $j = 0, \dots, m$, we regularize Y at these points using the definitions (2.3.1) and (2.3.3) as follows:

$$\widetilde{Y}(t_j) := \begin{pmatrix} Y_{11}(t_j) & \operatorname{Reg}_j\left(\frac{1}{2\pi i}Y_{11}(x)e^{-2n(x+1)}\right) \\ Y_{21}(t_j) & \operatorname{Reg}_j\left(\frac{1}{2\pi i}Y_{21}(x)e^{-2n(x+1)}\right) \end{pmatrix}. \tag{2.3.7}$$

From Proposition 2.3.1, we have

$$\widetilde{Y}_{k2}(t_j) = \lim_{z\to t_j} \alpha_j Y_{k2}(z) - c_j Y_{k1}(t_j)(z-t_j)^{\alpha_j}, \qquad k = 1, 2, \tag{2.3.8}$$

where the limit is taken along a path in the upper half plane non-tangential to the real line. For $j = 1, \dots, m$, c_j is given by

$$c_j = \frac{\pi\alpha_j}{\sin(\pi\alpha_j)} \frac{e^{-2n(t_j+1)}}{2\pi i} \omega_{t_j}(t_j)(e^{\pi i\beta_j} - e^{-\pi i\alpha_j}e^{-\pi i\beta_j}), \tag{2.3.9}$$

and for $j = 0$ we have

$$c_0 = \frac{\pi\alpha_0}{\sin(\pi\alpha_0)} \frac{-e^{-\pi i\alpha_0}}{2\pi i} \omega_{-1}(-1). \tag{2.3.10}$$

Note that $\det \widetilde{Y}(t_j)$ is not equal to 1, but instead we have

$$\det \widetilde{Y}(t_j) = \alpha_j, \qquad j = 0, 1, \ldots, m. \tag{2.3.11}$$

Proposition 2.3.2 *Let $p_0, p_1, \ldots$ be the family of OPs with respect to the weight $w(x) = \omega(x)e^{-2n(x+1)}$, whose leading coefficients are denoted by*

$$p_k(x) = \kappa_k(x^k + \eta_k x^{k-1} + \ldots). \tag{2.3.12}$$

Let $\nu \in \{\alpha_0, \alpha_1, \beta_1, \ldots, \alpha_m, \beta_m\}$ and let n, $\vec{\alpha}$ and $\vec{\beta}$ be such that $p_0, p_1, \ldots, p_n$ exist. We have the following identity:

$$\begin{aligned} \partial_\nu \log L_n(\vec{\alpha}, \vec{\beta}, 2(x+1), 0) &= -(n + \mathcal{A})\partial_\nu \log(\kappa_n \kappa_{n-1}) + 2n\partial_\nu \eta_n \\ &+ \sum_{j=0}^{m} \Big(\widetilde{Y}_{22}(t_j)\partial_\nu Y_{11}(t_j) - \widetilde{Y}_{12}(t_j)\partial_\nu Y_{21}(t_j) + Y_{11}(t_j)\widetilde{Y}_{22}(t_j)\partial_\nu \log(\kappa_n \kappa_{n-1}) \Big), \end{aligned} \tag{2.3.13}$$

where $\mathcal{A} = \sum_{j=0}^{m} \alpha_j$.

Remark 2.3.1 *We do not need an analogous formula for $\partial_\nu \log J_n(\vec{\alpha}, \vec{\beta}, 0, 0)$ as large n asymptotics of $J_n(\vec{\alpha}, \vec{\beta}, 0, 0)$ are already known from [9], see the outline.*

Proof The proof is an adaptation of [3, Subsection 3.1] where the author obtained a differential identity for $\partial_\nu \log G_n(\vec{\alpha}, \vec{\beta}, 2x^2, 0)$ (this proof was itself a generalization of [4,5]). Here, the proof is even slightly easier, due to the fact that the potential is a polynomial of degree 1 (and not of degree 2 as in [3–5]). Since we assume that $p_0, \ldots, p_n$ exist, we can use the following general identity, which was obtained in [4]

$$\partial_\nu \log L_n(\vec{\alpha}, \vec{\beta}, 2(x+1), 0) = -n\partial_\nu \log \kappa_{n-1} + \frac{\kappa_{n-1}}{\kappa_n}(I_1 - I_2), \tag{2.3.14}$$

where

$$I_1 = \int_{\mathcal{I}} p'_{n-1}(x)\partial_\nu p_n(x) w(x) dx, \quad \text{and} \quad I_2 = \int_{\mathcal{I}} p'_n(x)\partial_\nu p_{n-1}(x) w(x) dx. \tag{2.3.15}$$

Since $\Re \alpha_j > -1$ for all $j = 0, 1, \ldots, m$, we first note that

$$I_1 = \lim_{\epsilon \to 0_+} \int_{\mathcal{I}_\epsilon} p'_{n-1}(x)\partial_\nu p_n(x) w(x) dx, \tag{2.3.16}$$

where $\mathcal{I}_\epsilon$ is the union of $m+1$ intervals given by

$$\mathcal{I}_\epsilon = [t_0+\epsilon, t_1-\epsilon] \cup [t_1+\epsilon, t_2-\epsilon] \cup \ldots \cup [t_{m-1}+\epsilon, t_m-\epsilon] \cup [t_m+\epsilon, \infty).$$

Along each of these $m+1$ intervals, we integrate by parts (for each fixed and sufficiently small ϵ), using

$$w'(x) = \Big(-2n + \sum_{j=0}^{m} \frac{\alpha_j}{x-t_j} \Big) w(x), \qquad x \in (-1,\infty) \setminus \{t_1, \ldots, t_m\}. \tag{2.3.17}$$

Then, we simplify the expression by using the orthogonality relations (2.2.1). Finally, we substitute it in the limit (2.3.16) using (2.3.1) and (2.3.3), and we find

$$I_1 = -(n+\mathcal{A}) \frac{\partial_\nu \kappa_n}{\kappa_{n-1}} + 2n \frac{\kappa_n}{\kappa_{n-1}} \partial_\nu \eta_n - \sum_{j=0}^{m} \partial_\nu p_n(t_j) \mathrm{Reg}_j \left[p_{n-1}(x) e^{-2n(x+1)} \right]. \tag{2.3.18}$$

We proceed similarly to find the following expression for I_2 (the calculations are easier as several integrals can be identified as equal to 0 by using (2.2.1)):

$$I_2 = -\sum_{j=0}^{m} \partial_\nu p_{n-1}(t_j) \mathrm{Reg}_j \left[p_n(x) e^{-2n(x+1)} \right]. \tag{2.3.19}$$

By rewriting first I_1 and I_2 in terms of Y and $\widetilde{Y}$, then by substituting these expressions into (2.3.14), and finally by using (2.3.11), we obtain the claim. ■

2.3.2 A general differential identity

We recall here a differential identity that is valid for all three types of weights. In Section 2.7 and Section 2.8, we will use Proposition 2.3.3 below with $\nu = s$ or $\nu = t$ to obtain identities for the quantities in (2.1.55) (save the case of $\partial_\nu L_n(\vec{\alpha}, \vec{\beta}, 2(x+1), 0)$ for which we will use Proposition 2.3.2).

Proposition 2.3.3 *Let D_n be a Hankel determinant whose weight w depends smoothly on a parameter ν. Let us assume that the associated orthonormal polynomials $p_0,\ldots,p_n$ exist. Then we have*

$$\partial_\nu \log D_n = \frac{1}{2\pi i} \int_{\mathcal{I}} [Y^{-1}(x) Y'(x)]_{21} \partial_\nu w(x) dx, \tag{2.3.20}$$

where $\mathcal{I}$ is the support of w, and Y is given by (2.2.5).

Proof It suffices to start from the well-known [18] identity

$$D_n = \prod_{j=0}^{n-1} \kappa_j^{-2}, \tag{2.3.21}$$

take the log, differentiate with respect to ν, use the orthogonality relations and finally substitute Y in the expression. ■

2.4 Steepest descent analysis

In this section we will construct an asymptotic solution to the RH problem for Y through the Deift/Zhou steepest descent method, for Laguerre-type and Jacobi-type weights. The analysis goes via a series of transformations $Y \mapsto T \mapsto S \mapsto R$. The $Y \mapsto T$ transformation of Subsection 2.4.2 normalizes the RH problem at ∞ by means of a so-called g-function (whose properties are presented in Subsection 2.4.1). We proceed with the opening of the lenses $T \mapsto S$ in Subsection 2.4.3. As a preliminary to the last step $S \mapsto R$, we first construct approximations (called "parametrices") for S in different regions of the complex plane: a global parametrix in Subsection 2.4.4, local parametrices in the bulk around t_k in Subsection 2.4.5, and local parametrices at the edges ± 1 in Subsection 2.4.6 and Subsection 2.4.7. These parametrices are rather standard: our global parametrix is close to the one done in [3] and local parametrices in the bulk are built out of confluent hypergeometric functions (as in [5, 9, 33]), local parametrices at soft edges in terms of Airy functions (as in [34]) and at a hard edge, in terms of Bessel functions (as in [10]). Finally, the last step $S \mapsto R$ is carried out in Subsection 2.4.8.

2.4.1 Equilibrium measure and g-function

It is convenient for us to introduce the notation ρ for the density of μ_V:

$$d\mu_V(x) = \rho(x)dx = \begin{cases} \psi(x)\dfrac{\sqrt{1-x}}{\sqrt{1+x}}dx, & \text{for Laguerre-type weight,} \\ \psi(x)\dfrac{1}{\sqrt{1-x^2}}dx, & \text{for Jacobi-type weight,} \end{cases} \tag{2.4.1}$$

where we recall that by assumption $\psi : \mathcal{I} \to \mathbb{R}$ is analytic and positive on $[-1,1]$. Let U_V be the maximal open neighbourhood of $\mathcal{I}$ in which V is analytic, and U_W be an open neighbourhood of $[-1,1]$ in which W is analytic, sufficiently small such that $U_W \subset U_V$. The so-called g-function is defined by

$$g(z) = \int_{-1}^{1} \log(z-s)\rho(s)ds, \qquad \text{for } z \in \mathbb{C} \setminus (-\infty, 1], \tag{2.4.2}$$

where the principal branch is chosen for the logarithm. The g-function is analytic in $\mathbb{C}\setminus(-\infty,1]$ and has the following properties

$$g_+(x) + g_-(x) = 2\int_{-1}^{1} \log|x-s|\rho(s)ds, \qquad x \in \mathbb{R}, \tag{2.4.3}$$

$$g_+(x) - g_-(x) = 2\pi i, \qquad x \in (-\infty, -1), \tag{2.4.4}$$

$$g_+(x) - g_-(x) = 2\pi i \int_{x}^{1} \rho(s)ds, \qquad x \in [-1, 1]. \tag{2.4.5}$$

The Euler-Lagrange conditions (2.1.4)-(2.1.5) can be rewritten in terms of the g-function as follows:

$$g_+(x) + g_-(x) = V(x) - \ell, \qquad x \in [-1, 1], \tag{2.4.6}$$

$$2g(x) < V(x) - \ell, \qquad x \in \mathcal{I} \setminus [-1, 1]. \tag{2.4.7}$$

The above inequality is relevant only for Laguerre-type weight (since for Jacobi-type weight $\mathcal{I} \setminus [-1,1] = \emptyset$), and is strict since we assume that V is regular.

For $z \in U_V \setminus [-1,1]$, we define

$$\widetilde{\rho}(z) = \begin{cases} -i\psi(z)\dfrac{\sqrt{z-1}}{\sqrt{z+1}}, & \text{for Laguerre-type weight,} \\ i\psi(z)\dfrac{1}{\sqrt{z^2-1}}, & \text{for Jacobi-type weight,} \end{cases} \tag{2.4.8}$$

where the principal branches are chosen for $\sqrt{z-1}$ and $\sqrt{z+1}$. Note that for $x \in (-1,1)$ we have $\widetilde{\rho}_+(s) = -\widetilde{\rho}_-(s) = \rho(s)$. Let us also define

$$\xi(z) = -\pi i \int_{1}^{z} \widetilde{\rho}(s)ds, \qquad z \in U_V \setminus (-\infty, 1), \tag{2.4.9}$$

where the path of integration lies in $U_V \setminus (-\infty, 1)$. Since $\xi_+(x) + \xi_-(x) = 0$ for $x \in (-1, 1)$, by (2.4.5) and (2.4.6), we have

$$2\xi_\pm(x) = g_\pm(x) - g_\mp(x) = 2g_\pm(x) - V(x) + \ell. \tag{2.4.10}$$

By analytic continuation, we have

$$\xi(z) = g(z) + \frac{\ell}{2} - \frac{V(z)}{2}, \qquad z \in U_V \setminus (-\infty, 1). \tag{2.4.11}$$

Thus, the Euler-Lagrange inequality (2.4.7) can be simply rewritten as $2\xi(x) < 0$ for $x \in \mathcal{I} \setminus [-1, 1]$. Furthermore, since $g(z) \sim \log(z)$ as $z \to \infty$, we have that $(\xi_+(x)+\xi_-(x))/V(x) \to -1$ as $x \to +\infty$, $x \in \mathcal{I}$. Finally, by a standard and straightforward analysis of ξ, we conclude that there exists a small enough neighbourhood of $(-1, 1)$ such that, for z in this neighbourhood with $\Im z \neq 0$, we have $\Re\xi(z) > 0$.

We will also need later large z asymptotics of $e^{ng(z)}$ for the Laguerre-type potential $V(x) = 2(x + 1)$. In this case, we recall that $\psi(x) = \frac{1}{\pi}$, and after a straightforward calculation we obtain

$$e^{ng(z)} = z^n \Big(1 + \frac{n}{2z} + \mathcal{O}(z^{-2})\Big), \qquad \text{as } z \to \infty. \tag{2.4.12}$$

2.4.2 First transformation: $Y \mapsto T$

We normalize the RH problem for Y at ∞ by the standard transformation

$$T(z) := e^{\frac{n\ell}{2}\sigma_3} Y(z) e^{-ng(z)\sigma_3} e^{-\frac{n\ell}{2}\sigma_3}. \tag{2.4.13}$$

T satisfies the following RH problem.

RH problem for T

(a) $T : \mathbb{C} \setminus \mathcal{I} \to \mathbb{C}^{2\times 2}$ is analytic.

(b) The jumps for T follows from (2.4.4), (2.4.6) and (2.4.11). We obtain

$$T_+(x) = T_-(x)\begin{pmatrix} e^{-2n\xi_+(x)} & e^{W(x)}\omega(x) \\ 0 & e^{2n\xi_+(x)} \end{pmatrix}, \qquad \text{if } x \in (-1,1)\setminus\{t_1,\cdots,t_m\}, \tag{2.4.14}$$

$$T_+(x) = T_-(x)\begin{pmatrix} 1 & e^{W(x)}\omega(x)e^{2n\xi(x)} \\ 0 & 1 \end{pmatrix}, \qquad \text{if } x \in \mathcal{I}\setminus[-1,1]. \tag{2.4.15}$$

(c) As $z \to \infty$, $T(z) = I + \mathcal{O}(z^{-1})$.

(d) As $z \to t_j$, for $j = 0, 1, \ldots, m+1$, we have

$$T(z) = \begin{cases} \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) + \mathcal{O}((z-t_j)^{\alpha_j}) \\ \mathcal{O}(1) & \mathcal{O}(1) + \mathcal{O}((z-t_j)^{\alpha_j}) \end{pmatrix}, & \text{if } \Re\alpha_j \neq 0, \\ \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\log(z-t_j)) \\ \mathcal{O}(1) & \mathcal{O}(\log(z-t_j)) \end{pmatrix}, & \text{if } \Re\alpha_j = 0. \end{cases} \tag{2.4.16}$$

2.4.3 Second transformation: $T \mapsto S$

In this step, we will deform the contour of the RH problem. Therefore, we first consider the analytic continuations of the functions ω_{α_k} and ω_{β_k} from $\mathbb{R}\setminus\{t_k\}$ to $\mathbb{C}\setminus\{z : \Re(z) = t_k\}$. They are given by

$$\omega_{\alpha_k}(z) = \begin{cases} (t_k - z)^{\alpha_k}, & \text{if } \Re z < t_k, \\ (z-t_k)^{\alpha_k}, & \text{if } \Re z > t_k, \end{cases} \qquad \omega_{\beta_k}(z) = \begin{cases} e^{i\pi\beta_k}, & \text{if } \Re z < t_k, \\ e^{-i\pi\beta_k}, & \text{if } \Re z > t_k. \end{cases} \tag{2.4.17}$$

For $k = 0, \ldots, m+1$, we also define

$$\omega_{t_k}(z) = \prod_{\substack{0\le j\le m \\ j\neq k}} \omega_{\alpha_j}(z)\omega_{\beta_j}(z). \tag{2.4.18}$$

Note that for $x \in (-1,1)\setminus\{t_1,\ldots,t_m\}$ we have the following factorization for $J_T(x)$:

$$\begin{pmatrix} e^{-2n\xi_+(x)} & e^{W(x)}\omega(x) \\ 0 & e^{2n\xi_+(x)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ e^{-W(x)}\omega(x)^{-1}e^{-2n\xi_-(x)} & 1 \end{pmatrix}$$
$$\times \begin{pmatrix} 0 & e^{W(x)}\omega(x) \\ -e^{-W(x)}\omega(x)^{-1} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{-W(x)}\omega(x)^{-1}e^{-2n\xi_+(x)} & 1 \end{pmatrix}. \tag{2.4.19}$$

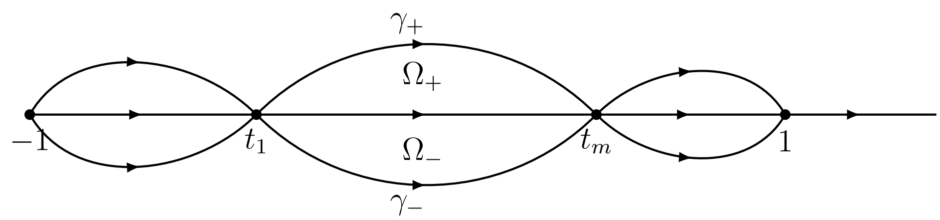

Figure 2.1. The jump contour for the RH problem for S with $m = 2$ and a Laguerre-type weight. For Jacobi-type weights, the jump contour for S is of the same shape, except that there are no jumps on $(1, +\infty)$.

Let γ_+ and γ_- be two curves (lying respectively in the upper and lower half plane) that join the points $-1, t_1, \ldots, t_m, 1$ as depicted in Figure 2.1. In order to be able to deform the contour of the RH problem, we choose them so that they both lie in U_W. In the constructions of the local parametrices, they will be required to make angles of $\frac{\pi}{4}$ with $\mathbb{R}$ at the points $t_1, \ldots, t_m$, and angles of $\frac{\pi}{3}$ with $\mathbb{R}$ at the points ± 1, and this is already shown in Figure 2.1. Also, we denote $\Omega_\pm$ for the open regions delimited by $\gamma_\pm$ and $\mathbb{R}$, see Figure 2.1. The next transformation is given by

$$S(z) = T(z) \begin{cases} \begin{pmatrix} 1 & 0 \\ -e^{-W(z)}\omega(z)^{-1}e^{-2n\xi(z)} & 1 \end{pmatrix}, & \text{if } z \in \Omega_+, \\ \begin{pmatrix} 1 & 0 \\ e^{-W(z)}\omega(z)^{-1}e^{-2n\xi(z)} & 1 \end{pmatrix}, & \text{if } z \in \Omega_-, \\ I, & \text{if } z \in \mathbb{C} \setminus \overline{(\Omega_+ \cup \Omega_- \cup (\mathcal{I} \setminus \mathcal{S}))}. \end{cases} \tag{2.4.20}$$

S satisfies the following RH problem.

RH problem for S

(a) $S : \mathbb{C} \setminus (\mathcal{I} \cup \gamma_+ \cup \gamma_-) \to \mathbb{C}^{2\times 2}$ is analytic.

(b) The jumps for S follows from those of T and from (2.4.19). They are given by

$$S_+(z) = S_-(z)\begin{pmatrix} 0 & e^{W(z)}\omega(z) \\ -e^{-W(z)}\omega(z)^{-1} & 0 \end{pmatrix}, \quad \text{if } z \in (-1,1)\setminus\{t_1,\dots,t_m\}, \tag{2.4.21}$$

$$S_+(z) = S_-(z)\begin{pmatrix} 1 & e^{W(z)}\omega(z)e^{2n\xi(z)} \\ 0 & 1 \end{pmatrix}, \quad \text{if } z \in \mathcal{I}\setminus[-1,1], \tag{2.4.22}$$

$$S_+(z) = S_-(z)\begin{pmatrix} 1 & 0 \\ e^{-W(z)}\omega(z)^{-1}e^{-2n\xi(z)} & 1 \end{pmatrix}, \quad \text{if } z \in \gamma_+ \cup \gamma_-. \tag{2.4.23}$$

(c) As $z \to \infty$, $S(z) = I + \mathcal{O}(z^{-1})$.

(d) As $z \to t_j$, for $j = 0, 1, \dots, m+1$, we have

$$S(z) = \begin{cases} \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix} & \text{if } \Re\alpha_j > 0, \quad z \in \mathbb{C}\setminus\overline{(\Omega_+\cup\Omega_-)}, \\ \begin{pmatrix} \mathcal{O}((z-t_j)^{-\alpha_j}) & \mathcal{O}(1) \\ \mathcal{O}((z-t_j)^{-\alpha_j}) & \mathcal{O}(1) \end{pmatrix} & \text{if } \Re\alpha_j > 0, \quad z \in \Omega_+\cup\Omega_-, \\ \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}((z-t_j)^{\alpha_j}) \\ \mathcal{O}(1) & \mathcal{O}((z-t_j)^{\alpha_j}) \end{pmatrix} & \text{if } \Re\alpha_j < 0, \quad z \notin \Gamma_S, \\ \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\log(z-t_j)) \\ \mathcal{O}(1) & \mathcal{O}(\log(z-t_j)) \end{pmatrix} & \text{if } \Re\alpha_j = 0, \quad z \in \mathbb{C}\setminus\overline{(\Omega_+\cup\Omega_-)}, \\ \begin{pmatrix} \mathcal{O}(\log(z-t_j)) & \mathcal{O}(\log(z-t_j)) \\ \mathcal{O}(\log(z-t_j)) & \mathcal{O}(\log(z-t_j)) \end{pmatrix} & \text{if } \Re\alpha_j = 0, \quad z \in \Omega_+\cup\Omega_-. \end{cases} \tag{2.4.24}$$

Now, the rest of the steepest descent analysis consists of finding good approximations to S in different regions of the complex plane. If z is away from neighbourhoods of -1, t_1, ..., t_m, 1, then the jumps for S are uniformly exponentially close to the identity matrix, except those on $(-1,1)$ (see the discussion at the end of Section 2.4.1). By ignoring the jumps that tend to the identity matrix, we are left with an RH problem that does not depend on n, and whose solution will be a good approximation of S away from -1, t_1, ..., t_m, 1. This approximation is called the global parametrix, denoted by $P^{(\infty)}$, and will be given in Section 2.4.4 below. Near the points -1, t_1, ..., t_m, 1 we need to construct local approximations to S (also called local parametrices and denoted in the present paper by $P^{(-1)}$, $P^{(t_1)}$, $\ldots$, $P^{(1)}$). Let $\delta > 0$, independent of n, be such that

$$\delta \leq \min_{0 \leq k \neq j \leq m+1} |t_j - t_k|. \tag{2.4.25}$$

The local parametrix $P^{(t_k)}$ (for $k \in \{0, 1, \ldots, m, m+1\}$) solves an RH problem with the same jumps as S, but on a domain which is a disk $\mathcal{D}_{t_k}$ centered at t_k of radius $\leq \delta/3$. Furthermore, we require the following matching condition with $P^{(\infty)}$ on the boundary $\partial\mathcal{D}_{t_k}$. As $n \to \infty$, uniformly for $z \in \partial\mathcal{D}_{t_k}$, we have

$$P^{(t_k)}(z) = (I + o(1))P^{(\infty)}(z). \tag{2.4.26}$$

Again, these constructions are standard and well-known: near a FH singularity in the bulk, the local parametrix is given in terms of hypergeometric functions, near a soft edge in terms of Airy functions, and near a hard edge in terms of Bessel functions. The local parametrices are presented in Section 2.4.5, Section 2.4.6 and Section 2.4.7.

2.4.4 Global parametrix

By disregarding the jump conditions on the lenses $\gamma_+ \cup \gamma_-$ and on $\mathcal{I} \setminus [-1, 1]$, we are left with the following RH problem for $P^{(\infty)}$ (condition (d) below ensures uniqueness of the RH problem and can not be seen from the RH problem for S).

RH problem for $P^{(\infty)}$

(a) $P^{(\infty)} : \mathbb{C} \setminus [-1, 1] \to \mathbb{C}^{2\times 2}$ is analytic.

(b) The jumps for $P^{(\infty)}$ are given by

$$P_+^{(\infty)}(z) = P_-^{(\infty)}(z) \begin{pmatrix} 0 & e^{W(z)}\omega(z) \\ -e^{-W(z)}\omega(z)^{-1} & 0 \end{pmatrix}, \quad \text{if } z \in (-1, 1) \setminus \{t_1, \ldots, t_m\}.$$

(c) As $z \to \infty$, $P^{(\infty)}(z) = I + P_1^{(\infty)} z^{-1} + \mathcal{O}(z^{-2})$.

(d) As $z \to t_j$, for $j = 1, \ldots, m$, we require

$$P^{(\infty)}(z) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix} (z - t_j)^{-(\frac{\alpha_j}{2} + \beta_j)\sigma_3}. \tag{2.4.27}$$

As $z \to t_j$ with $j \in \{0, m+1\}$ (we recall that $t_0 = -1$ and $t_{m+1} = 1$, and that $\alpha_{m+1} = 0$ for Laguerre-type weight), we have

$$P^{(\infty)}(z) = \begin{pmatrix} \mathcal{O}((z-t_j)^{-\frac{1}{4}}) & \mathcal{O}((z-t_j)^{-\frac{1}{4}}) \\ \mathcal{O}((z-t_j)^{-\frac{1}{4}}) & \mathcal{O}((z-t_j)^{-\frac{1}{4}}) \end{pmatrix} (z - t_j)^{-\frac{\alpha_j}{2}\sigma_3}. \tag{2.4.28}$$

Remark 2.4.1 *Note that this RH problem is the same regardless of the weight, the only exception being that $\alpha_{m+1} = 0$ for Laguerre-type weight (and not necessarily for Jacobi-type weight).*

This RH problem was solved first in [34] with $W \equiv 0$ and $\omega \equiv 0$. In [10], the authors explain how to construct the solution to the above RH problem for general W and ω by using Szegö functions. Our RH problem for $P^{(\infty)}$ is close to the one obtained in [3] for Gaussian-type weights. The solution is given by

$$P^{(\infty)}(z) = D_\infty^{\sigma_3} \begin{pmatrix} \frac{1}{2}(a(z) + a(z)^{-1}) & \frac{1}{2i}(a(z) - a(z)^{-1}) \\ -\frac{1}{2i}(a(z) - a(z)^{-1}) & \frac{1}{2}(a(z) + a(z)^{-1}) \end{pmatrix} D(z)^{-\sigma_3}, \tag{2.4.29}$$

where $a(z) = \sqrt[4]{\frac{z-1}{z+1}}$ is analytic on $\mathbb{C} \setminus [-1,1]$ and $a(z) \sim 1$ as $z \to \infty$. The Szegö function D is given by $D(z) = D_\alpha(z) D_\beta(z) D_W(z)$, where

$$D_W(z) = \exp\left(\frac{\sqrt{z^2-1}}{2\pi} \int_{-1}^{1} \frac{W(x)}{\sqrt{1-x^2}} \frac{dx}{z-x} \right), \tag{2.4.30}$$

$$D_\alpha(z) = \prod_{j=0}^{m+1} \exp\left(\frac{\sqrt{z^2-1}}{2\pi} \int_{-1}^{1} \frac{\log \omega_{\alpha_j}(x)}{\sqrt{1-x^2}} \frac{dx}{z-x} \right) = \left(z + \sqrt{z^2-1} \right)^{-\frac{\mathcal{A}}{2}} \prod_{j=0}^{m+1} (z - t_j)^{\frac{\alpha_j}{2}}, \tag{2.4.31}$$

$$D_\beta(z) = \prod_{j=1}^{m} \exp\left(\frac{\sqrt{z^2-1}}{2\pi} \int_{-1}^{1} \frac{\log \omega_{\beta_j}(x)}{\sqrt{1-x^2}} \frac{dx}{z-x} \right) = e^{\frac{i\pi\mathcal{B}}{2}} \prod_{j=1}^{m} \left(\frac{z t_j - 1 - i\sqrt{(z^2-1)(1-t_j^2)}}{z - t_j} \right)^{\beta_j}, \tag{2.4.32}$$

where $\mathcal{A} = \sum_{j=0}^{m+1} \alpha_j$ and $\mathcal{B} = \sum_{j=1}^{m} \beta_j$. The simplified forms of (2.4.31) and (2.4.32) were found in [10] and [5], respectively. Also, $D_\infty = \lim_{z\to\infty} D(z)$ appearing in (2.4.29) is given by

$$D_\infty = 2^{-\frac{\mathcal{A}}{2}} \exp\left(i \sum_{j=1}^{m} \beta_j \arcsin t_j\right) \exp\left(\frac{1}{2\pi} \int_{-1}^{1} \frac{W(x)}{\sqrt{1-x^2}} dx\right). \tag{2.4.33}$$

The following asymptotic expressions were obtained in [3, Section 4.4] with $\alpha_0 = \alpha_{m+1} = 0$. It is straightforward to adapt them for general α_0 and α_{m+1}. As $z \to t_k$, with $k \in \{1, \ldots, m\}$ and $\Im z > 0$, we have

$$D_\alpha(z) = e^{-i\frac{\mathcal{A}}{2}\arccos t_k} \left(\prod_{0\le j\ne k\le m+1} |t_k - t_j|^{\frac{\alpha_j}{2}} \prod_{j=k+1}^{m} e^{\frac{i\pi\alpha_j}{2}} \right) (z-t_k)^{\frac{\alpha_k}{2}} (1 + \mathcal{O}(z-t_k)), \tag{2.4.34}$$

$$D_\beta(z) = e^{-\frac{i\pi}{2}(\mathcal{B}_k + \beta_k)} \left(\prod_{1\le j\ne k\le m} T_{kj}^{\beta_j} \right) (1-t_k^2)^{-\beta_k} 2^{-\beta_k} (z-t_k)^{\beta_k} (1 + \mathcal{O}(z-t_k)), \tag{2.4.35}$$

where

$$\mathcal{B}_k = \sum_{j=1}^{k-1} \beta_j - \sum_{j=k+1}^{m} \beta_j, \qquad T_{kj} = \frac{1 - t_k t_j - \sqrt{(1-t_k^2)(1-t_j^2)}}{|t_k - t_j|}. \tag{2.4.36}$$

Let us also define the following quantities:

$$\widetilde{\mathcal{B}}_1 = 2i \sum_{j=1}^{m} \sqrt{\frac{1+t_j}{1-t_j}} \beta_j, \qquad \widetilde{\mathcal{B}}_{-1} = 2i \sum_{j=1}^{m} \sqrt{\frac{1-t_j}{1+t_j}} \beta_j. \tag{2.4.37}$$

As $z \to 1$, we have

$$D_\alpha^2(z) \prod_{j=0}^{m+1} (z-t_j)^{-\alpha_j} = 1 - \sqrt{2}\mathcal{A}\sqrt{z-1} + \mathcal{A}^2(z-1) + \mathcal{O}((z-1)^{3/2}), \tag{2.4.38}$$

$$D_\beta^2(z) e^{i\pi\mathcal{B}} = 1 + \sqrt{2}\widetilde{\mathcal{B}}_1\sqrt{z-1} + \widetilde{\mathcal{B}}_1^2(z-1) + \mathcal{O}((z-1)^{3/2}). \tag{2.4.39}$$

As $z \to -1$, $\Im z > 0$, we have

$$D_\alpha^2(z) \prod_{j=0}^{m+1} (t_j - z)^{-\alpha_j} = 1 + i\sqrt{2}\mathcal{A}\sqrt{z+1} - \mathcal{A}^2(z+1) + \mathcal{O}((z+1)^{3/2}), \tag{2.4.40}$$

$$D_\beta^2(z) e^{-i\pi\mathcal{B}} = 1 + i\sqrt{2}\widetilde{\mathcal{B}}_{-1}\sqrt{z+1} - \widetilde{\mathcal{B}}_{-1}^2(z+1) + \mathcal{O}((z+1)^{3/2}). \tag{2.4.41}$$

As $z \to \infty$, with $W \equiv 0$ and recalling that $\beta_0 = \beta_{m+1} = 0$, we have

$$P_1^{(\infty)} = \begin{pmatrix} \sum_{j=0}^{m+1} \left(\frac{\alpha_j t_j}{2} + i\sqrt{1-t_j^2}\beta_j \right) & \frac{i}{2} D_\infty^2 \\ -\frac{i}{2} D_\infty^{-2} & -\sum_{j=0}^{m+1} \left(\frac{\alpha_j t_j}{2} + i\sqrt{1-t_j^2}\beta_j \right) \end{pmatrix}. \tag{2.4.42}$$

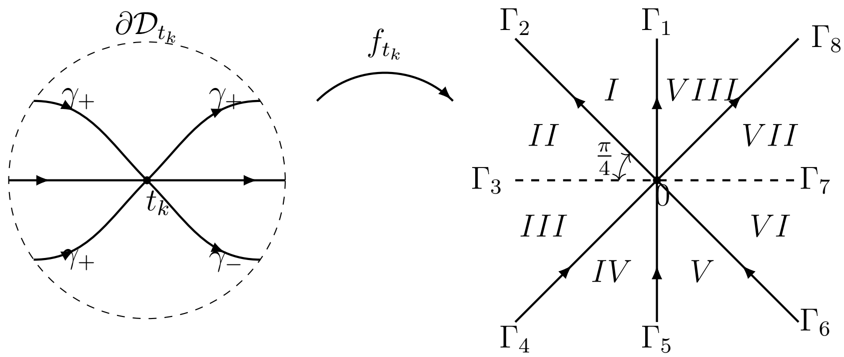

Figure 2.2. The neighborhood $\mathcal{D}_{t_k}$ and its image under the mapping f_{t_k}.

2.4.5 Local parametrix near t_k, $1 \leq k \leq m$

It is well-known [5, 9, 33] that $P^{(t_k)}$ can be written in terms of hypergeometric functions. In [3], the local parametrix was obtained for Gaussian-type weights, and it is straightforward to adapt the construction for Laguerre-type and Jacobi-type weights, the only difference being in the definition of ξ. Let us define the function f_{t_k} by

$$f_{t_k}(z) = -2 \begin{cases} \xi(z) - \xi_+(t_k), & \Im z > 0, \\ -(\xi(z) - \xi_-(t_k)), & \Im z < 0, \end{cases} = 2\pi i \int_{t_k}^{z} \rho(s) ds, \tag{2.4.43}$$

where in the above expression ρ is the analytic continuation on $U_V \setminus ((-\infty, -1) \cup (1, +\infty))$ of the density of the equilibrium measure (ρ was previously only defined on $[-1, 1]$). This is a conformal map from $\mathcal{D}_{t_k}$ to a neighbourhood of 0, and its expansion as $z \to t_k$ is given by

$$f_{t_k}(z) = 2\pi i \rho(t_k)(z - t_k)(1 + \mathcal{O}(z - t_k)), \qquad \text{as } z \to t_k. \tag{2.4.44}$$

The lenses in a neighbourhood of t_k are chosen such that $f_{t_k}(\gamma_+ \cap \mathcal{D}_{t_k}) \subset \Gamma_4 \cup \Gamma_2$ and $f_{t_k}(\gamma_- \cap \mathcal{D}_{t_k}) \subset \Gamma_6 \cup \Gamma_8$, see Figure 2.2. Let us define $Q_{+,k}^R = f_{t_k}^{-1}(II) \cap \mathcal{D}_{t_k}$, that is, it is the subset of $\mathcal{D}_{t_k}$ that lies outside the lenses in the upper half plane and which is mapped by f_{t_k} into a subset of II. All we need is to find the expression of $P^{(t_k)}$ in the region $Q_{+,k}^R$. This was done in [3, equation (4.48) and below (5.2)] for Gaussian-type weights. It is straightforward to adapt the construction in our situations, and we omit the details. For $z \in Q^R_{+,k}$, $P^{(t_k)}(z)$ is given by

$$P^{(t_k)}(z) = E_{t_k}(z)\times \begin{pmatrix} \frac{\Gamma(1+\frac{\alpha_k}{2}-\beta_k)}{\Gamma(1+\alpha_k)} G(\frac{\alpha_k}{2}+\beta_k, \alpha_k; nf_{t_k}(z))e^{-\frac{i\pi\alpha_k}{2}} & -\frac{\Gamma(1+\frac{\alpha_k}{2}-\beta_k)}{\Gamma(\frac{\alpha_k}{2}+\beta_k)} H(1+\frac{\alpha_k}{2}-\beta_k, \alpha_k; nf_{t_k}(z)e^{-\pi i}) \\ \frac{\Gamma(1+\frac{\alpha_k}{2}+\beta_k)}{\Gamma(1+\alpha_k)} G(1+\frac{\alpha_k}{2}+\beta_k, \alpha_k; nf_{t_k}(z))e^{-\frac{i\pi\alpha_k}{2}} & H(\frac{\alpha_k}{2}-\beta_k, \alpha_k; nf_{t_k}(z)e^{-\pi i}) \end{pmatrix}$$
$$\times (z-t_k)^{-\frac{\alpha_k}{2}\sigma_3} e^{\frac{\pi i\alpha_k}{4}\sigma_3} e^{-n\xi(z)\sigma_3} e^{-\frac{W(z)}{2}\sigma_3} \omega_{t_k}(z)^{-\frac{\sigma_3}{2}}, \quad (2.4.45)$$

where G and H are given in terms of the Whittaker functions (see [35, Chapter 13]):

$$G(a,\alpha;z) = \frac{M_{\kappa,\mu}(z)}{\sqrt{z}}, \quad H(a,\alpha;z) = \frac{W_{\kappa,\mu}(z)}{\sqrt{z}}, \quad \mu = \frac{\alpha}{2}, \quad \kappa = \frac{1}{2}+\frac{\alpha}{2}-a. \quad (2.4.46)$$

The function E_{t_k} is analytic in $\mathcal{D}_{t_k}$ (see [3, (4.49)-(4.51)]) and its value at t_k is given by

$$E_{t_k}(t_k) = \frac{D_\infty^{\sigma_3}}{2\sqrt[4]{1-t_k^2}} \begin{pmatrix} e^{-\frac{\pi i}{4}}\sqrt{1+t_k} + e^{\frac{\pi i}{4}}\sqrt{1-t_k} & i\left(e^{-\frac{\pi i}{4}}\sqrt{1+t_k} - e^{\frac{\pi i}{4}}\sqrt{1-t_k}\right) \\ -i\left(e^{-\frac{\pi i}{4}}\sqrt{1+t_k} - e^{\frac{\pi i}{4}}\sqrt{1-t_k}\right) & e^{-\frac{\pi i}{4}}\sqrt{1+t_k} + e^{\frac{\pi i}{4}}\sqrt{1-t_k} \end{pmatrix} \Lambda_k^{\sigma_3}, \quad (2.4.47)$$

where

$$\Lambda_k = e^{\frac{W(t_k)}{2}} D_{W,+}(t_k)^{-1} e^{i\frac{\lambda_k}{2}} (4\pi\rho(t_k)n(1-t_k^2))^{\beta_k} \prod_{1\le j\ne k\le m} T_{kj}^{-\beta_j}, \quad (2.4.48)$$

and

$$\lambda_k = \mathcal{A}\arccos t_k - \frac{\pi}{2}\alpha_k - \sum_{j=k+1}^{m+1} \pi\alpha_j + 2\pi n\int_{t_k}^1 \rho(s)ds. \quad (2.4.49)$$

Also, we need a more detailed knowledge of the asymptotics (2.4.26). By [3, equation (4.52)], we have

$$P^{(t_k)}(z)P^{(\infty)}(z)^{-1} = I + \frac{v_k}{nf_{t_k}(z)} E_{t_k}(z) \begin{pmatrix} -1 & \tau(\alpha_k,\beta_k) \\ -\tau(\alpha_k,-\beta_k) & 1 \end{pmatrix} E_{t_k}(z)^{-1} + \mathcal{O}(n^{-2+2|\Re\beta_k|}), \quad (2.4.50)$$

uniformly for $z \in \partial\mathcal{D}_{t_k}$ as $n\to\infty$, where $v_k = \beta_k^2 - \frac{\alpha_k^2}{4}$ and $\tau(\alpha_k,\beta_k) = \frac{-\Gamma(\frac{\alpha_k}{2}-\beta_k)}{\Gamma(\frac{\alpha_k}{2}+\beta_k+1)}$.

2.4.6 Local parametrix near 1

The local parametrix near 1 cannot be treated for both Laguerre-type and Jacobi-type weights simultaneously, since 1 is a soft edge for Laguerre-type weights, and a hard edge for

Jacobi-type weights. At a soft edge, the construction relies on the Airy model RH problem (whose solution is denoted Φ_{Ai}), and at a hard edge on the Bessel model RH problem (whose solution is denoted Φ_{Be}). For the reader's convenience, we recall these model RH problems in the appendix.

Laguerre-type weights

Let us define $f_1(z) = (-\frac{3}{2}\xi(z))^{2/3}$. This is a conformal map in $\mathcal{D}_1$ whose expansion as $z \to 1$ is given by

$$f_1(z) = \left(\frac{\pi\psi(1)}{\sqrt{2}}\right)^{2/3}(z-1)\left(1 - \frac{1}{10}\left(1 - 4\frac{\psi'(1)}{\psi(1)}\right)(z-1) + \mathcal{O}((z-1)^2)\right). \tag{2.4.51}$$

The lenses γ_+ and γ_- in a neighborhood of 1 are chosen such that $f_1(\gamma_+ \cap \mathcal{D}_1) \subset e^{\frac{2\pi i}{3}}\mathbb{R}^+$ and $f_1(\gamma_- \cap \mathcal{D}_1) \subset e^{-\frac{2\pi i}{3}}\mathbb{R}^+$. The local parametrix is given by

$$P^{(1)}(z) = E_1(z)\Phi_{\mathrm{Ai}}(n^{2/3}f_1(z))\omega(z)^{-\frac{\sigma_3}{2}}e^{-n\xi(z)\sigma_3}e^{-\frac{W(z)}{2}\sigma_3}, \tag{2.4.52}$$

where E_1 is analytic in $\mathcal{D}_1$ and given by

$$E_1(z) = P^{(\infty)}(z)e^{\frac{W(z)}{2}\sigma_3}\omega(z)^{\frac{\sigma_3}{2}}N^{-1}f_1(z)^{\frac{\sigma_3}{4}}n^{\frac{\sigma_3}{6}}, \qquad N = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \tag{2.4.53}$$

and $\Phi_{\mathrm{Ai}}(z)$ is the solution to the Airy model RH problem presented in the appendix (see Subsection C.1). Using (C.1.2), we obtain a more detailed description of the matching condition (2.4.26):

$$P^{(1)}(z)P^{(\infty)}(z)^{-1} = I + \frac{P^{(\infty)}(z)e^{\frac{W(z)}{2}\sigma_3}\omega(z)^{\frac{\sigma_3}{2}}}{8nf_1(z)^{3/2}}\begin{pmatrix} \frac{1}{6} & i \\ i & -\frac{1}{6} \end{pmatrix}\omega(z)^{-\frac{\sigma_3}{2}}e^{-\frac{W(z)}{2}\sigma_3}P^{(\infty)}(z)^{-1} + \mathcal{O}(n^{-2}) \tag{2.4.54}$$

uniformly for $z \in \partial\mathcal{D}_1$ as $n \to \infty$.

Jacobi-type weights

In this case we define $f_1(z) = \xi(z)^2/4$. This is a conformal map in $\mathcal{D}_1$ whose expansion as $z \to 1$ is given by

$$f_1(z) = \left(\frac{\pi}{\sqrt{2}}\psi(1)\right)^2 (z-1)\Big(1 + \Big(\frac{2}{3}\frac{\psi'(1)}{\psi(1)} - \frac{1}{6}\Big)(z-1) + \mathcal{O}((z-1)^2)\Big). \tag{2.4.55}$$

The lenses γ_+ and γ_- in a neighborhood of 1 are again chosen such that $f_1(\gamma_+ \cap \mathcal{D}_1) \subset e^{\frac{2\pi i}{3}}\mathbb{R}^+$ and $f_1(\gamma_- \cap \mathcal{D}_1) \subset e^{-\frac{2\pi i}{3}}\mathbb{R}^+$. The local parametrix is given by

$$P^{(1)}(z) = E_1(z)\Phi_{\mathrm{Be}}(n^2 f_1(z); \alpha_{m+1})\omega_1(z)^{-\frac{\sigma_3}{2}}(z-1)^{-\frac{\alpha_{m+1}}{2}\sigma_3} e^{-n\xi(z)\sigma_3} e^{-\frac{W(z)}{2}\sigma_3}, \tag{2.4.56}$$

where the principal branch is taken for $(z-1)^{\frac{\alpha_{m+1}}{2}}$, $\Phi_{\mathrm{Be}}(z)$ is the solution to the Bessel model RH problem presented in Subsection C.2, and E_1 is analytic in $\mathcal{D}_1$ and given by

$$E_1(z) = P^{(\infty)}(z) e^{\frac{W(z)}{2}\sigma_3}(z-1)^{\frac{\alpha_{m+1}}{2}\sigma_3}\omega_1(z)^{\frac{\sigma_3}{2}} N^{-1}(2\pi n f(z)^{1/2})^{\frac{\sigma_3}{2}}. \tag{2.4.57}$$

In this case, using (C.2.2), the matching condition (2.4.26) can be written as

$$\begin{aligned} P^{(1)}(z)P^{(\infty)}(z)^{-1} &= I + \frac{P^{(\infty)}(z)e^{\frac{W(z)}{2}\sigma_3}\omega_1(z)^{\frac{\sigma_3}{2}}(z-1)^{\frac{\alpha_{m+1}}{2}\sigma_3}}{16 n f_1(z)^{1/2}} \\ &\times \begin{pmatrix} -(1+4\alpha_{m+1}^2) & -2i \\ -2i & 1+4\alpha_{m+1}^2 \end{pmatrix} (z-1)^{-\frac{\alpha_{m+1}}{2}\sigma_3}\omega_1(z)^{-\frac{\sigma_3}{2}} e^{-\frac{W(z)}{2}\sigma_3} P^{(\infty)}(z)^{-1} + \mathcal{O}(n^{-2}), \end{aligned} \tag{2.4.58}$$

uniformly for $z \in \partial\mathcal{D}_1$ as $n \to \infty$.

2.4.7 Local parametrix near -1

Since Laguerre-type and Jacobi-type weights both have a hard edge at -1, the construction of this local parametrix can be treated simultaneously for both cases, the only difference being in the conformal map. This map is defined by $f_{-1}(z) = -(\xi(z) - \pi i)^2/4$, and its expansion as $z \to -1$ is given by

$$f_{-1}(z) = \begin{cases} \left(\sqrt{2}\pi\psi(-1)\right)^2 (z+1)\Big(1 + \Big(\frac{2}{3}\frac{\psi'(-1)}{\psi(-1)} - \frac{1}{6}\Big)(z+1) + \mathcal{O}((z+1)^2)\Big), & \text{for Laguerre-type weights,} \\ \left(\frac{\pi}{\sqrt{2}}\psi(-1)\right)^2 (z+1)\Big(1 + \Big(\frac{2}{3}\frac{\psi'(-1)}{\psi(-1)} + \frac{1}{6}\Big)(z+1) + \mathcal{O}((z+1)^2)\Big), & \text{for Jacobi-type weights.} \end{cases} \tag{2.4.59}$$

The local parametrix is given by

$$P^{(-1)}(z) = E_{-1}(z)\sigma_3\Phi_{\mathrm{Be}}(-n^2 f_{-1}(z);\alpha_0)\sigma_3\omega_{-1}(z)^{-\frac{\sigma_3}{2}}(-z-1)^{-\frac{\alpha_0}{2}\sigma_3}e^{-n\xi(z)\sigma_3}e^{-\frac{W(z)}{2}\sigma_3}, \quad (2.4.60)$$

where the principal branch is chosen for $(-z-1)^{-\frac{\alpha_0}{2}\sigma_3}$, and E_{-1} is analytic in $\mathcal{D}_{-1}$ and given by

$$E_{-1}(z) = (-1)^n P^{(\infty)}(z)e^{\frac{W(z)}{2}\sigma_3}\omega_{-1}(z)^{\frac{\sigma_3}{2}}(-z-1)^{\frac{\alpha_0}{2}\sigma_3}N(2\pi n(-f_{-1}(z))^{1/2})^{\frac{\sigma_3}{2}}. \quad (2.4.61)$$

For Laguerre-type weights with $W \equiv 0$, by taking the limit $z \to -1$ in (2.4.61) (from e.g. the upper half plane) and using the asymptotics (2.4.40)–(2.4.41) we have

$$E_{-1}(-1) = (-1)^n D_\infty^{\sigma_3}\left(N + \begin{pmatrix} 0 & \frac{i}{\sqrt{2}}(\mathcal{A}+\widetilde{\mathcal{B}}_{-1}) \\ 0 & \frac{-1}{\sqrt{2}}(\mathcal{A}+\widetilde{\mathcal{B}}_{-1}) \end{pmatrix}\right)(4\pi^2\psi(-1)n)^{\frac{\sigma_3}{2}}. \quad (2.4.62)$$

Furthermore, as $n \to \infty$, we have

$$P^{(-1)}(z)P^{(\infty)}(z)^{-1} = I + \frac{P^{(\infty)}(z)e^{\frac{W(z)}{2}\sigma_3}\omega_{-1}(z)^{\frac{\sigma_3}{2}}(-z-1)^{\frac{\alpha_0}{2}\sigma_3}}{16n(-f_{-1}(z))^{1/2}}$$
$$\times \begin{pmatrix} -(1+4\alpha_0^2) & 2i \\ 2i & 1+4\alpha_0^2 \end{pmatrix}(-z-1)^{-\frac{\alpha_0}{2}\sigma_3}\omega_{-1}(z)^{-\frac{\sigma_3}{2}}e^{-\frac{W(z)}{2}\sigma_3}P^{(\infty)}(z)^{-1} + \mathcal{O}(n^{-2}), \quad (2.4.63)$$

uniformly for $z \in \partial\mathcal{D}_{-1}$.

2.4.8 Small norm RH problem

We are now in a position to do the last transformation. We recall that the disks are nonoverlapping. Using the parametrices, we define the matrix valued function R as

$$R(z) = \begin{cases} S(z)P^{(\infty)}(z)^{-1}, & \text{if } z \in \mathbb{C} \setminus \cup_{j=0}^{m+1}\overline{\mathcal{D}_{t_j}}, \\ S(z)P^{(t_j)}(z)^{-1}, & \text{if } z \in \mathcal{D}_{t_j}, \quad j = 0, \ldots, m+1. \end{cases} \quad (2.4.64)$$

We recall that the local parametrices have the same jumps as S inside the disks and also that the global parametrix has the same jumps as S on $(-1,1)$, hence R has jumps only on the contour Σ_R depicted in Figure 2.3, where the orientation of the jump contour on $\partial\mathcal{D}_{t_j}$ is chosen to be clockwise. Since $P^{(t_j)}$ and S have the same asymptotic behavior near t_j, $j = 0, \ldots, m+1$, R is bounded at these points. Therefore, it satisfies the following RH problem.

RH problem for R

(a) $R : \mathbb{C} \setminus \Sigma_R \to \mathbb{C}^{2\times 2}$ is analytic.

(b) R satisfies $R_+(z) = R_-(z)J_R(z)$ for z on $\Sigma_R \setminus \{\text{intersection points of } \Sigma_R\}$ with

$$J_R(z) = \begin{cases} P^{(t_j)}(z)P^{(\infty)}(z)^{-1} & z \in \partial\mathcal{D}_{t_j}, \\ P^{(\infty)}(z)J_S(z)P^{(\infty)}(z)^{-1} & z \in \Sigma_R \setminus \cup_{j=0}^{m+1}\partial\mathcal{D}_{t_j}, \end{cases} \tag{2.4.65}$$

where $J_S(z) := S_-^{-1}(z)S_+(z)$ is given in (2.4.21)–(2.4.23).

(c) As $z \to \infty$, $R(z) = I + R_1 z^{-1} + \mathcal{O}(z^{-2})$ for a certain matrix R_1 independent of z.

As $z \to z_\star \in \{\text{intersections points of } \Sigma_R\}$, $R(z)$ is bounded.

We recall that outside fixed neighbourhoods of t_j, $j = 0, \ldots, m+1$, the jumps for S on $\gamma_+ \cup \gamma_-$ and on $\mathcal{I} \setminus [-1, 1]$ are exponentially and uniformly close to the identity matrix (see the discussion at the end of Subsection 2.4.3). Therefore, from (2.4.50), (2.4.54),(2.4.58), (2.4.63) and (2.4.65), as $n \to \infty$ we have

$$J_R(z) = \begin{cases} I + \mathcal{O}(e^{-cn}), & \text{uniformly for } z \in \Sigma_R \cap (\gamma^+ \cup \gamma^- \cup \mathbb{R}), \\ I + \mathcal{O}(n^{-1}), & \text{uniformly for } z \in \partial\mathcal{D}_1 \cup \partial\mathcal{D}_{-1}, \\ I + \mathcal{O}(n^{-1+2|\Re\beta_k|}), & \text{uniformly for } z \in \partial\mathcal{D}_{t_k}, k = 1, \ldots, m, \end{cases} \tag{2.4.66}$$

for a positive constant c. By standard theory of small-norm RH problems (see e.g. [34, 36]), R exists for sufficiently large n (we also refer to [3–5, 14] for very similar situations with more details provided). Furthermore, for any $r \in \mathbb{N}$, as $n \to \infty$, R has an expansion given by

$$R(z) = I + \sum_{j=1}^{r} \frac{R^{(j)}(z)}{n^j} + R_R^{(r+1)}(z)n^{-r-1}, \tag{2.4.67}$$

$$R^{(j)}(z) = \mathcal{O}(n^{2\beta_{\max}}), \quad R^{(j)}(z)' = \mathcal{O}(n^{2\beta_{\max}}) \quad R_R^{(r+1)}(z) = \mathcal{O}(n^{2\beta_{\max}}), \quad R_R^{(r+1)}(z)' = \mathcal{O}(n^{2\beta_{\max}}),$$

uniformly for $z \in \mathbb{C} \setminus \Sigma_R$, uniformly for $(\vec{\alpha}, \vec{\beta})$ in any fixed compact set, and uniformly in $\vec{t}$ if there exists $\delta > 0$, independent of n, such that

$$\min_{j \neq k}\{|t_j - t_k|, |t_j - 1|, |t_j + 1|\} \geq \delta. \tag{2.4.68}$$

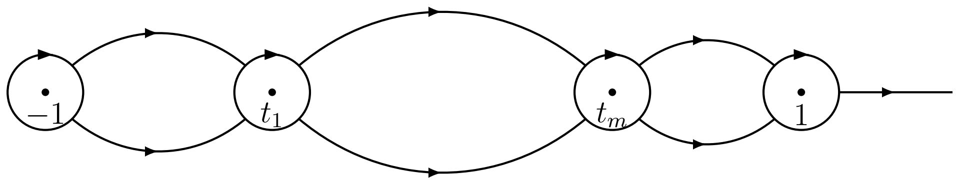

Figure 2.3. Jump contour Σ_R for the RH problem for R for Laguerre-type weights with $m = 2$. For Jacobi-type weights, Σ_R is of the same shape except that there are no jumps on $(1, \infty) \setminus \overline{\mathcal{D}_1}$.

Furthermore, in the way as done in [3], we show that

$$\partial_\nu R^{(j)}(z) = \mathcal{O}(n^{2\beta_{max}} \log n), \qquad \partial_\nu R_R^{(r+1)}(z) = \mathcal{O}(n^{2\beta_{max}} \log n) \tag{2.4.69}$$

for $\nu \in \{\alpha_0, \alpha_1, \ldots, \alpha_{m+1}, \beta_1, \ldots, \beta_m\}$. From (2.4.50), (2.4.54),(2.4.58), (2.4.63), we show that J_R admits an expansion as $n \to +\infty$ of the form

$$J_R(z) = I + \sum_{j=1}^{r} \frac{J_R^{(j)}(z)}{n^j} + \mathcal{O}(n^{-r-1+2\beta_{max}}), \qquad J_R^{(j)}(z) = \mathcal{O}(n^{2\beta_{\max}}), \tag{2.4.70}$$

uniformly for $z \in \cup_{j=0}^{m+1} \partial\mathcal{D}_{t_j}$. The matrices $R^{(j)}$ are obtained in a recursive way via the Plemelj-Sokhotski formula (for instance see [10]), in particular one has

$$R^{(1)}(z) = \sum_{j=0}^{m+1} \frac{1}{2\pi i} \int_{\partial\mathcal{D}_{t_j}} \frac{J_R^{(1)}(s)}{s-z} ds, \tag{2.4.71}$$

where we recall that the orientation on $\partial\mathcal{D}_{t_j}$ is clockwise. The goal for the rest of this section is to explicitly compute $R^{(1)}$ in the case $W \equiv 0$ for Laguerre-type and Jacobi-type weights.

Laguerre-type weights

From (2.4.50), (2.4.54), and (2.4.63) we easily show that $J_R^{(1)}$ has a double pole at 1 and a simple pole at t_j, $j = 0, \ldots, m$. Therefore $R^{(1)}(z)$ can be explicitly computed from (2.4.71) via a residue calculation. For $z \in \mathbb{C} \setminus \cup_{j=0}^{m+1} \overline{\mathcal{D}_{t_j}}$, we have

$$\begin{aligned} R^{(1)}(z) \;&=\; \sum_{j=1}^{m} \frac{1}{z-t_j} \operatorname{Res}\big(J_R^{(1)}(s), s = t_j\big) + \frac{1}{z+1} \operatorname{Res}\big(J_R^{(1)}(s), s = -1\big) \\ &\quad + \frac{1}{z-1} \operatorname{Res}\big(J_R^{(1)}(s), s = 1\big) + \frac{1}{(z-1)^2} \operatorname{Res}\big((s-1) J_R^{(1)}(s), s = 1\big). \end{aligned} \tag{2.4.72}$$

The residue at t_k can be computed from (2.4.50) (in the same way as in [3, eq (4.82)])

$$\operatorname{Res}\bigl(J_R^{(1)}(z), z=t_k\bigr) = \frac{v_k D_\infty^{\sigma_3}}{2\pi\rho(t_k)\sqrt{1-t_k^2}} \begin{pmatrix} t_k+\widetilde{\Lambda}_{I,k} & -i-i\widetilde{\Lambda}_{R,2,k} \\ -i+i\widetilde{\Lambda}_{R,1,k} & -t_k-\widetilde{\Lambda}_{I,k} \end{pmatrix} D_\infty^{-\sigma_3}, \tag{2.4.73}$$

where

$$\widetilde{\Lambda}_{I,k} = \frac{\tau(\alpha_k,\beta_k)\Lambda_k^2 - \tau(\alpha_k,-\beta_k)\Lambda_k^{-2}}{2i}, \tag{2.4.74}$$

$$\widetilde{\Lambda}_{R,1,k} = \frac{\tau(\alpha_k,\beta_k)\Lambda_k^2 e^{i\arcsin t_k} + \tau(\alpha_k,-\beta_k)\Lambda_k^{-2}e^{-i\arcsin t_k}}{2}, \tag{2.4.75}$$

$$\widetilde{\Lambda}_{R,2,k} = \frac{\tau(\alpha_k,\beta_k)\Lambda_k^2 e^{-i\arcsin t_k} + \tau(\alpha_k,-\beta_k)\Lambda_k^{-2}e^{i\arcsin t_k}}{2}. \tag{2.4.76}$$

Furthermore, we note the following relation:

$$\widetilde{\Lambda}_{R,1,k} - \widetilde{\Lambda}_{R,2,k} = -2t_k\widetilde{\Lambda}_{I,k}. \tag{2.4.77}$$

Now let us compute the other terms in (2.4.72). We compute the residue at -1 from (2.4.29), (2.4.40), (2.4.41), (2.4.59) and (2.4.63), and we find

$$\operatorname{Res}\bigl(J_R^{(1)}(z), z=-1\bigr) = \frac{1-4\alpha_0^2}{2^5\pi\psi(-1)} D_\infty^{\sigma_3} \begin{pmatrix} -1 & -i \\ -i & 1 \end{pmatrix} D_\infty^{-\sigma_3}. \tag{2.4.78}$$

Similarly, from (2.4.29), (2.4.38), (2.4.39), (2.4.51) and (2.4.54) we obtain

$$\operatorname{Res}\bigl((z-1)J_R^{(1)}(z), z=1\bigr) = \frac{5}{2^4 3\pi\psi(1)} D_\infty^{\sigma_3} \begin{pmatrix} -1 & i \\ i & 1 \end{pmatrix} D_\infty^{-\sigma_3}, \tag{2.4.79}$$

and

$$\operatorname{Res}\bigl(J_R^{(1)}(z), z=1\bigr) = \frac{D_\infty^{\sigma_3}}{2^5\pi\psi(1)} \times$$
$$\begin{pmatrix} -4(\mathcal{A}-\widetilde{\mathcal{B}_1})^2 + 1 + 2\frac{\psi'(1)}{\psi(1)} & 4i\left((\mathcal{A}-\widetilde{\mathcal{B}_1})^2 + 2(\mathcal{A}-\widetilde{\mathcal{B}_1}) + \frac{11}{12} - \frac{1}{2}\frac{\psi'(1)}{\psi(1)}\right) \\ 4i\left((\mathcal{A}-\widetilde{\mathcal{B}_1})^2 - 2(\mathcal{A}-\widetilde{\mathcal{B}_1}) + \frac{11}{12} - \frac{1}{2}\frac{\psi'(1)}{\psi(1)}\right) & 4(\mathcal{A}-\widetilde{\mathcal{B}_1})^2 - 1 - 2\frac{\psi'(1)}{\psi(1)} \end{pmatrix} D_\infty^{-\sigma_3}. \tag{2.4.80}$$

The quantity $R^{(1)}(-1)$ will also play an important role in Section 2.6. From another residue calculation, we obtain

$$\begin{aligned} R^{(1)}(-1) &= \sum_{j=1}^{m} \frac{-1}{1+t_j} \mathrm{Res}\big(J_R^{(1)}(s), s=t_j\big) - \mathrm{Res}\big(\frac{J_R^{(1)}(s)}{s+1}, s=-1\big) \\ & -\frac{1}{2}\mathrm{Res}\big(J_R^{(1)}(s), s=1\big) + \frac{1}{4}\mathrm{Res}\big((s-1)J_R^{(1)}(s), s=1\big). \end{aligned} \tag{2.4.81}$$

In (2.4.73), (2.4.79) and (2.4.80) we have already computed the above residues at $t_1, \ldots, t_m$ and at 1, the other residue at -1 can be computed from (2.4.29), (2.4.40)–(2.4.41), (2.4.59) and (2.4.63) from which we obtain:

$$\mathrm{Res}\big(\frac{J_R^{(1)}(s)}{s+1}, s=-1\big) = \frac{D_\infty^{\sigma_3}}{2^3 3\pi\psi(-1)} \begin{pmatrix} \frac{3}{2}(\mathcal{A}+\mathcal{B}_{-1})^2 - 2\alpha_0^2 - 1 + \frac{1-4\alpha_0^2}{4}\frac{\psi'(-1)}{\psi(-1)} & i\Big(\frac{3}{2}(\mathcal{A}+\mathcal{B}_{-1})^2 - 3(\mathcal{A}+\mathcal{B}_{-1}) + \alpha_0^2 + \frac{5}{4} + \frac{1-4\alpha_0^2}{4}\frac{\psi'(-1)}{\psi(-1)}\Big) \\ \cdots \quad i\Big(\frac{3}{2}(\mathcal{A}+\mathcal{B}_{-1})^2 + 3(\mathcal{A}+\mathcal{B}_{-1}) + \alpha_0^2 + \frac{5}{4} + \frac{1-4\alpha_0^2}{4}\frac{\psi'(-1)}{\psi(-1)}\Big) & -\frac{3}{2}(\mathcal{A}+\mathcal{B}_{-1})^2 + 2\alpha_0^2 + 1 - \frac{1-4\alpha_0^2}{4}\frac{\psi'(-1)}{\psi(-1)} \end{pmatrix} D_\infty^{-\sigma_3}. \tag{2.4.82}$$

Jacobi-type weights

In this case $J_R^{(1)}(z)$ has simple poles at all t_j, $j = 0, 1, \ldots, m+1$ as can be seen from (2.4.50), (2.4.58), and (2.4.63). For z outside all of the disks $\mathcal{D}_{t_j}$, $j = 0, 1, \ldots, m+1$, we have

$$\begin{aligned} R^{(1)}(z) &= \sum_{j=1}^{m} \frac{1}{z-t_j} \mathrm{Res}\big(J_R^{(1)}(s), s=t_j\big) + \frac{1}{z+1}\mathrm{Res}\big(J_R^{(1)}(s), s=-1\big) \\ & + \frac{1}{z-1}\mathrm{Res}\big(J_R^{(1)}(s), s=1\big). \end{aligned} \tag{2.4.83}$$

Here the residue at t_k is again given by (2.4.73) (with ρ given by (2.4.1)). The residues at -1 can be computed from (2.4.29), (2.4.40), (2.4.41), (2.4.59) and (2.4.63) and is given by

$$\mathrm{Res}\big(J_R^{(1)}(z), z=-1\big) = \frac{1-4\alpha_0^2}{2^4\pi\psi(-1)} D_\infty^{\sigma_3} \begin{pmatrix} -1 & -i \\ -i & 1 \end{pmatrix} D_\infty^{-\sigma_3}. \tag{2.4.84}$$

Similarly, from (2.4.29), (2.4.38), (2.4.39), (2.4.55) and (2.4.58) we obtain the residue at 1:

$$\mathrm{Res}\big(J_R^{(1)}(z), z=1\big) = \frac{1-4\alpha_{m+1}^2}{2^4\pi\psi(1)} D_\infty^{\sigma_3} \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} D_\infty^{-\sigma_3}. \tag{2.4.85}$$

2.5 Starting points of integration

Since we will find large n asymptotics only for the logarithmic derivative of Hankel determinant, we still face the classical problem of finding a good starting point for the integration. It turns out that in our case, it can be obtained by a direct computation, using some known results in the literature concerning standard Laguerre and Jacobi polynomials, and using the formula (2.3.21).

Lemma 2.5.1 *As $n \to \infty$, we have*

$$\log L_n((\alpha_0, 0, \dots, 0), \vec{0}, 2(x+1), 0) = \left(-\frac{3}{2} - \log 2\right) n^2 + (\log(2\pi) - \alpha_0(1 + \log 2))\, n + \left(\frac{\alpha_0^2}{2} - \frac{1}{6}\right) \log n + \frac{\alpha_0}{2} \log(2\pi) + 2\zeta'(-1) - \log G(1+\alpha_0) + \mathcal{O}(n^{-1}). \quad (2.5.1)$$

As $n \to \infty$, we have

$$\log J_n((\alpha_0, 0, \dots, 0, \alpha_{m+1}), \vec{0}, 0, 0) = -n^2 \log 2 + [(1-\alpha_0-\alpha_{m+1}) \log 2 + \log \pi] n + \frac{2\alpha_0^2 + 2\alpha_{m+1}^2 - 1}{4} \log n - \log(G(1+\alpha_0)G(1+\alpha_{m+1})) + 3\zeta'(-1) + \left(\frac{1}{12} - \frac{(\alpha_0 + \alpha_{m+1})^2}{2}\right) \log 2 + \frac{\alpha_0 + \alpha_{m+1}}{2} \log(2\pi) + \mathcal{O}(n^{-1}). \quad (2.5.2)$$

Proof From [18, equations (5.1.1) and (5.1.8)], the orthonormal polynomials of degree k with respect to the weight $e^{-x}x^{\alpha_0}$ (supported on $(0, \infty)$) has a leading coefficient given by

$$\frac{(-1)^k}{\sqrt{k!\,\Gamma(k+\alpha_0+1)}}.$$

Therefore, by a simple change of variables, the degree k orthonormal polynomials with respect to the weight $(x+1)^{\alpha_0} e^{-2n(x+1)}$ (supported on $(-1, \infty)$) has a leading coefficient given by

$$\frac{(-1)^k (2n)^{k + \frac{1+\alpha_0}{2}}}{\sqrt{k!\,\Gamma(k+\alpha_0+1)}}.$$

By applying formula (2.3.21) for this weight, one obtains that

$$\begin{aligned} L_n((\alpha_0, 0, \dots, 0), \vec{0}, 2(x+1), 0) &= (2n)^{-n(n+\alpha_0)} \prod_{k=1}^{n} \Gamma(k+\alpha_0)\Gamma(k) \\ &= (2n)^{-n(n+\alpha_0)} \frac{G(n+1)G(n+\alpha_0+1)}{G(1+\alpha_0)}, \end{aligned} \quad (2.5.3)$$

where we have used $G(z+1) = \Gamma(z)G(z)$. The Barne's G-function has a known asymptotics for large argument (see [35, eq (5.17.5)]). As $z \to \infty$ with $|\arg z| < \pi$, we have

$$\log G(z+1) = \frac{z^2}{4} + z\log\Gamma(z+1) - \left(\frac{z(z+1)}{2} + \frac{1}{12}\right)\log z - \frac{1}{12} + \zeta'(-1) + \mathcal{O}(z^{-2}). \quad (2.5.4)$$

The asymptotics of $\log\Gamma(z)$ is given by

$$\log\Gamma(z) = (z - \tfrac{1}{2})\log z - z + \tfrac{1}{2}\log(2\pi) + \frac{1}{12z} + \mathcal{O}(z^{-3}), \qquad \text{as } z \to \infty, \quad |\arg z| < \pi, \quad (2.5.5)$$

(see [35, eq (5.11.1)]). We obtain (2.5.1) by using the above asymptotic formulas in (2.5.3). Similarly, from [18, equations (4.3.3) and (4.21.6)], the degree k orthonormal polynomial with respect to the weight $(1-x)^{\alpha_{m+1}}(1+x)^{\alpha_0}$ has a leading coefficient given by

$$\frac{2^{-k}\sqrt{2k+\alpha_0+\alpha_{m+1}+1}\;\Gamma(2k+\alpha_0+\alpha_{m+1}+1)}{\sqrt{2^{\alpha_0+\alpha_{m+1}+1}\Gamma(k+1)\Gamma(k+\alpha_0+1)\Gamma(k+\alpha_{m+1}+1)\Gamma(k+\alpha_0+\alpha_{m+1}+1)}}$$

By applying formula (2.3.21) to this weight, one obtains

$$J_n((\alpha_0, 0, \ldots, 0, \alpha_{m+1}), \vec{0}, 0, 0) = 2^{n^2+n(\alpha_0+\alpha_{m+1})} \times \prod_{k=0}^{n-1} \frac{\Gamma(k+1)\Gamma(k+\alpha_0+1)\Gamma(k+\alpha_{m+1}+1)\Gamma(k+\alpha_0+\alpha_{m+1}+1)}{\Gamma(2k+\alpha_0+\alpha_{m+1}+1)\Gamma(2k+\alpha_0+\alpha_{m+1}+2)}.$$

Using the functional equation $G(z+1) = \Gamma(z)G(z)$ we can simplify the above product. We obtain

$$J_n((\alpha_0, 0, \ldots, 0, \alpha_{m+1}), \vec{0}, 0, 0) = 2^{n^2+n(\alpha_0+\alpha_{m+1})} \times \frac{G(n+1)G(n+\alpha_0+1)G(n+\alpha_{m+1}+1)G(n+\alpha_0+\alpha_{m+1}+1)}{G(1+\alpha_0)G(1+\alpha_{m+1})G(2n+\alpha_0+\alpha_{m+1}+1)}. \quad (2.5.6)$$

We obtain (2.5.2) by expanding (2.5.6) as $n \to +\infty$, using the asymptotic formulas (2.5.4) and (2.5.5). ■

As mentioned in the outline, large n asymptotics for $J_n(\vec{\alpha}, \vec{\beta}, 0, 0)$ are known in the literature, and we reproduce the precise statement here.

Theorem 2.5.2 *(Deift-Its-Krasovsky [9]). As $n \to \infty$, we have*

$$\log \frac{J_n(\vec{\alpha},\vec{\beta},0,0)}{J_n(\vec{0},\vec{0},0,0)} = \left[2i\sum_{j=1}^{m}\beta_j \arcsin t_j - \mathcal{A}\log 2\right]n + \left[\frac{\alpha_0^2+\alpha_{m+1}^2}{2} + \sum_{j=1}^{m}\left(\frac{\alpha_j^2}{4}-\beta_j^2\right)\right]\log n$$
$$+ i\mathcal{A}\sum_{j=1}^{m}\beta_j \arcsin t_j + \frac{i\pi}{2}\sum_{0\le j<k\le m+1}(\alpha_k\beta_j - \alpha_j\beta_k) + \frac{\alpha_0+\alpha_{m+1}}{2}\log(2\pi) - \frac{\alpha_0^2+\alpha_{m+1}^2}{2}\log 2$$
$$+ \sum_{0\le j<k\le m+1}\log\left(\frac{\left(1-t_jt_k-\sqrt{(1-t_j^2)(1-t_k^2)}\right)^{2\beta_j\beta_k}}{2^{\frac{\alpha_j\alpha_k}{2}}|t_j-t_k|^{\frac{\alpha_j\alpha_k}{2}+2\beta_j\beta_k}}\right) + \sum_{j=1}^{m}\log\frac{G(1+\frac{\alpha_j}{2}+\beta_j)G(1+\frac{\alpha_j}{2}-\beta_j)}{G(1+\alpha_j)}$$
$$- \sum_{j=1}^{m}\left(\frac{\alpha_j^2}{4}+\beta_j^2\right)\log(\sqrt{1-t_j^2}) - \log(G(1+\alpha_0)G(1+\alpha_{m+1})) - \sum_{j=1}^{m}2\beta_j^2\log 2 + \mathcal{O}\left(\frac{\log n}{n^{1-2\beta_{\max}}}\right), \tag{2.5.7}$$

where $\mathcal{A} = \sum_{j=0}^{m+1}\alpha_j$.

Remark 2.5.3 *The asymptotics* (2.5.7) *with $\vec{\beta}=\vec{0}$ and $\alpha_1 = \ldots = \alpha_m = 0$ is consistent with* (2.5.2)*.*

Remark 2.5.4 *Our notation differs slightly from the one used in [9]: α_j and β_j in our paper corresponds to $2\alpha_{m+1-j}$ and β_{m+1-j} in the paper [9].*

The goal of the next section is to obtain a similar formula as (2.5.7) for $L_n(\vec{\alpha},\vec{\beta},2(x+1),0)$.

2.6 Integration in $\vec{\alpha}$ and $\vec{\beta}$ for the Laguerre weight

In this section, we specialize to the classical Laguerre weight with FH singularities

$$w(x) = e^{-2n(x+1)}\omega(x), \tag{2.6.1}$$

supported on $\mathcal{I} = [-1,+\infty)$. In this case, we recall that $\ell = 2+2\log 2$ and $\psi(x) = \frac{1}{\pi}$. We will find large n asymptotics for the differential identity (2.3.13), and then integrate in the parameters $\alpha_0,\ldots,\alpha_m,\beta_1,\ldots,\beta_m$. We first focus on finding large n asymptotics for $\widetilde{Y}(t_k)$, $k=0,\ldots,m$.

Proposition 2.6.1 *For $k \in \{1,\dots,m\}$, as $n \to +\infty$, we have*

$$\widetilde{Y}(t_k) = e^{-\frac{n\ell}{2}\sigma_3}(I + \mathcal{O}(n^{-1+2\beta_{max}}))E_{t_k}(t_k)\begin{pmatrix}\Phi_{k,11} & \Phi_{k,12}\\ \Phi_{k,21} & \Phi_{k,22}\end{pmatrix}e^{n(t_k+1)\sigma_3}, \tag{2.6.2}$$

where

$$\Phi_{k,11} = \frac{\Gamma(1+\frac{\alpha_k}{2}-\beta_k)}{\Gamma(1+\alpha_k)}\left(2n\frac{\sqrt{1-t_k}}{\sqrt{1+t_k}}\right)^{\frac{\alpha_k}{2}}\omega_k^{-\frac{1}{2}}(t_k), \quad \Phi_{k,12} = \frac{-\alpha_k\Gamma(\alpha_k)}{\Gamma(\frac{\alpha_k}{2}+\beta_k)}\left(2n\frac{\sqrt{1-t_k}}{\sqrt{1+t_k}}\right)^{-\frac{\alpha_k}{2}}\omega_k^{\frac{1}{2}}(t_k),$$
$$\Phi_{k,21} = \frac{\Gamma(1+\frac{\alpha_k}{2}+\beta_k)}{\Gamma(1+\alpha_k)}\left(2n\frac{\sqrt{1-t_k}}{\sqrt{1+t_k}}\right)^{\frac{\alpha_k}{2}}\omega_k^{-\frac{1}{2}}(t_k), \quad \Phi_{k,22} = \frac{\alpha_k\Gamma(\alpha_k)}{\Gamma(\frac{\alpha_k}{2}-\beta_k)}\left(2n\frac{\sqrt{1-t_k}}{\sqrt{1+t_k}}\right)^{-\frac{\alpha_k}{2}}\omega_k^{\frac{1}{2}}(t_k). \tag{2.6.3}$$

As $n \to +\infty$, we have

$$\widetilde{Y}(-1) = e^{-\frac{n\ell}{2}\sigma_3}\Big(I + \frac{R^{(1)}(-1)}{n} + \mathcal{O}(n^{-2+2\beta_{\max}})\Big)E_{-1}(-1)\begin{pmatrix}\Phi_{0,11} & \Phi_{0,12}\\ \Phi_{0,21} & \Phi_{0,22}\end{pmatrix}, \tag{2.6.4}$$

where $R^{(1)}(-1)$ is given explicitly in (2.4.81) and

$$\Phi_{0,11} = \frac{1}{\Gamma(1+\alpha_0)}\left(\sqrt{2}n\right)^{\alpha_0}\omega_{-1}^{-\frac{1}{2}}(-1), \qquad \Phi_{0,12} = -\frac{i\alpha_0\Gamma(\alpha_0)}{2\pi}\left(\sqrt{2}n\right)^{-\alpha_0}\omega_{-1}^{\frac{1}{2}}(-1),$$
$$\Phi_{0,21} = -\frac{\pi i\alpha_0}{\Gamma(1+\alpha_0)}\left(\sqrt{2}n\right)^{\alpha_0}\omega_{-1}^{-\frac{1}{2}}(-1), \quad \Phi_{0,22} = \frac{\alpha_0^2\Gamma(\alpha_0)}{2}\left(\sqrt{2}n\right)^{-\alpha_0}\omega_{-1}^{\frac{1}{2}}(-1). \tag{2.6.5}$$

Proof For fixed $1 \le k \le m$, let $z \in \mathcal{D}_{t_k} \cap Q_{+,k}^R$ be outside the lenses. By inverting the RH transformations $Y \mapsto T \mapsto S \mapsto R$, we obtain

$$Y(z) = e^{-\frac{n\ell}{2}\sigma_3}R(z)P^{(t_k)}(z)e^{ng(z)\sigma_3}e^{\frac{n\ell}{2}\sigma_3} \tag{2.6.6}$$

where $P^{(t_k)}(z)$ is given by (2.4.45). From [35, Section 13.14(iii)], we have

$$G(a,\alpha_k;z) = z^{\frac{\alpha_k}{2}}(1+\mathcal{O}(z)), \qquad z \to 0, \tag{2.6.7}$$

and, if $\alpha_k \neq 0$, and $a - \frac{\alpha_k}{2} \pm \frac{\alpha_k}{2} \neq 0, -1, -2, \dots$, as $z \to 0$ we have

$$H(a,\alpha_k;z) = \begin{cases} \frac{\Gamma(\alpha_k)}{\Gamma(a)}z^{-\frac{\alpha_k}{2}} + \mathcal{O}(z^{1-\frac{\Re\alpha_k}{2}}) + \mathcal{O}(z^{\frac{\Re\alpha_k}{2}}) & \Re\alpha_k > 0,\\ \frac{\Gamma(-\alpha_k)}{\Gamma(a-\alpha_k)}z^{\frac{\alpha_k}{2}} + \frac{\Gamma(\alpha_k)}{\Gamma(a)}z^{-\frac{\alpha_k}{2}} + \mathcal{O}(z^{1+\frac{\Re\alpha_k}{2}}) & -1 < \Re\alpha_k \le 0.\end{cases} \tag{2.6.8}$$

Conditions $a - \frac{\alpha_k}{2} \pm \frac{\alpha_k}{2} \neq 0, -1, -2, \dots$ for $a = \frac{\alpha_k}{2} - \beta_k$ and $a = 1 + \frac{\alpha_k}{2} - \beta_k$ reduce to $-\beta_k \pm \frac{\alpha_k}{2} \neq 0, -1, -2, \dots$. Recalling that $V(x) = 2(x+1)$ and $\psi(x) = \dfrac{1}{\pi}$, and using (2.4.44), we find that the leading terms of $E_{t_k}^{-1}(z)P^{(t_k)}(z)e^{n\xi(z)\sigma_3}$ as $z \to t_k$ for $\alpha_k \neq 0$, $-\beta_k \pm \frac{\alpha_k}{2} \neq 0, -1, -2, \dots$ are given by

$$\begin{pmatrix} \Phi_{k,11} & \alpha_k^{-1}\left(\Phi_{k,12} + \tilde{c_k}\Phi_{k,11}(z-t_k)^{\alpha_k}\right) \\ \Phi_{k,21} & \alpha_k^{-1}\left(\Phi_{k,22} + \tilde{c_k}\Phi_{k,21}(z-t_k)^{\alpha_k}\right) \end{pmatrix}, \tag{2.6.9}$$

where

$$\tilde{c_k} = \alpha_k \frac{\Gamma(-\alpha_k)\Gamma(1+\alpha_k)e^{-\frac{\pi i \alpha_k}{2}}\omega_k(t_k)}{\Gamma(-\frac{\alpha_k}{2} - \beta_k)\Gamma(1 + \frac{\alpha_k}{2} + \beta_k)} = \frac{\pi\alpha_k}{\sin(\pi\alpha_k)} \frac{e^{i\pi\beta_k} - e^{-i\pi\alpha_k}e^{-i\pi\beta_k}}{2\pi i}\omega_k(t_k) = e^{2n(t_k+1)}c_k \tag{2.6.10}$$

and c_k is given by (2.3.9). The claim (2.6.2) for $\alpha_k \neq 0$, $-\beta_k \pm \frac{\alpha_k}{2} \neq 0, -1, -2, \dots$ follows from (2.3.8), (2.3.9), (2.4.11), (2.4.67), (2.6.6) and (2.6.9). We extend it for general parameters α_k and β_k (still subject to the constraint $\Re\alpha_k > -1$ and $\Re\beta_k \in (-\frac{1}{2}, \frac{1}{2})$) by continuity of $\widetilde{Y}(t_k)$ in α_k and β_k (this can be shown by a simple contour deformation, see e.g. [4, eq (29) and below]). Now we turn to the proof of (2.6.4). For $z \in \mathcal{D}_{-1} \setminus \overline{(\Omega_+ \cup \Omega_-)}$, from Section 2.4, we have

$$Y(z) = e^{-\frac{n\ell}{2}\sigma_3} R(z) P^{(-1)}(z) e^{ng(z)\sigma_3} e^{\frac{n\ell}{2}\sigma_3}. \tag{2.6.11}$$

In this region, by (2.4.60) and (C.2.4), $P^{(-1)}(z)$ is given by

$$P^{(-1)}(z) = E_{-1}(z)\sigma_3 \begin{pmatrix} I_{\alpha_0}(2n(-f_{-1}(z))^{\frac{1}{2}}) & \frac{i}{\pi}K_{\alpha_0}(2n(-f_{-1}(z))^{\frac{1}{2}}) \\ 2\pi i n(-f_{-1}(z))^{\frac{1}{2}} I'_{\alpha_0}(2n(-f_{-1}(z))^{\frac{1}{2}}) & -2n(-f_{-1}(z))^{\frac{1}{2}} K'_{\alpha_0}(2n(-f_{-1}(z))^{\frac{1}{2}}) \end{pmatrix}$$
$$\times \sigma_3 \omega_{-1}(z)^{-\frac{\sigma_3}{2}}(-z-1)^{-\frac{\alpha_0}{2}\sigma_3} e^{-n\xi(z)\sigma_3}.$$

From [35, Section 10.30(i)], we have the following asymptotic behaviors as $z \to 0$ for the modified Bessel functions

$$I_{\alpha_0}(z) = \frac{1}{\Gamma(\alpha_0+1)}\left(\frac{z}{2}\right)^{\alpha_0}(1 + \mathcal{O}(z^2)),$$

$$K_{\alpha_0}(z) = \begin{cases} \frac{\Gamma(\alpha_0)}{2}(\frac{z}{2})^{-\alpha_0} + \mathcal{O}(z^{1-\Re\alpha_0}) + \mathcal{O}(z^{\Re\alpha_0}), & \text{if } \Re\alpha_0 \geq 0, \alpha_0 \neq 0, \\ \frac{\Gamma(-\alpha_0)}{2}(\frac{z}{2})^{\alpha_0} + \frac{\Gamma(\alpha_0)}{2}(\frac{z}{2})^{-\alpha_0} + \mathcal{O}(z^{2+\Re\alpha_0}), & \text{if } -1 < \Re\alpha_0 < 0. \end{cases}$$

Using (2.4.59), for $\alpha_k \neq 0$, we find that the leading terms of $E_{-1}^{-1}(z)P^{(-1)}(z)e^{n\xi(z)\sigma_3}$ as $z \to -1$ are given by

$$\begin{pmatrix} \Phi_{0,11} & \alpha_0^{-1}\left(\Phi_{0,12} + \tilde{c_0}\Phi_{0,11}(-z-1)^{\alpha_0}\right) \\ \Phi_{0,21} & \alpha_0^{-1}\left(\Phi_{0,22} + \tilde{c_0}\Phi_{0,21}(-z-1)^{\alpha_0}\right) \end{pmatrix}, \tag{2.6.12}$$

where

$$\tilde{c_0} = \frac{i\alpha_0}{2\sin(\pi\alpha_0)}\omega_{-1}(-1) = e^{\pi i\alpha_0}c_0 \tag{2.6.13}$$

and where we recall that c_0 is defined in (2.3.10). This proves (2.6.4) for $\alpha_0 \neq 0$. The case $\alpha_0 = 0$ follows by continuity of $\widetilde{Y}(-1)$. ■

2.6.1 Asymptotics for $\partial_\nu \log L_n(\vec{\alpha}, \vec{\beta}, 2(x+1), 0)$, $\nu \in \{\alpha_0, \ldots, \alpha_m, \beta_1, \ldots, \beta_m\}$

From (2.2.5) and (2.3.12), we have

$$\kappa_{n-1}^2 = \lim_{z\to\infty} \frac{iY_{21}(z)}{2\pi z^{n-1}}, \qquad \kappa_n^{-2} = -2\pi i \lim_{z\to\infty} z^{n+1}Y_{12}(z), \qquad \eta_n = \lim_{z\to\infty} \frac{Y_{11}(z) - z^n}{z^{n-1}}. \tag{2.6.14}$$

Inverting the transformations $Y \mapsto T \mapsto S \mapsto R$ for $z \in \mathbb{C} \setminus \overline{\left(\Omega_+ \cup \Omega_- \cup (\mathcal{I} \setminus \mathcal{S}) \cup_{j=0}^{m+1} \mathcal{D}_j\right)}$ (i.e. outside the lenses and outside the disks) gives

$$Y(z) = e^{-\frac{n\ell}{2}\sigma_3}R(z)P^{(\infty)}(z)e^{ng(z)\sigma_3}e^{\frac{n\ell}{2}\sigma_3}. \tag{2.6.15}$$

From (2.4.12), (2.4.42), (2.4.67), (2.4.72) and (2.6.14), we find large n asymptotic for $\kappa_{n-1}^2, \kappa_n^2$ and η_n. As $n \to +\infty$, we have

$$\kappa_{n-1}^2 = e^{2n}2^{2(n-1)+\mathcal{A}}\pi^{-1}\exp\left(-2i\sum_{j=1}^m \beta_j \arcsin t_j\right)\left(1 + \frac{R_{1,21}^{(1)}}{nP_{1,21}^{(\infty)}} + \mathcal{O}(n^{-2+2\beta_{max}})\right), \tag{2.6.16}$$

where $\mathcal{A} = \alpha_0 + \alpha_1 + \ldots + \alpha_m$ and

$$\frac{R_{1,21}^{(1)}}{P_{1,21}^{(\infty)}} = \sum_{j=1}^m \frac{v_j(1 - \widetilde{\Lambda}_{R,1,j})}{1 - t_j} + \frac{1 - 4\alpha_0^2}{16} - \frac{1}{4}\left((\mathcal{A} - \widetilde{\mathcal{B}_1})^2 - 2(\mathcal{A} - \widetilde{\mathcal{B}_1}) + \frac{11}{12}\right). \tag{2.6.17}$$

Similarly, for κ_n^2 we find

$$\kappa_n^2 = e^{2n}2^{2n+\mathcal{A}}\pi^{-1}\exp\left(-2i\sum_{j=1}^m \beta_j \arcsin t_j\right)\left(1 - \frac{R_{1,12}^{(1)}}{nP_{1,12}^{(\infty)}} + \mathcal{O}(n^{-2+2\beta_{max}})\right), \tag{2.6.18}$$

as $n \to +\infty$, where by (2.4.72) we have

$$-\frac{R^{(1)}_{1,12}}{P^{(\infty)}_{1,12}} = \sum_{j=1}^{m} \frac{v_j(1+\widetilde{\Lambda}_{R,2,j})}{1-t_j} + \frac{1-4\alpha_0^2}{16} - \frac{1}{4}\Big((\mathcal{A}-\widetilde{\mathcal{B}_1})^2 + 2(\mathcal{A}-\widetilde{\mathcal{B}_1}) + \frac{11}{12}\Big). \tag{2.6.19}$$

Finally, for η_n we obtain

$$\eta_n = \frac{n}{2} + P^{(\infty)}_{1,11} + \frac{R^{(1)}_{1,11}}{n} + \mathcal{O}(n^{-2+2\beta_{max}}), \qquad \text{as } n \to +\infty, \tag{2.6.20}$$

where $P^{(\infty)}_{1,11}$ is given by (2.4.42) and $R^{(1)}_{1,11}$ can be computed from (2.4.72) and is given by

$$R^{(1)}_{1,11} = \sum_{j=1}^{m} \frac{v_j(t_j+\widetilde{\Lambda}_{I,j})}{2(1-t_j)} - \frac{1-4\alpha_0^2}{32} + \frac{1-4(\mathcal{A}-\widetilde{\mathcal{B}_1})^2}{32}. \tag{2.6.21}$$

Let $\nu \in \{\alpha_0, \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_m\}$. Then, from (2.4.69), (2.6.16), (2.6.18) and (2.6.20), we find that the large n asymptotics of the first part of the differential identity (2.3.13) are given by

$$\begin{aligned} -(n+\mathcal{A})\partial_\nu \log(\kappa_n\kappa_{n-1}) + 2n\partial_\nu\eta_n = \partial_\nu\Big(2\log D_\infty - \alpha_0 + \sum_{j=1}^{m} t_j\alpha_j + 2i\sum_{j=1}^{m}\sqrt{1-t_j^2}\beta_j\Big)n + \\ 2\mathcal{A}\partial_\nu \log D_\infty + \partial_\nu\Big(\frac{\alpha_0^2}{2}\Big) + \partial_\nu \sum_{j=1}^{m} v_j(\widetilde{\Lambda}_{I,j}-1) + \mathcal{O}\left(\frac{\log n}{n^{1-4\beta_{\max}}}\right). \end{aligned} \tag{2.6.22}$$

Now we compute the second part of the differential identity (2.3.13). First, we compute the contributions from t_j, $j = 1, \ldots, m$ using (2.4.47), (2.4.69), (2.6.2) and (2.6.3). We obtain

$$\begin{aligned} &\sum_{j=1}^{m}\Big(\widetilde{Y}_{22}(t_j)\partial_\nu Y_{11}(t_j) - \widetilde{Y}_{12}(t_j)\partial_\nu Y_{21}(t_j) + Y_{11}(t_j)\widetilde{Y}_{22}(t_j)\partial_\nu \log(\kappa_n\kappa_{n-1})\Big) = \\ &-(\mathcal{A}-\alpha_0)\partial_\nu \log D_\infty + \sum_{j=1}^{m}\Big(\Phi_{j,22}\partial_\nu\Phi_{j,11} - \Phi_{j,12}\partial_\nu\Phi_{j,21} - 2\beta_j\partial_\nu \log\Lambda_j\Big) + \mathcal{O}\Big(\frac{\log n}{n^{1-4\beta_{\max}}}\Big). \end{aligned} \tag{2.6.23}$$

Note that $E_{-1}(-1) = \mathcal{O}(n^{\frac{\sigma_3}{2}})$ as $n \to +\infty$, while $E_{t_k}(t_k) = \mathcal{O}(n^{\beta_k\sigma_3})$, $k = 1, \ldots, m$. This makes the computations for the contribution from -1 more involved. From (2.4.62), (2.4.69) (2.6.4) and (2.6.5), we obtain

$$\begin{aligned} &\widetilde{Y}_{22}(-1)\partial_\nu Y_{11}(-1) - \widetilde{Y}_{12}(-1)\partial_\nu Y_{21}(-1) + Y_{11}(-1)\widetilde{Y}_{22}(-1)\partial_\nu \log(\kappa_n\kappa_{n-1}) = \\ &+\partial_\nu\Big(R^{(1)}_{11}(-1) - R^{(1)}_{22}(-1) + iD_\infty^{-2}R^{(1)}_{12}(-1) + iD_\infty^{2}R^{(1)}_{21}(-1) + iD_\infty^{-2}R^{(1)}_{1,12} + iD_\infty^{2}R^{(1)}_{1,21}\Big) \\ &-\alpha_0\partial_\nu \log D_\infty + \Phi_{0,22}\partial_\nu\Phi_{0,11} - \Phi_{0,12}\partial_\nu\Phi_{0,21} + \mathcal{O}\Big(\frac{\log n}{n^{1-4\beta_{\max}}}\Big). \end{aligned} \tag{2.6.24}$$

We observe significant simplifications using (2.4.73), (2.4.77), (2.4.78), (2.4.80), (2.4.81), and (2.4.82):

$$R_{11}^{(1)}(-1)-R_{22}^{(1)}(-1)+iD_\infty^{-2}R_{12}^{(1)}(-1)+iD_\infty^{2}R_{21}^{(1)}(-1)+iD_\infty^{-2}R_{1,12}^{(1)}+iD_\infty^{2}R_{1,21}^{(1)} = -\sum_{j=1}^{m} v_j\widetilde{\Lambda}_{I,j}. \tag{2.6.25}$$

Adding (2.6.22), (2.6.23) and (2.6.24) yields

$$\partial_\nu \log L_n(\vec{\alpha},\vec{\beta},2(x+1),0) = \partial_\nu\Big(2\log D_\infty - \alpha_0 + \sum_{j=1}^m t_j\alpha_j + 2i\sum_{j=1}^m \sqrt{1-t_j^2}\beta_j\Big)n + \mathcal{A}\partial_\nu \log D_\infty$$
$$+\,\partial_\nu\Big(\frac{\alpha_0^2}{2}\Big) + \sum_{j=0}^m \Big(\Phi_{j,22}\partial_\nu\Phi_{j,11} - \Phi_{j,12}\partial_\nu\Phi_{j,21}\Big) - \sum_{j=1}^m(\partial_\nu v_j + 2\beta_j\partial_\nu\log\Lambda_j) + \mathcal{O}\Big(\frac{\log n}{n^{1-4\beta_{\max}}}\Big), \tag{2.6.26}$$

as $n \to +\infty$. Now, we perform some computations to make the above asymptotic formula more explicit. From (2.6.5) and using the identity $z\Gamma(z) = \Gamma(z+1)$ we have

$$\Phi_{0,22}\partial_\nu\Phi_{0,11} - \Phi_{0,12}\partial_\nu\Phi_{0,21} =$$
$$\frac{\alpha_0}{2}\partial_\nu\log\Big(\frac{\alpha_0}{\Gamma(1+\alpha_0)^2}\Big) + \alpha_0\log(\sqrt{2}n)\partial_\nu\alpha_0 - \frac{\alpha_0}{2}\partial_\nu\Big(\sum_{\ell=1}^m \alpha_\ell\log(1+t_\ell) + i\pi\sum_{\ell=1}^m\beta_\ell\Big). \tag{2.6.27}$$

And from (2.6.3), after a long computation, for $1 \le j \le m$ we obtain

$$\Phi_{j,22}\partial_\nu\Phi_{j,11} - \Phi_{j,12}\partial_\nu\Phi_{j,21} = \frac{\alpha_j}{2}\partial_\nu\log\frac{\Gamma(1+\frac{\alpha_j}{2}-\beta_j)\Gamma(1+\frac{\alpha_j}{2}+\beta_j)}{\Gamma(1+\alpha_j)^2} + \frac{\alpha_j}{2}\log\Big(2n\frac{\sqrt{1-t_j}}{\sqrt{1+t_j}}\Big)\partial_\nu\alpha_j$$
$$+\,\beta_j\partial_\nu\log\frac{\Gamma(1+\frac{\alpha_j}{2}+\beta_j)}{\Gamma(1+\frac{\alpha_j}{2}-\beta_j)} - \frac{\alpha_j}{2}\partial_\nu\Big(\sum_{\substack{\ell=0\\ \ell\neq j}}^m \alpha_\ell\log|t_\ell - t_j| - i\pi\sum_{\ell=1}^{j-1}\beta_\ell + i\pi\sum_{\ell=j+1}^m\beta_\ell\Big). \tag{2.6.28}$$

Also, from (2.4.48) and (2.4.49), we have

$$\partial_\nu\log\Lambda_j = \partial_\nu\Big(\frac{i\mathcal{A}}{2}\arccos t_j - \frac{\pi i}{4}\alpha_j - \frac{\pi i}{2}\sum_{\ell=j+1}^m\alpha_\ell + \beta_j\log(4\pi\rho(t_j)n(1-t_j^2)) - \sum_{\substack{\ell=1\\ \ell\neq j}}^m\beta_\ell\log T_{j\ell}\Big). \tag{2.6.29}$$

Substituting (2.6.27)–(2.6.29) into (2.6.26), and using the expression for D_∞ and v_j given by (2.4.33) and below (2.4.50), we obtain

$$\partial_\nu \log L_n(\vec{\alpha},\vec{\beta},2(x+1),0) = \partial_\nu\bigg(\sum_{j=0}^{m}(t_j-\log 2)\alpha_j + 2i\sum_{j=1}^{m}\beta_j\big(\arcsin t_j + \sqrt{1-t_j^2}\,\big)\bigg)n$$
$$+\mathcal{A}\partial_\nu\Big(i\sum_{j=1}^{m}\beta_j\arcsin t_j - \frac{\mathcal{A}}{2}\log 2\Big)+\partial_\nu\Big(\frac{\alpha_0^2}{2}\Big)+\frac{\alpha_0}{2}\partial_\nu\log\frac{\alpha_0}{\Gamma(1+\alpha_0)^2}+\alpha_0\log(\sqrt{2}n)\partial_\nu\alpha_0$$
$$-\sum_{j=0}^{m}\frac{\alpha_j}{2}\partial_\nu\bigg(\sum_{\substack{\ell=0\\ \ell\neq j}}^{m}\alpha_\ell\log|t_\ell-t_j| - i\pi\sum_{\ell=1}^{j-1}\beta_\ell + i\pi\sum_{\ell=j+1}^{m}\beta_\ell\bigg)+\sum_{j=1}^{m}\frac{\alpha_j}{2}\log\Big(2n\frac{\sqrt{1-t_j}}{\sqrt{1+t_j}}\Big)\partial_\nu\alpha_j$$
$$+\sum_{j=1}^{m}\frac{\alpha_j}{2}\partial_\nu\log\frac{\Gamma(1+\frac{\alpha_j}{2}-\beta_j)\Gamma(1+\frac{\alpha_j}{2}+\beta_j)}{\Gamma(1+\alpha_j)^2}+\sum_{j=1}^{m}\beta_j\partial_\nu\log\frac{\Gamma(1+\frac{\alpha_j}{2}+\beta_j)}{\Gamma(1+\frac{\alpha_j}{2}-\beta_j)}+\sum_{j=1}^{m}\partial_\nu\Big(\frac{\alpha_j^2}{4}-\beta_j^2\Big)$$
$$-\sum_{j=1}^{m}2\beta_j\partial_\nu\bigg(\frac{i\mathcal{A}}{2}\arccos t_j-\frac{\pi i}{4}\alpha_j-\frac{\pi i}{2}\sum_{\ell=j+1}^{m}\alpha_\ell+\beta_j\log(4\pi\rho(t_j)n(1-t_j^2))-\sum_{\substack{\ell=1\\ \ell\neq j}}^{m}\beta_\ell\log T_{j\ell}\bigg)$$
$$+\mathcal{O}\Big(\frac{\log n}{n^{1-4\beta_{\max}}}\Big),\qquad \text{as } n\to+\infty, \tag{2.6.30}$$

where we recall that $t_0=-1$. From the discussion in Subsection 2.4.8, the above error term is uniform *for all* $(\vec{\alpha},\vec{\beta})$ in a given compact set Ω, and uniform in $\vec{t}$ such that (2.4.68) holds. However, as stated in Proposition 2.3.2, the identity (2.6.30) itself is valid on the subset $\Omega\setminus\widetilde{\Omega}$ for which $p_0,\dots,p_n$ exist. From the determinantal representation of orthogonal polynomials, $\widetilde{\Omega}$ is locally finite and we can extend (2.6.30) *for all* $(\vec{\alpha},\vec{\beta})\in\Omega$ by continuity (for n large enough such that the r.h.s. exists). We refer to [3–5, 9] for very similar situations, with more details provided. Our goal for the rest of this section is to prove Proposition 2.6.2 below.

Proposition 2.6.2 *As $n\to\infty$, we have*

$$\log\frac{L_n(\vec{\alpha},\vec{\beta},2(x+1),0)}{L_n(\vec{\alpha_0},\vec{0},2(x+1),0)} = 2in\sum_{j=1}^{m}\beta_j\Big(\arcsin t_j+\sqrt{1-t_j^2}\Big)+\sum_{j=1}^{m}(t_j-\log 2)\alpha_j n+\sum_{j=1}^{m}\frac{\alpha_j^2}{4}\log\Big(n\frac{\sqrt{1-t_j}}{\sqrt{1+t_j}}\Big)$$
$$-\sum_{j=1}^{m}\beta_j^2\log(4\pi\rho(t_j)n(1-t_j^2))+\sum_{j=1}^{m}\log\frac{G(1+\frac{\alpha_j}{2}+\beta_j)G(1+\frac{\alpha_j}{2}-\beta_j)}{G(1+\alpha_j)}+\frac{i\pi}{2}\sum_{0\le j<k\le m}(\alpha_k\beta_j-\alpha_k\beta_j)$$
$$+i\mathcal{A}\sum_{j=1}^{m}\beta_j\arcsin t_j+2\sum_{1\le j<k\le m}\beta_j\beta_k\log T_{jk}-\frac{\log 2}{2}\sum_{0\le j<k\le m}\alpha_j\alpha_k-\sum_{0\le j<k\le m}\frac{\alpha_j\alpha_k}{2}\log|t_k-t_j|+\mathcal{O}\Big(\frac{\log n}{n^{1-4\beta_{\max}}}\Big), \tag{2.6.31}$$

where $\log L_n(\vec{\alpha_0},\vec{0},2(x+1),0)$ *is given by* (2.5.1).

2.6.2 Integration in α_0

In this short subsection, we make a consistency check with (2.5.1). Let us set $\alpha_1 = \ldots = \alpha_m = 0 = \beta_1 = \ldots = \beta_m$ and $\nu = \alpha_0$ in (2.6.30). With the notations $\vec{\alpha}_0 = (\alpha_0, 0, \ldots, 0) \in \mathbb{C}^{m+1}$ and $\vec{0} = (0, \ldots, 0) \in \mathbb{C}^m$, this gives

$$\partial_{\alpha_0} \log L_n(\vec{\alpha}_0, \vec{0}, 2(x+1), 0) = -(1+\log 2)n - \frac{\log 2}{2}\alpha_0 + \alpha_0 + \frac{\alpha_0}{2}\partial_{\alpha_0} \log \frac{\alpha_0}{\Gamma(1+\alpha_0)^2} + \alpha_0 \log(\sqrt{2}n) + \mathcal{O}\Big(\frac{\log n}{n}\Big) \tag{2.6.32}$$

as $n \to +\infty$. Integrating (2.6.32) from $\alpha_0 = 0$ to an arbitrary α_0, we obtain

$$\log \frac{L_n(\vec{\alpha}_0, \vec{0}, 2(x+1), 0)}{L_n(\vec{0}, \vec{0}, 2(x+1), 0)} = -(1+\log 2)\alpha_0 n + \frac{\alpha_0^2}{2}\Big(1 - \frac{\log 2}{2}\Big) + \int_0^{\alpha_0} \frac{x}{2}\partial_x \log \frac{x}{\Gamma(1+x)^2} dx + \frac{\alpha_0^2}{2}\log(\sqrt{2}n) + \mathcal{O}\Big(\frac{\log n}{n}\Big). \tag{2.6.33}$$

From [35, formula 5.17.4], we have

$$\int_0^z \log \Gamma(1+x)dx = \frac{z}{2}\log 2\pi - \frac{z(z+1)}{2} + z\log\Gamma(z+1) - \log G(z+1), \tag{2.6.34}$$

where G is Barnes' G-function. Therefore, after an integration by parts, (2.6.33) can be rewritten as

$$\log \frac{L_n(\vec{\alpha}_0, \vec{0}, 2(x+1), 0)}{L_n(\vec{0}, \vec{0}, 2(x+1), 0)} = -(1+\log 2)\alpha_0 n + \frac{\alpha_0^2}{2}\log n + \frac{\alpha_0}{2}\log 2\pi - \log G(1+\alpha_0) + \mathcal{O}\Big(\frac{\log n}{n}\Big),$$

which is consistent with (2.5.1).

2.6.3 Integration in $\alpha_1, \ldots, \alpha_m$

We set $\alpha_2 = \ldots = \alpha_m = 0 = \beta_1 = \ldots = \beta_m$ and $\nu = \alpha_1$ in (2.6.30). With the notation $\vec{\alpha}_1 = (\alpha_0, \alpha_1, 0, \ldots, 0) \in \mathbb{C}^{m+1}$, we obtain

$$\partial_{\alpha_1} \log L_n(\vec{\alpha}_1, \vec{0}, 2(x+1), 0) = (t_1 - \log 2)n - \frac{\log 2}{2}\alpha_0 - \frac{\alpha_0}{2}\log|t_1 - t_0| + \frac{\alpha_1}{2}\log\Big(n\frac{\sqrt{1-t_1}}{\sqrt{1+t_1}}\Big) + \alpha_1\partial_{\alpha_1}\log\frac{\Gamma(1+\frac{\alpha_1}{2})}{\Gamma(1+\alpha_1)} + \frac{\alpha_1}{2} + \mathcal{O}\Big(\frac{\log n}{n}\Big), \tag{2.6.35}$$

as $n \to +\infty$. Using integration by parts and (2.6.34) we obtain, we obtain the following relation

$$\int_0^z x\partial_x \log \frac{\Gamma(1+\frac{x}{2})}{\Gamma(1+x)} dx = -\frac{z^2}{4} + \log \frac{G(1+\frac{z}{2})^2}{G(1+z)}. \tag{2.6.36}$$

Using (2.6.36), we integrate (2.6.35) from $\alpha_1 = 0$ to an arbitrary α_1. We get

$$\log \frac{L_n(\vec{\alpha}_1, \vec{0}, 2(x+1), 0)}{L_n(\vec{\alpha}_0, \vec{0}, 2(x+1), 0)} = (t_1 - \log 2)\alpha_1 n - \frac{\log 2}{2}\alpha_0\alpha_1 - \frac{\alpha_0\alpha_1}{2}\log|t_1 - t_0| \\ + \frac{\alpha_1^2}{4}\log\left(n\frac{\sqrt{1-t_1}}{\sqrt{1+t_1}}\right) + \log\frac{G(1+\frac{\alpha_1}{2})^2}{G(1+\alpha_1)} + \mathcal{O}\Big(\frac{\log n}{n}\Big), \qquad \text{as } n \to +\infty. \tag{2.6.37}$$

We proceed in a similar way for the other variables, by integrating successively in $\alpha_2, \alpha_3, \ldots, \alpha_m$. At the last step, setting $\beta_1 = \ldots = \beta_m = 0$ and $\nu = \alpha_m$ in (2.6.30), we obtain

$$\partial_{\alpha_m} \log L_n(\vec{\alpha}, \vec{0}, 2(x+1), 0) = (t_m - \log 2)n - \frac{\log 2}{2}(\mathcal{A} - \alpha_m) - \sum_{j=0}^{m-1}\frac{\alpha_j}{2}\log|t_m - t_j| \\ + \frac{\alpha_m}{2}\log\left(n\frac{\sqrt{1-t_m}}{\sqrt{1+t_m}}\right) + \alpha_m\partial_{\alpha_m}\log\frac{\Gamma(1+\frac{\alpha_m}{2})}{\Gamma(1+\alpha_m)} + \frac{\alpha_m}{2} + \mathcal{O}\Big(\frac{\log n}{n}\Big). \tag{2.6.38}$$

Integrating (2.6.38) from $\alpha_m = 0$ to an arbitrary α_m using again (2.6.36), and with the notation $\vec{\alpha}_{m-1} = (\alpha_0, \ldots, \alpha_{m-1}, 0)$, we obtain

$$\log \frac{L_n(\vec{\alpha}, \vec{0}, 2(x+1), 0)}{L_n(\vec{\alpha}_{m-1}, \vec{0}, 2(x+1), 0)} = (t_m - \log 2)\alpha_m n - \frac{\log 2}{2}\sum_{j=0}^{m-1}\alpha_j\alpha_m - \sum_{j=0}^{m-1}\frac{\alpha_j\alpha_m}{2}\log|t_m - t_j| \\ + \frac{\alpha_m^2}{4}\log\left(n\frac{\sqrt{1-t_m}}{\sqrt{1+t_m}}\right) + \log\frac{G(1+\frac{\alpha_m}{2})^2}{G(1+\alpha_m)} + \mathcal{O}\Big(\frac{\log n}{n}\Big), \tag{2.6.39}$$

as $n \to +\infty$. Summing the contributions of each step, we arrive at

$$\log \frac{L_n(\vec{\alpha}, \vec{0}, 2(x+1), 0)}{L_n(\vec{\alpha}_0, \vec{0}, 2(x+1), 0)} = \sum_{j=1}^{m}(t_j - \log 2)\alpha_j n - \frac{\log 2}{2}\sum_{0\le j<k\le m}\alpha_j\alpha_k \\ - \sum_{0\le j<k\le m}\frac{\alpha_j\alpha_k}{2}\log|t_k - t_j| + \sum_{j=1}^{m}\frac{\alpha_j^2}{4}\log\left(n\frac{\sqrt{1-t_j}}{\sqrt{1+t_j}}\right) + \sum_{j=1}^{m}\log\frac{G(1+\frac{\alpha_j}{2})^2}{G(1+\alpha_j)} + \mathcal{O}\Big(\frac{\log n}{n}\Big), \tag{2.6.40}$$

as $n \to +\infty$.

2.6.4 Integration in $\beta_1, \ldots, \beta_m$

For convenience, we introduce the notation

$$\mathcal{A}_k = \sum_{j=0}^{k-1} \alpha_j - \sum_{j=k+1}^{m} \alpha_j, \qquad k = 0, 1, \ldots, m. \tag{2.6.41}$$

We set $\beta_2 = \ldots = \beta_m = 0$ and $\nu = \beta_1$ in (2.6.30). With the notation $\vec{\beta}_1 = (\beta_1, 0, \ldots, 0)$, we have

$$\begin{aligned}\partial_{\beta_1} \log L_n(\vec{\alpha}, \vec{\beta}_1, 2(x+1), 0) &= 2i\big(\arcsin t_1 + \sqrt{1-t_1^2}\big)n + i\mathcal{A} \arcsin t_1 - \frac{i\pi}{2}\mathcal{A}_1 \\ &+ \frac{\alpha_1}{2}\partial_{\beta_1} \log \Gamma(1+\tfrac{\alpha_1}{2}-\beta_1)\Gamma(1+\tfrac{\alpha_1}{2}+\beta_1) + \beta_1 \partial_{\beta_1} \log \frac{\Gamma(1+\frac{\alpha_1}{2}+\beta_1)}{\Gamma(1+\frac{\alpha_1}{2}-\beta_1)} - 2\beta_1 \\ &- 2\beta_1 \log\big(4\pi\rho(t_1)n(1-t_1^2)\big) + \mathcal{O}\Big(\frac{\log n}{n^{1-4\beta_{\max}}}\Big).\end{aligned} \tag{2.6.42}$$

After some computations using (2.6.34), we obtain

$$\begin{aligned}\int_0^{\beta_1} \left(\frac{\alpha_1}{2}\partial_x \log \Gamma(1+\tfrac{\alpha_1}{2}-x)\Gamma(1+\tfrac{\alpha_1}{2}+x) + x\partial_x \log \frac{\Gamma(1+\frac{\alpha_1}{2}+x)}{\Gamma(1+\frac{\alpha_1}{2}-x)} - 2x\right)dx \\ = \log \frac{G(1+\frac{\alpha_1}{2}+\beta_1)G(1+\frac{\alpha_1}{2}-\beta_1)}{G(1+\frac{\alpha_1}{2})^2}.\end{aligned} \tag{2.6.43}$$

Integrating (2.6.42) from $\beta_1 = 0$ to an arbitrary β_1 and using (2.6.43), we obtain

$$\begin{aligned}\log \frac{L_n(\vec{\alpha}, \vec{\beta}_1, 2(x+1), 0)}{L_n(\vec{\alpha}, \vec{0}, 2(x+1), 0)} &= 2i\beta_1\big(\arcsin t_1 + \sqrt{1-t_1^2}\big)n + i\mathcal{A}\beta_1 \arcsin t_1 - \frac{i\pi}{2}\mathcal{A}_1\beta_1 \\ &+ \log \frac{G(1+\frac{\alpha_1}{2}+\beta_1)G(1+\frac{\alpha_1}{2}-\beta_1)}{G(1+\frac{\alpha_1}{2})^2} - \beta_1^2 \log(4\pi\rho(t_1)n(1-t_1^2)) + \mathcal{O}\Big(\frac{\log n}{n^{1-4\beta_{\max}}}\Big).\end{aligned} \tag{2.6.44}$$

We integrate successively in $\beta_2, \ldots, \beta_m$. At the last step, we set $\nu = \beta_m$ in (2.6.30), which gives

$$\begin{aligned}\partial_{\beta_m} \log L_n(\vec{\alpha}, \vec{\beta}, 2(x+1), 0) &= 2i\big(\arcsin t_m + \sqrt{1-t_m^2}\big)n + i\mathcal{A} \arcsin t_m - \frac{i\pi}{2}\mathcal{A}_m \\ &+ \frac{\alpha_m}{2}\partial_{\beta_m} \log \Gamma(1+\tfrac{\alpha_m}{2}-\beta_m)\Gamma(1+\tfrac{\alpha_m}{2}+\beta_m) + \beta_m \partial_{\beta_m} \log \frac{\Gamma(1+\frac{\alpha_m}{2}+\beta_m)}{\Gamma(1+\frac{\alpha_m}{2}-\beta_m)} - 2\beta_m \\ &+ \sum_{j=1}^{m-1} 2\beta_j \log T_{jm} - 2\beta_m \log\big(4\pi\rho(t_m)n(1-t_m^2)\big) + \mathcal{O}\Big(\frac{\log n}{n^{1-4\beta_{\max}}}\Big),\end{aligned} \tag{2.6.45}$$

as $n \to +\infty$. Integrating (2.6.45) from $\beta_m = 0$ to an arbitrary β_m, using the notation $\vec{\beta}_{m-1} = (\beta_1, \ldots, \beta_{m-1}, 0)$, we obtain

$$\begin{aligned} \log \frac{L_n(\vec{\alpha}, \vec{\beta}_1, 2(x+1), 0)}{L_n(\vec{\alpha}, \vec{\beta}_{m-1}, 2(x+1), 0)} \quad = \quad & 2i\beta_m\Big(\arcsin t_m + \sqrt{1-t_m^2}\,\Big)n + i\mathcal{A}\beta_m \arcsin t_m - \frac{i\pi}{2}\mathcal{A}_m\beta_m \\ & + \log \frac{G(1+\frac{\alpha_m}{2}+\beta_m)G(1+\frac{\alpha_m}{2}-\beta_m)}{G(1+\frac{\alpha_m}{2})^2} - \beta_m^2 \log(4\pi\rho(t_m)n(1-t_m^2)) \\ & + \sum_{j=1}^{m-1} 2\beta_j\beta_m \log T_{jm} + \mathcal{O}\Big(\frac{\log n}{n^{1-4\beta_{\max}}}\Big). \end{aligned} \tag{2.6.46}$$

Summing all the contributions, as $n \to +\infty$ we obtain

$$\begin{aligned} \log \frac{L_n(\vec{\alpha}, \vec{\beta}, 2(x+1), 0)}{L_n(\vec{\alpha}, \vec{0}, 2(x+1), 0)} = 2in \sum_{j=1}^{m} \beta_j \Big(\arcsin t_j + \sqrt{1-t_j^2}\Big) + i\mathcal{A} \sum_{j=1}^{m} \beta_j \arcsin t_j \\ - \frac{i\pi}{2}\sum_{j=1}^{m} \mathcal{A}_j\beta_j + \sum_{j=1}^{m} \log \frac{G(1+\frac{\alpha_j}{2}+\beta_j)G(1+\frac{\alpha_j}{2}-\beta_j)}{G(1+\frac{\alpha_j}{2})^2} - \sum_{j=1}^{m} \beta_j^2 \log(4\pi\rho(t_j)n(1-t_j^2)) \\ + 2 \sum_{1\le j<k\le m} \beta_j\beta_k \log T_{jk} + \mathcal{O}\Big(\frac{\log n}{n^{1-4\beta_{\max}}}\Big). \end{aligned} \tag{2.6.47}$$

The claim of Proposition 2.6.2 follows now by summing (2.6.40) and (2.6.47) using the definition of $\mathcal{A}_j$ given in (2.6.41).

2.7 Integration in V

In this section, we obtain asymptotics for general Laguerre-type and Jacobi-type weights by means of a deformation parameter s and by using the analysis of Section 2.4 for the weight

$$w_s(x) = e^{-nV_s(x)}\omega(x), \tag{2.7.1}$$

where we emphasize in the notation the dependence in s. We specify in Subsection 2.7.1 the exact deformations we consider. In Subsection 2.7.2, we adapt several identities from [15] (that are valid for Gaussian-type weights) for our situations. Finally, we proceed with the integration in s for Laguerre-type and Jacobi-type weights in Subsection 2.7.3 and Subsection 2.7.4, respectively.

2.7.1 Deformation parameters s

Inspired by [3, 15], for each $s \in [0,1]$, we define

$$V_s(x) = (1-s)2(x+1) + sV(x), \qquad \text{for Laguerre-type weights,} \tag{2.7.2}$$

$$V_s(x) = sV(x), \qquad \text{for Jacobi-type weights.} \tag{2.7.3}$$

If $s = 0$, we already know large n asymptotics for the associated Hankel determinants (from Section 2.6 and the result of [9], see Proposition 2.6.2 and Theorem 2.5.2). It follows easily from (2.1.4)-(2.1.5) that V_s is one-cut regular for each $s \in [0,1]$, and the associated density ψ_s and Euler-Lagrange constant ℓ_s are given by

$$\psi_s(x) = (1-s)\frac{1}{\pi} + s\psi(x), \qquad \ell_s = (1-s)(2 + 2\log 2) + s\ell, \tag{2.7.4}$$

$$\psi_s(x) = (1-s)\frac{1}{\pi} + s\psi(x), \qquad \ell_s = (1-s)2\log 2 + s\ell, \tag{2.7.5}$$

where the first and second lines read for Laguerre-type and Jacobi-type weights respectively. We will use the differential identities

$$\partial_s \log L_n(\vec{\alpha}, \vec{\beta}, V_s, 0) = \frac{1}{2\pi i}\int_{-1}^{+\infty} [Y^{-1}(x)Y'(x)]_{21}\partial_s w_s(x)dx, \tag{2.7.6}$$

$$\partial_s \log J_n(\vec{\alpha}, \vec{\beta}, V_s, 0) = \frac{1}{2\pi i}\int_{-1}^{1} [Y^{-1}(x)Y'(x)]_{21}\partial_s w_s(x)dx, \tag{2.7.7}$$

which were obtained in Proposition 2.3.3. Our objective in this section is to compute asymptotics of these differential identities, and finally integrate them in the parameter s from 0 to 1.

2.7.2 Some identities

We generalize here several formulas of [15] (valid only for Gaussian-type potentials) for all three-types of canonical one-cut regular potentials. Most of the proofs are minor modifications of those done in [15].

Lemma 2.7.1 *For $t \in [-1,1]$, we have*

$$\fint_{-1}^{1} \frac{V'(x)\sqrt{1-x^2}}{x-t} dx = -2\pi + 2\pi^2\sqrt{1-t^2}\rho(t), \tag{2.7.8}$$

$$\int_t^1 \rho(x)dx = \frac{\sqrt{1-t^2}}{2\pi^2}\fint_{-1}^{1} \frac{V(x)}{t-x}\frac{dx}{\sqrt{1-x^2}} + \frac{1}{\pi}\arccos t. \tag{2.7.9}$$

Proof The proof goes as in [15, Lemma 5.8]. Let $H : \mathbb{C}\setminus[-1,1] \to \mathbb{C}$ be defined by

$$H(z) = 2\pi\sqrt{z-1}\sqrt{z+1}\int_{-1}^{1}\frac{\rho(x)}{x-z}dx + \int_{-1}^{1}\frac{V'(x)\sqrt{1-x^2}}{x-z}dx \tag{2.7.10}$$

where the principal branches are chosen for $\sqrt{z-1}$ and $\sqrt{z+1}$. For $t \in (-1,1)$, one can check that $H_+(t) = H_-(t)$. Also H is bounded at ± 1 and $H(\infty) = -2\pi$; so Liouville's theorem implies that $H(z) = -2\pi$. Considering $H_+(t) + H_-(t)$ for $t \in (-1,1)$ yields (2.7.8). Now, (2.7.9) follows from (2.7.8) and the following identity which is proved in [15, eq (5.18) and below]

$$\sqrt{1-t^2}\fint_{-1}^{1}\frac{V(x)}{t-x}\frac{dx}{\sqrt{1-x^2}} = \int_t^1 \frac{1}{\sqrt{1-x^2}}\left(\fint_{-1}^{1}\frac{V'(y)}{y-x}\sqrt{1-y^2}dy\right)dx. \tag{2.7.11}$$

■

Lemma 2.7.2 *Let $\mathcal{C}$ be a closed curve surrounding $[-1,1]$ in the clockwise direction, let $a(z) = \sqrt[4]{\frac{z-1}{z+1}}$ be analytic on $\mathbb{C}\setminus[-1,1]$ such that $a(z)\sim 1$ as $z\to\infty$, and let f be analytic in a neighbourhood of $[-1,1]$. We have*

$$\frac{1}{2\pi i}\int_{\mathcal{C}}\left[\frac{a^2(z)}{a_+^2(t_j)}+\frac{a_+^2(t_j)}{a^2(z)}\right]\frac{f(z)}{(z-t_j)^2}dz = \frac{2}{\pi i\sqrt{1-t_j^2}}\fint_{-1}^{1} f'(x)\frac{\sqrt{1-x^2}}{x-t_j}dx, \tag{2.7.12}$$

$$\frac{1}{2\pi i}\int_{\mathcal{C}}\left[\frac{a^2(z)}{a_+^2(t_j)}-\frac{a_+^2(t_j)}{a^2(z)}\right]\frac{f(z)}{(z-t_j)^2}dz = \frac{2}{\pi i\sqrt{1-t_j^2}}\fint_{-1}^{1} \frac{f(x)}{(t_j-x)\sqrt{1-x^2}}dx. \tag{2.7.13}$$

Proof The proof is the same as in [15, equations (5.22)-(5.23) and above]. ■

Applying Lemma 2.7.2 to $f=\partial_s V_s$ (with V_s given by (2.7.2)–(2.7.3)), and then simplifying using Lemma 2.7.1, we obtain

$$\int_{\mathcal{C}}\left[\frac{a^2(z)}{a_+^2(t_j)}+\frac{a_+^2(t_j)}{a^2(z)}\right]\frac{\partial_s V_s(z)}{(z-t_j)^2}dz = \begin{cases} 8\pi^2\big(\psi(t_j)-\frac{1}{\pi}\big)\dfrac{\sqrt{1-t_j}}{\sqrt{1+t_j}}, & \text{for Laguerre-type potentials} \\ 8\pi^2\big(\psi(t_j)-\frac{1}{\pi}\big)\dfrac{1}{\sqrt{1-t_j^2}}, & \text{for Jacobi-type potentials} \end{cases} \tag{2.7.14}$$

and

$$\int_{\mathcal{C}}\left[\frac{a^2(z)}{a_+^2(t_j)}-\frac{a_+^2(t_j)}{a^2(z)}\right]\frac{\partial_s V_s(z)}{(z-t_j)^2}dz = \begin{cases} \dfrac{8\pi^2}{1-t_j^2}\displaystyle\int_{t_j}^{1}\big(\psi(x)-\frac{1}{\pi}\big)\frac{\sqrt{1-x}}{\sqrt{1+x}}dx, & \text{for Laguerre-type potentials} \\ \dfrac{8\pi^2}{1-t_j^2}\displaystyle\int_{t_j}^{1}\big(\psi(x)-\frac{1}{\pi}\big)\frac{1}{\sqrt{1-x^2}}dx, & \text{for Jacobi-type potentials} \end{cases} \tag{2.7.15}$$

Lemma 2.7.3 *Let $\mathcal{C}$ be a closed curve surrounding $[-1,1]$ in the clockwise direction, let $a(z) = \sqrt[4]{\frac{z-1}{z+1}}$ be analytic on $\mathbb{C}\setminus[-1,1]$ such that $a(z)\sim 1$ as $z\to\infty$, and let f be analytic in a neighbourhood of $[-1,1]$. We have*

$$\int_{\mathcal{C}}\frac{a(z)^2}{(z-1)^2}f(z)dz = 2i\int_{-1}^{1}\frac{f'(x)\sqrt{1-x^2}}{x-1}dx, \tag{2.7.16}$$

$$\int_{\mathcal{C}}\frac{a(z)^3}{(z-1)^3}f(z)dz = -\frac{2i}{3}\int_{-1}^{1}\frac{f'(x)\sqrt{1-x^2}}{x-1}dx+\frac{2i}{3}\frac{d}{dt}\Big|_{t=1}\fint_{-1}^{1}\frac{f'(x)\sqrt{1-x^2}}{x-t}dx, \tag{2.7.17}$$

$$\int_{\mathcal{C}}\frac{a(z)^{-2}}{(z-1)^3}f(z)dz = \frac{2i}{3}\int_{-1}^{1}\frac{f'(x)\sqrt{1-x^2}}{x-1}dx+\frac{4i}{3}\frac{d}{dt}\Big|_{t=1}\fint_{-1}^{1}\frac{f'(x)\sqrt{1-x^2}}{x-t}dx, \tag{2.7.18}$$

$$\int_{\mathcal{C}}\frac{a(z)^{-2}}{(z+1)^2}f(z)dz = -2i\int_{-1}^{1}\frac{f'(x)\sqrt{1-x^2}}{x+1}dx. \tag{2.7.19}$$

Proof The proof of (2.7.16)–(2.7.18) is done in [15, Lemma 5.10], and the proof for (2.7.19) is similar. ■

Applying Lemma 2.7.3 to $f(x) = \partial_s V_s = V(x) - 2(x+1)$ with V_s given by (2.7.2) for Laguerre-type potentials, and then simplifying using Lemma 2.7.1, we obtain

$$\int_{\mathcal{C}} \frac{a(z)^2}{(z-1)^2} \partial_s V_s(z) dz = 0, \tag{2.7.20}$$

$$\int_{\mathcal{C}} \frac{a(z)^2}{(z-1)^3} \partial_s V_s(z) dz = -\frac{4\pi^2 i}{3}\Big(\psi(1) - \frac{1}{\pi}\Big), \tag{2.7.21}$$

$$\int_{\mathcal{C}} \frac{a(z)^{-2}}{(z-1)^2} \partial_s V_s(z) dz = -\frac{8\pi^2 i}{3}\Big(\psi(1) - \frac{1}{\pi}\Big), \tag{2.7.22}$$

$$\int_{\mathcal{C}} \frac{a(z)^{-2}}{(z-1)^2} \partial_s V_s(z) dz = -8\pi^2 i\Big(\psi(-1) - \frac{1}{\pi}\Big). \tag{2.7.23}$$

Similarly, for Jacobi-type weights with $f(x) = \partial_s V_s = V(x)$ with V_s given by (2.7.3) for Jacobi-type potentials, we obtain

$$\int_{\mathcal{C}} \frac{a(z)^2}{(z-1)^2} \partial_s V_s(z) dz = 4\pi^2 i\Big(\psi(1) - \frac{1}{\pi}\Big), \tag{2.7.24}$$

$$\int_{\mathcal{C}} \frac{a(z)^{-2}}{(z+1)^2} \partial_s V_s(z) dz = -4\pi^2 i\Big(\psi(-1) - \frac{1}{\pi}\Big). \tag{2.7.25}$$

2.7.3 Integration in s for Laguerre-type weights

In this subsection we prove Proposition 2.7.1 below.

Proposition 2.7.1 *As $n \to +\infty$, we have*

$$\begin{aligned}\log \frac{L_n(\vec{\alpha}, \vec{\beta}, V, 0)}{L_n(\vec{\alpha}, \vec{\beta}, 2(x+1), 0)} &= -\frac{n^2}{2}\int_{-1}^{1} \big(V(x) - 2(x+1)\big)\Big(\frac{1}{\pi} + \psi(x)\Big)\sqrt{\frac{1-x}{1+x}} dx \\ &+ n\sum_{j=0}^{m} \frac{\alpha_j}{2}\big(V(t_j) - 2(1+t_j)\big) - \frac{n\mathcal{A}}{2\pi}\int_{-1}^{1} \frac{V(x) - 2(1+x)}{\sqrt{1-x^2}} dx - 2\pi n \sum_{j=1}^{m} i\beta_j \int_{t_j}^{1} \Big(\psi(x) - \frac{1}{\pi}\Big)\sqrt{\frac{1-x}{1+x}} dx \\ &+ \sum_{j=1}^{m}\left(\frac{\alpha_j^2}{4} - \beta_j^2\right)\log\left(\pi\psi(t_j)\right) - \frac{1}{24}\log\left(\pi\psi(1)\right) - \frac{1-4\alpha_0^2}{8}\log(\pi\psi(-1)) + \mathcal{O}\Big(n^{-1+4\beta_{\max}}\Big).\end{aligned} \tag{2.7.26}$$

Let $\mathcal{C}$ be a closed contour surrounding $[-1,1]$ and the lenses $\gamma_+ \cup \gamma_-$, which is oriented clockwise and passes through $-1-\varepsilon$ and $1+\varepsilon$ for a certain $\varepsilon > 0$. Using the jumps for Y given by (2.2.2), we rewrite the differential identity (2.7.6) as follows

$$\partial_s \log L_n(\vec{\alpha}, \vec{\beta}, V_s, 0) = \int_{1+\varepsilon}^{+\infty} [Y^{-1}(x)Y'(x)]_{21} \partial_s w_s(x) \frac{dx}{2\pi i} - \frac{1}{2\pi i} \int_{\mathcal{C}} [Y^{-1}(z)Y'(z)]_{11} \partial_s \log w_s(z) \frac{dz}{2\pi i}. \tag{2.7.27}$$

From (2.4.3), (2.4.7) and by inverting the transformations $Y \mapsto T \mapsto S \mapsto R$ outside the lenses and outside the disks, we conclude that the first integral in the r.h.s. of (2.7.27) is of order $\mathcal{O}(e^{-cn})$ as $n \to +\infty$, for a positive constant c, and that the integral over $\mathcal{C}$ can be decomposed into three integrals:

$$\begin{aligned} \partial_s \log L_n(\vec{\alpha}, \vec{\beta}, V_s, 0) &= I_{1,s} + I_{2,s} + I_{3,s} + \mathcal{O}(e^{-cn}), \qquad \text{as } n \to \infty, \\ I_{1,s} &= \frac{-n}{2\pi i} \int_{\mathcal{C}} g'(z) \partial_s \log w_s(z) dz, \\ I_{2,s} &= \frac{-1}{2\pi i} \int_{\mathcal{C}} [P^{(\infty)}(z)^{-1} P^{(\infty)}(z)']_{11} \partial_s \log w_s(z) dz, \\ I_{3,s} &= \frac{-1}{2\pi i} \int_{\mathcal{C}} [P^{(\infty)}(z)^{-1} R^{-1}(z) R'(z) P^{(\infty)}(z)]_{11} \partial_s \log w_s(z) dz. \end{aligned} \tag{2.7.28}$$

In exactly the same way as in [3,15], we show from a detailed analysis of the Cauchy operator associated to R that the estimates in (2.4.67) hold uniformly for $(\vec{\alpha}, \vec{\beta})$ in any fixed compact set Ω, and uniformly in $s \in [0,1]$. However, from Proposition 2.3.3, the identity (2.7.28) itself is not valid for the values of $(\vec{\alpha}, \vec{\beta}, s)$ for which at least one of the polynomials $p_0, \ldots, p_n$ does not exist. From [3, beginning of Section 3], this set is locally finite except possible some accumulation points at $s = 0$ and $s = 1$. As in [3], we extend (2.7.28) *for all* $(\vec{\alpha}, \vec{\beta}, s) \in \Omega \times [0,1]$ (for sufficiently large n) using the continuity of the l.h.s. of (2.7.28). A similar reasoning holds also for (2.7.46) below.

Note from (2.7.1) and (2.7.2) that $\partial_s \log w_s(z) = -n \partial_s V_s(z) = -n(V(x) - 2(x+1))$. Using the definition of g given by (2.4.2) and switching the order of integration, we get

$$I_{1,s} = -n^2 \int_{-1}^{1} \rho_s(x) \partial_s V_s(x) dx = -n^2 \int_{-1}^{1} (V(x) - 2(x+1)) \Big((1-s)\frac{1}{\pi} + s\psi(x) \Big) \frac{\sqrt{1-x}}{\sqrt{1+x}} dx. \tag{2.7.29}$$

Therefore, we have

$$\int_0^1 I_{1,s} ds = -\frac{n^2}{2} \int_{-1}^{1} (V(x) - 2(x+1)) \Big(\frac{1}{\pi} + \psi(x) \Big) \frac{\sqrt{1-x}}{\sqrt{1+x}} dx. \tag{2.7.30}$$

From (2.4.29), (2.4.31), (2.4.32) and a contour deformation, we obtain the following expression for $I_{2,s}$:

$$I_{2,s} = n \sum_{j=0}^{m} \frac{\alpha_j}{2} \Big(V(t_j) - 2(1+t_j) \Big) - \frac{n\mathcal{A}}{2\pi} \int_{-1}^{1} \frac{V(x) - 2(1+x)}{\sqrt{1-x^2}} dx \\ + n \sum_{j=1}^{m} \frac{i\beta_j}{\pi} \sqrt{1-t_j^2} \fint_{-1}^{1} \frac{V(x) - 2(1+x)}{\sqrt{1-x^2}(x-t_j)} dx. \tag{2.7.31}$$

We simplify the last integral of (2.7.31) using (2.7.9):

$$\sqrt{1-t_j^2} \fint_{-1}^{1} \frac{V(x) - 2(1+x)}{\sqrt{1-x^2}(x-t_j)} dx = -2\pi^2 \int_{t_j}^{1} \Big(\psi(x) - \frac{1}{\pi} \Big) \sqrt{\frac{1-x}{1+x}} dx. \tag{2.7.32}$$

Then, integrating in s (note that $I_{2,s}$ is in fact independent of s), we obtain

$$\int_0^1 I_{2,s} ds = n \sum_{j=0}^{m} \frac{\alpha_j}{2} \Big(V(t_j) - 2(1+t_j) \Big) - \frac{n\mathcal{A}}{2\pi} \int_{-1}^{1} \frac{V(x) - 2(1+x)}{\sqrt{1-x^2}} dx \\ - 2\pi n \sum_{j=1}^{m} i\beta_j \int_{t_j}^{1} \Big(\psi(x) - \frac{1}{\pi} \Big) \sqrt{\frac{1-x}{1+x}} dx. \tag{2.7.33}$$

Using the expansion of R given by (2.4.67), we have

$$I_{3,s} = \frac{1}{2\pi i} \int_{\mathcal{C}} [P^{(\infty)}(z)^{-1} R^{(1)}(z)' P^{(\infty)}(z)]_{11} \partial_s V_s(z) dz + \mathcal{O}\big(n^{-1+4\beta_{\max}}\big), \qquad \text{as } n \to \infty, \tag{2.7.34}$$

The leading term of $I_{3,s}$ can be written down more explicitly using the definition of $P^{(\infty)}$ given by (2.4.29), and we obtain

$$I_{3,s} = \frac{1}{2\pi i} \int_{\mathcal{C}} \Bigg(\frac{a(z)^2 + a(z)^{-2}}{4} [R_{11}^{(1)}(z)' - R_{22}^{(1)}(z)'] + \frac{1}{2} [R_{11}^{(1)}(z)' + R_{22}^{(1)}(z)'] \\ + i \frac{a(z)^2 - a(z)^{-2}}{4} [R_{12}^{(1)}(z)' D_\infty^{-2} + R_{21}^{(1)}(z)' D_\infty^{2}] \Bigg) \Big(V(z) - 2(z+1) \Big) dz + \mathcal{O}\big(n^{-1+4\beta_{\max}}\big). \tag{2.7.35}$$

From (2.4.72), (2.4.73), (2.4.78), (2.4.79) and (2.4.80) we have

$$R_{11}^{(1)\prime}(z) - R_{22}^{(1)\prime}(z) = \sum_{j=1}^{m} \frac{1}{(z-t_j)^2} \frac{-2v_j(t_j+\widetilde{\Lambda}_{I,j})}{2\pi\rho_s(t_j)\sqrt{1-t_j^2}} + \frac{1}{(z-1)^3}\frac{5}{2^2 3\pi\psi_s(1)}$$
$$+ \frac{1}{(z-1)^2}\frac{(\mathcal{A}-\widetilde{\mathcal{B}}_1)^2 - \frac{1}{4} - \frac{1}{2}\frac{\psi_s'(1)}{\psi_s(1)}}{2^2\pi\psi_s(1)} + \frac{1}{(z+1)^2}\frac{1-4\alpha_0^2}{2^4\pi\psi_s(-1)}, \tag{2.7.36}$$

$$R_{11}^{(1)\prime}(z) + R_{22}^{(1)\prime}(z) = 0, \tag{2.7.37}$$

$$i[R_{12}^{(1)\prime}(z)D_\infty^{-2} + R_{21}^{(1)\prime}(z)D_\infty^2] = \sum_{j=1}^{m} \frac{1}{(z-t_j)^2}\frac{v_j(-2+\widetilde{\Lambda}_{R,1,j}-\widetilde{\Lambda}_{R,2,j})}{2\pi\rho_s(t_j)\sqrt{1-t_j^2}} + \frac{1}{(z-1)^3}\frac{5}{2^2 3\pi\psi_s(1)}$$
$$+ \frac{1}{(z-1)^2}\frac{(\mathcal{A}-\widetilde{\mathcal{B}}_1)^2 + \frac{11}{12} - \frac{1}{2}\frac{\psi_s'(1)}{\psi_s(1)}}{2^2\pi\psi_s(1)} + \frac{1}{(z+1)^2}\frac{-(1-4\alpha_0^2)}{2^4\pi\psi_s(-1)}. \tag{2.7.38}$$

Therefore, from (2.7.35)–(2.7.38) and using the connection formula (2.4.77), we obtain

$$I_{3,s} = \sum_{j=1}^{m} I_{3,s,t_j} + I_{3,s,1} + I_{3,s,-1} + \mathcal{O}\big(n^{-1+4\beta_{\max}}\big), \qquad \text{as } n \to \infty, \tag{2.7.39}$$

where

$$I_{3,s,t_k} = \frac{-v_k}{8\pi^2\rho_s(t_k)}\int_{\mathcal{C}}\left[\frac{a^2(z)}{a_+^2(t_k)} + \frac{a_+^2(t_k)}{a^2(z)} + \widetilde{\Lambda}_{I,k}\left(\frac{a^2(z)}{a_+^2(t_k)} - \frac{a_+^2(t_k)}{a^2(z)}\right)\right]\frac{\partial_s V_s(z)}{(z-t_k)^2}dz,$$

$$I_{3,s,1} = \int_{\mathcal{C}}\left[\frac{a^2(z)}{4\pi\psi_s(1)}\left(\frac{2(\mathcal{A}-\widetilde{\mathcal{B}}_1)^2+\frac{2}{3}-\frac{\psi_s'(1)}{\psi_s(1)}}{2^2(z-1)^2} + \frac{5}{6(z-1)^3}\right) + \frac{a^{-2}(z)}{4(z-1)^2}\frac{-\frac{7}{6}}{2^2\pi\psi_s(1)}\right]\partial_s V_s(z)\frac{dz}{2\pi i},$$

$$I_{3,s,-1} = \int_{\mathcal{C}}\left[\frac{a^{-2}(z)}{4(z+1)^2}\frac{1-4\alpha_0^2}{2^3\pi\psi_s(-1)}\right]\partial_s V_s(z)\frac{dz}{2\pi i}.$$

Formulas (2.7.14) and (2.7.15) allow us to simplify I_{3,s,t_k} as follows:

$$I_{3,s,t_k} = -\frac{v_k}{\psi_s(t_k)}\Big(\psi(t_k) - \frac{1}{\pi}\Big) - \frac{v_k\widetilde{\Lambda}_{I,k}}{\rho_s(t_k)(1-t_k^2)}\int_{t_k}^{1}\Big(\psi(x)-\frac{1}{\pi}\Big)\sqrt{\frac{1-x}{1+x}}dx. \tag{2.7.40}$$

Integrating the above from $s=0$ to $s=1$, we have

$$\int_0^1 I_{3,s,t_k}ds = -v_k\log(\pi\psi(t_k)) - \frac{v_k}{1-t_k^2}\int_{t_k}^{1}\Big(\psi(x)-\frac{1}{\pi}\Big)\sqrt{\frac{1-x}{1+x}}dx\int_0^1\frac{\widetilde{\Lambda}_{I,k}}{\rho_s(t_k)}ds. \tag{2.7.41}$$

By the same argument as the one given in [3, equations (6.23) and (6.24)], the second term in the r.h.s of (2.7.41) is of order $\mathcal{O}(n^{-1+2|\Re\beta_k|})$ as $n\to+\infty$, that is,

$$\int_0^1 I_{3,s,t_k}ds = -v_k\log(\pi\psi(t_k)) + \mathcal{O}(n^{-1+2|\Re\beta_k|}). \tag{2.7.42}$$

We can also simplify the expression for $I_{3,s,1}$. Using the formulas (2.7.20)–(2.7.22), we obtain

$$I_{3,s,1} = -\frac{1}{24}\frac{\psi(1)-\frac{1}{\pi}}{\psi_s(1)}, \qquad \text{and then} \qquad \int_0^1 I_{3,s,1}ds = -\frac{1}{24}\log(\pi\psi(1)). \tag{2.7.43}$$

Similarly, using (2.7.23) we get

$$I_{3,s,-1} = -\frac{1-4\alpha_0^2}{8}\frac{\psi(-1)-\frac{1}{\pi}}{\psi_s(-1)}, \quad \text{and then} \quad \int_0^1 I_{3,s,-1}ds = -\frac{1-4\alpha_0^2}{8}\log\big(\pi\psi(-1)\big). \tag{2.7.44}$$

This finishes the proof of Proposition 2.7.1.

2.7.4 Jacobi-type weights

We prove here the analogue of Proposition 2.7.1 for Jacobi-type weights.

Proposition 2.7.2 *As $n \to \infty$, we have*

$$\begin{aligned}\log\frac{J_n(\vec{\alpha},\vec{\beta},V,0)}{J_n(\vec{\alpha},\vec{\beta},0,0)} &= -\frac{n^2}{2}\int_{-1}^{1}\frac{V(x)}{\sqrt{1-x^2}}\Big(\frac{1}{\pi}+\psi(x)\Big)dx + n\sum_{j=0}^{m+1}\frac{\alpha_j}{2}V(t_j) \\ &-\frac{n\mathcal{A}}{2\pi}\int_{-1}^{1}\frac{V(x)}{\sqrt{1-x^2}}dx - 2\pi n\sum_{j=1}^{m} i\beta_j\int_{t_j}^{1}\Big(\psi(x)-\frac{1}{\pi}\Big)\frac{dx}{\sqrt{1-x^2}} + \sum_{j=1}^{m}\left(\frac{\alpha_j^2}{4}-\beta_j^2\right)\log\left(\pi\psi(t_j)\right) \\ &-\frac{1-4\alpha_{m+1}^2}{8}\log\left(\pi\psi(1)\right) - \frac{1-4\alpha_0^2}{8}\log(\pi\psi(-1)) + \mathcal{O}\big(n^{-1+4\beta_{\max}}\big).\end{aligned} \tag{2.7.45}$$

The computations of this subsection are organised similarly to those done in Subsection 2.7.3, and we provide less details. Let $\mathcal{C}$ be a closed contour surrounding $[-1,1]$ and the lenses $\gamma_+\cup\gamma_-$, which is oriented clockwise and passes through $-1-\varepsilon$ and $1+\varepsilon$ for a certain $\varepsilon>0$. Using the jumps for Y (2.2.2), we rewrite the differential identity (2.7.7) as follows

$$\partial_s \log J_n(\vec{\alpha},\vec{\beta},V_s,0) = -\frac{1}{2\pi i}\int_{\mathcal{C}}[Y^{-1}(z)Y'(z)]_{11}\partial_s\log w_s(z)\frac{dz}{2\pi i}, \tag{2.7.46}$$

where from (2.7.1) and (2.7.3), we have $\partial_s\log w_s(z) = -n\partial_s V_s(z) = -nV(z)$. In the same way as done in (2.7.28), by inverting the transformations $Y\mapsto T\mapsto S\mapsto R$ in the region outside the lenses and outside the disks, we have

$$\partial_s \log J_n(\vec{\alpha},\vec{\beta},V_s,0) = I_{1,s}+I_{2,s}+I_{3,s}, \tag{2.7.47}$$

where $I_{1,s}$, $I_{2,s}$ and $I_{3,s}$ are given as in (2.7.28). For $I_{1,s}$, a simple calculation implies

$$I_{1,s} = -n^2 \int_{-1}^{1} \rho_s(x)\partial_s V_s(x)dx = -n^2 \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} V(x)\Big((1-s)\frac{1}{\pi} + s\psi(x)\Big)dx, \qquad (2.7.48)$$

which gives

$$\int_0^1 I_{1,s}ds = -\frac{n^2}{2}\int_{-1}^{1} \frac{V(x)}{\sqrt{1-x^2}}\Big(\frac{1}{\pi} + \psi(x)\Big)dx. \qquad (2.7.49)$$

The computations of $I_{2,s}$ are similar to those done for [3, equations (6.10)–(6.15)] and for (2.7.31). We obtain

$$\int_0^1 I_{2,s}ds = n\sum_{j=0}^{m+1} \frac{\alpha_j}{2} V(t_j) - \frac{n\mathcal{A}}{2\pi}\int_{-1}^{1} \frac{V(x)}{\sqrt{1-x^2}}dx - 2\pi n \sum_{j=1}^{m} i\beta_j \int_{t_j}^{1} \Big(\psi(x) - \frac{1}{\pi}\Big)\frac{1}{\sqrt{1-x^2}}dx. \qquad (2.7.50)$$

For $I_{3,s}$, similar to (2.7.35) we get

$$\begin{aligned} I_{3,s} = \frac{1}{2\pi i}\int_{\mathcal{C}} \Bigg(& \frac{a(z)^2 + a(z)^{-2}}{4}[R_{11}^{(1)}(z)' - R_{22}^{(1)}(z)'] + \frac{1}{2}[R_{11}^{(1)}(z)' + R_{22}^{(1)}(z)'] \\ & + i\frac{a(z)^2 - a(z)^{-2}}{4}[R_{12}^{(1)}(z)' D_\infty^{-2} + R_{21}^{(1)}(z)' D_\infty^{2}]\Bigg) V(z)dz + \mathcal{O}\big(n^{-1+4\beta_{\max}}\big). \qquad (2.7.51) \end{aligned}$$

The quantities involving $R^{(1)}$ are made explicit using (2.4.83), we obtain

$$R_{11}^{(1)\prime}(z) - R_{22}^{(1)\prime}(z) = \sum_{j=1}^{m} \frac{1}{(z-t_j)^2}\frac{-2v_j(t_j + \widetilde{\Lambda}_{I,j})}{2\pi\rho_s(t_j)\sqrt{1-t_j^2}} + \frac{1}{(z-1)^2}\frac{4\alpha_{m+1}^2 - 1}{2^3\pi\psi_s(1)} + \frac{1}{(z+1)^2}\frac{1-4\alpha_0^2}{2^3\pi\psi_s(-1)},$$

$$R_{11}^{(1)\prime}(z) + R_{22}^{(1)\prime}(z) = 0,$$

$$\begin{aligned} i[R_{12}^{(1)\prime}(z)D_\infty^{-2} + R_{21}^{(1)\prime}(z)D_\infty^{2}] = & \sum_{j=1}^{m} \frac{1}{(z-t_j)^2}\frac{v_j(-2 + \widetilde{\Lambda}_{R,1,j} - \widetilde{\Lambda}_{R,2,j})}{2\pi\rho_s(t_j)\sqrt{1-t_j^2}} + \frac{1}{(z-1)^2}\frac{-(1-4\alpha_{m+1}^2)}{2^3\pi\psi_s(1)} \\ & + \frac{1}{(z+1)^2}\frac{-(1-4\alpha_0^2)}{2^3\pi\psi_s(-1)}. \end{aligned}$$

As in Subsection 2.7.3, we rewrite $I_{3,s}$ in the form

$$I_{3,s} = \sum_{j=1}^{m} I_{3,s,t_j} + I_{3,s,1} + I_{3,s,-1} + \mathcal{O}\big(n^{-1+4\beta_{\max}}\big), \qquad \text{as } n \to \infty, \qquad (2.7.52)$$

where

$$I_{3,s,t_k} = \frac{-v_k}{8\pi^2\rho_s(t_k)} \int_{\mathcal{C}} \left[\frac{a^2(z)}{a_+^2(t_k)} + \frac{a_+^2(t_k)}{a^2(z)} + \widetilde{\Lambda}_{I,k} \left(\frac{a^2(z)}{a_+^2(t_k)} - \frac{a_+^2(t_k)}{a^2(z)} \right) \right] \frac{\partial_s V_s(z)}{(z-t_k)^2} dz,$$

$$I_{3,s,1} = \frac{4\alpha_{m+1}^2 - 1}{2^5\pi^2 i\psi_s(1)} \int_{\mathcal{C}} \frac{a^2(z)}{(z-1)^2} \partial_s V_s(z) dz,$$

$$I_{3,s,-1} = \frac{1-4\alpha_0^2}{2^5\pi^2 i\psi_s(-1)} \int_{\mathcal{C}} \frac{a^{-2}(z)}{(z+1)^2} \partial_s V_s(z) dz.$$

From (2.7.14) and (2.7.15), I_{3,s,t_k} simplifies to

$$I_{3,s,t_k} = -\frac{v_k}{\psi_s(t_k)} \Big(\psi(t_k) - \frac{1}{\pi} \Big) - \frac{v_k \widetilde{\Lambda}_{I,k}}{\rho_s(t_k)(1-t_k^2)} \int_{t_k}^1 \Big(\psi(x) - \frac{1}{\pi} \Big) \frac{dx}{\sqrt{1-x^2}} \tag{2.7.53}$$

and hence, similarly to (2.7.41)–(2.7.42), as $n \to +\infty$ we have

$$\int_0^1 I_{3,s,t_k} ds = -v_k \log(\pi\psi(t_k)) + \mathcal{O}(n^{-1+2|\Re\beta_k|}). \tag{2.7.54}$$

Also, from (2.7.24)–(2.7.25), we have

$$I_{3,s,1} = -\frac{1-4\alpha_{m+1}^2}{8\psi_s(1)} \Big(\psi(1) - \frac{1}{\pi} \Big) \qquad \text{and} \qquad I_{3,s,-1} = -\frac{1-4\alpha_0^2}{8\psi_s(-1)} \Big(\psi(-1) - \frac{1}{\pi} \Big), \tag{2.7.55}$$

and hence

$$\int_0^1 I_{3,s,1} ds = -\frac{1-4\alpha_{m+1}^2}{8} \log(\pi\psi(1)) \qquad \text{and} \qquad \int_0^1 I_{3,s,-1} ds = -\frac{1-4\alpha_0^2}{8} \log\big(\pi\psi(-1)\big). \tag{2.7.56}$$

This concludes the proof of proposition 2.7.2.

2.8 Integration in W

The main result of this section is the following.

Proposition 2.8.1 *As $n \to \infty$, we have*

$$\log \frac{D_n(\vec{\alpha}, \vec{\beta}, V, W)}{D_n(\vec{\alpha}, \vec{\beta}, V, 0)} = n \int_{-1}^1 W(x)\rho(x) dx - \frac{1}{4\pi^2} \int_{-1}^1 \frac{W(y)}{\sqrt{1-y^2}} \Big(\fint_{-1}^1 \frac{W'(x)\sqrt{1-x^2}}{x-y} dx \Big) dy$$
$$+ \frac{\mathcal{A}}{2\pi} \int_{-1}^1 \frac{W(x)}{\sqrt{1-x^2}} dx - \sum_{j=0}^{m+1} \frac{\alpha_j}{2} W(t_j) + \sum_{j=1}^{m} \frac{i\beta_j}{\pi} \sqrt{1-t_j^2} \fint_{-1}^1 \frac{W(x)}{\sqrt{1-x^2}(t_j - x)} dx + \mathcal{O}\big(n^{-1+2\beta_{\max}}\big). \tag{2.8.1}$$

where D_n stands for either L_n or J_n.

Remark 2.8.1 *The difference between Laguerre-type and Jacobi-type weights in the r.h.s. of* (2.8.1) *is only reflected in the definitions of* ρ *and* $\mathcal{A}$*.*

The proof of Proposition 2.8.1 goes in a similar way as in [3]. For each $t \in [0,1]$, we define

$$W_t(z) = \log\left(1 - t + te^{W(z)}\right), \tag{2.8.2}$$

where the principal branch is taken for the log. For every $t \in [0,1]$, W_t is analytic on a neighbourhood of $[-1,1]$ (independent of t) and is still Hölder continuous on $\mathcal{I}$. This deformation is the same as the one used in [3,9,15]. Therefore, we can and do use the steepest descent analysis of Section 2.4 applied to the weight

$$w_t(x) = e^{-nV(x)}e^{W_t(x)}\omega(x). \tag{2.8.3}$$

From Proposition 2.3.3, we have the following differential identities

$$\partial_t \log L_n(\vec{\alpha}, \vec{\beta}, V, W_t) = \frac{1}{2\pi i}\int_{-1}^{+\infty} [Y^{-1}(x)Y'(x)]_{21}\partial_t w_t(x)dx, \tag{2.8.4}$$

$$\partial_t \log J_n(\vec{\alpha}, \vec{\beta}, V, W_t) = \frac{1}{2\pi i}\int_{-1}^{1} [Y^{-1}(x)Y'(x)]_{21}\partial_t w_t(x)dx. \tag{2.8.5}$$

The rest of the proof consists of inverting the transformations $Y \mapsto T \mapsto S \mapsto R$ and evaluating certain integrals by contour deformations. These computations are identical to those done in [3, Section 7] for Gaussian-type weights and we omit them here.

3. A RIEMANN-HILBERT APPROACH TO ASYMPTOTIC ANALYSIS OF TOEPLITZ + HANKEL DETERMINANTS

Abstract. In this chapter we will formulate a 4×4 RH problem for Toeplitz+Hankel determinants. We will develop a nonlinear steepest descent method for analysing this problem in the case where the symbols are smooth (no Fisher-Hartwig singularities). We will finally introduce a model problem and will present its solution requiring certain conditions on the ratio of Hankel and Toeplitz symbols, which allows us to find the asymptotics of the norm h_n of the corresponding orthogonal polynomials. We will explain how this solvable case is related to the recent operator-theoretic approach in [37] to Toeplitz+Hankel determinants. At the end we will present a number of interesting problems related to the asymptotics of Toeplitz+Hankel determinants and will discuss the prospects of future work in each direction within the 4×4 Riemann-Hilbert framework introduced in this chapter. This is a joint work with A. Its.

Notation. In this chapter we will frequently use the notation $\tilde{f}(z)$, to denote $f(z^{-1})$.

3.1 Introduction and preliminaries

The work in this chapter is intended to develop a Riemann-Hilbert approach to the study of the large-n asymptotics of Toeplitz+Hankel determinants

$$D_n(\phi, w; r, s) := \det \begin{pmatrix} \phi_r + w_s & \phi_{r-1} + w_{s+1} & \cdots & \phi_{r-n+1} + w_{s+n-1} \\ \phi_{r+1} + w_{s+1} & \phi_r + w_{s+2} & \cdots & \phi_{r-n+2} + w_{s+n} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{r+n-1} + w_{s+n-1} & \phi_{r+n-2} + w_{s+n} & \cdots & \phi_r + w_{s+2n-2} \end{pmatrix}, \qquad r, s \in \mathbb{Z}. \tag{3.1.1}$$

In [9] the Riemann-Hilbert technique which has already been proven very effective to study the asymptotics of Toeplitz and Hankel detreminants was extended for the first time to the determinants of Toelpitz + Hankel matrices generated by the same symbol $w(z) = \phi(z)$, where the Hankel weight is supported on $\mathbb{T}$. In that work the symbol was assumed to be of Fisher-Hartwig type and it was further required that the symbol be even, i.e. $w(z) = \tilde{w}(z)$. Also in the very recent work [37], via operator-theoretic methods, the authors managed to generalize the results obtained in [9] in the following sense; they assumed that

$$\phi(z) = c(z)\phi_0(z) \qquad \text{and} \qquad w(z) = c(z)d(z)w_0(z) \tag{3.1.2}$$

where $c(z)$ and $d(z)$ are supposed to be smooth and nonvanishing with zero winding number. Neither $c(z)$ or $d(z)$ were assumed to be even functions but it was further required that $d(z)$ satisfies the conditions $d(z)\tilde{d}(z) = 1$ and $d(\pm 1) = 1$. Furthermore, $\phi_0(z)$ is an even function of FH type and $w_0(z)$ is related to $\phi_0(z)$ in one of the following ways: a) $w_0(z) = \pm\phi_0(z)$, b) $w_0(z) = z\phi_0(z)$ and c) $w_0(z) = -z^{-1}\phi_0(z)$.

Perhaps the most important motivation behind studying Toeplitz+Hankel determinants is to study the large n asymptotics of the eigenvalues of the Hankel matrix $H_n[w]$ associated to the symbol $w(z)$, simply because the characteristic polynomial $\det(H_n[w] - \lambda I)$ of the Hankel matrix $H_n[w]$ is indeed a particular *Toeplitz+Hankel* determinant, with $\phi(z) \equiv -\lambda$.[1] Clearly in the case of characteristic polynomial of a Hankel determinant, there is no relationship between $\phi(z)$ and $w(z)$, so to study the asymptotics of this determinant, one can not refer to the works [9] or [37] mentioned above. So there is a methodological issue which has to be addressed at a fundamental level: formulation of a suitable Riemann-Hilbert problem.

In this chapter, we are proposing a version of the Riemann-Hilbert formalism for the asymptotic analysis of Toeplitz+Hankel determinants based on a certain 4×4 Riemann-Hilbert problem. We also show that in the case where the symbols are smooth, nonzero and have zero winding number on the unit circle, one can proceed with a 4×4 analogue of

[1] Unlike the characteristic polynomial of a Hankel matrix, the key feature which allows an effective asymptotic spectral analysis of Toeplitz matrices and, in particular, the use of the Riemann-Hilbert method, is that the characteristic polynomial of a Toeplitz matrix is again a Toeplitz determinant with the symbol of the general Fisher-Hartwig type. For example see [38].

Deift-Zhou steepest descent method and arrive at a 4×4 model Riemann-Hilbert problem on the unit circle which does not contain the parameter n and hence plays the role of "global parametrix" in our analysis. We have been able to solve the model problem for the class of symbols (3.1.2) considered in [37], where there is no Fisher-Hartwig singularity. It should be noticed that in our approach we do not need the condition that $d(\pm 1) = 1$. Solving the model problem allows us to find the asymptotics of the norm h_n of the associated orthogonal polymomials (see (3.2.2)). The following theorem is our main result in this chapter.

Theorem 3.1.1 *Suppose that $\phi(e^{i\theta})$ is smooth, nonzero, and has zero winding number on the unit circle. Let $w = d\phi$, where d satisfies all the properties of ϕ in addition to $d(e^{i\theta})d(e^{-i\theta}) = 1$, for all $\theta \in [0, 2\pi)$. Then the asymptotics of*

$$h_{n-1} \equiv \frac{D_n(\phi, w, 1, 1)}{D_{n-1}(\phi, w, 1, 1)},$$

is given by

$$h_{n-1} = \frac{\mathcal{E}(n)}{\mathcal{E}(n-1)}(1 + \mathcal{O}(e^{-2cn})), \qquad n \to \infty, \tag{3.1.3}$$

where

$$\mathcal{E}(n) := (-\alpha(0))^n \left(\frac{2}{\alpha(0)} R_{1,43}(0; n) - C_\rho(0) R_{1,23}(0; n) \right), \tag{3.1.4}$$

$$R_{1,23}(z; n) = \frac{1}{2\pi i} \int_{\Gamma_i'} \frac{\mu^n g_{23}(\mu)}{\mu - z} d\mu, \qquad R_{1,43}(z; n) = \frac{1}{2\pi i} \int_{\Gamma_i'} \frac{\mu^n g_{43}(\mu)}{\mu - z} d\mu, \tag{3.1.5}$$

$$g_{23}(z) = -\frac{\alpha(0)\tilde{w}(z)\beta(z)}{\tilde{\phi}(z)\tilde{\alpha}(z)}, \qquad g_{43}(z) = -\alpha^2(0) \left(\frac{\alpha(z)\tilde{\beta}(z)}{\tilde{\phi}(z)} + \frac{\beta(z)\tilde{w}(z)C_\rho(z)}{\tilde{\alpha}(z)\tilde{\phi}(z)} \right), \tag{3.1.6}$$

$$\mathcal{C}_\rho(z) = -\frac{1}{2\pi i} \int_{\mathbb{T}} \frac{1}{\tilde{\beta}_-(\tau)\beta_+(\tau)\tilde{\alpha}_-(\tau)\alpha_+(\tau)(\tau - z)} d\tau, \tag{3.1.7}$$

and finally

$$\alpha(z) = \exp\left[\frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\ln(\phi(\tau))}{\tau - z} d\tau \right], \qquad \beta(z) = \exp\left[\frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\ln(d(\tau))}{\tau - z} d\tau \right]. \tag{3.1.8}$$

In (3.1.5), the contour Γ_i' is a circle with radius $r < 1$, so that the functions ϕ and d are analytic in the annulus $\{z : r < |z| < 1\}$.

3.2 Toeplitz + Hankel determinants: Hankel weight supported on $\mathbb{T}$

We want to study

$$D_n(\phi, w; r, s) \equiv D_n := \det(T_n[\phi; r] + H_n[w; s]), \quad r, s \in \mathbb{Z}. \tag{3.2.1}$$

A key observation is that the determinant (3.2.1) is related in the usual way to the system of monic polynomials, $\{\mathcal{P}_n(z)\}$, determined by the orthogonal relations

$$\int_{\mathbb{T}} \mathcal{P}_n(z) z^{-k-r} \phi(z) \frac{dz}{2\pi i z} + \int_{\mathbb{T}} \mathcal{P}_n(z) z^{k+s} \tilde{w}(z) \frac{dz}{2\pi i z} = h_n \delta_{n,k}, \quad k = 0, 1, \cdots, n. \tag{3.2.2}$$

These polynomials exist and are unique if the Toeplitz+Hankel determinants

$$D_n = \det \begin{pmatrix} \phi_r + w_s & \phi_{r-1} + w_{s+1} & \cdots & \phi_{r-n+1} + w_{s+n-1} \\ \phi_{r+1} + w_{s+1} & \phi_r + w_{s+2} & \cdots & \phi_{r-n+2} + w_{s+n} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{r+n-1} + w_{s+n-1} & \phi_{r+n-2} + w_{s+n} & \cdots & \phi_r + w_{s+2n-2} \end{pmatrix} \tag{3.2.3}$$

are non-zero. The uniqueness of the polynomial $\mathcal{P}_n(z) = z^n + a_{n-1} z^{n-1} + \cdots + a_0$ satisfying (3.2.2), simply follows from the fact that one has the following linear system for the coefficients $a_j, 1 \le j \le n-1$:

$$(T_{n+1}[\phi; r] + H_{n+1}[w; s]) \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ h_n \end{pmatrix}. \tag{3.2.4}$$

So if $D_{n+1} \neq 0$, the coefficients a_j and hence $\mathcal{P}_n$, can be uniquely determined by inverting the Toeplitz+Hankel matrix in (3.2.4). Expectedly the polynomials $\mathcal{P}_n$ can be written as the following determinants

$$\mathcal{P}_n(z) := \frac{1}{D_n} \det \begin{pmatrix} \phi_r + w_s & \phi_{r-1} + w_{s+1} & \cdots & \phi_{r-n+1} + w_{s+n-1} & \phi_{r-n} + w_{s+n} \\ \phi_{r+1} + w_{s+1} & \phi_r + w_{s+2} & \cdots & \phi_{r-n+2} + w_{s+n} & \phi_{r-n+1} + w_{s+n+1} \\ \vdots & \vdots & \ddots & \vdots & \\ \phi_{r+n-1} + w_{s+n-1} & \phi_{r+n-2} + w_{s+n} & \cdots & \phi_r + w_{s+2n-2} & \phi_{r-1} + w_{s+2n-1} \\ 1 & z & \cdots & z^{n-1} & z^n \end{pmatrix}. \tag{3.2.5}$$

Indeed for the polynomials defined by (3.2.5) we have that

$$\int_{\mathbb{T}} \mathcal{P}_n(z) z^{k+s} \tilde{w}(z) \frac{dz}{2\pi i z} = \frac{1}{D_n} \det \begin{pmatrix} \phi_r + w_s & \phi_{r-1} + w_{s+1} & \cdots & \phi_{r-n+1} + w_{s+n-1} & \phi_{r-n} + w_{s+n} \\ \phi_{r+1} + w_{s+1} & \phi_r + w_{s+2} & \cdots & \phi_{r-n+2} + w_{s+n} & \phi_{r-n+1} + w_{s+n+1} \\ \vdots & \vdots & \ddots & \vdots & \\ \phi_{r+n-1} + w_{s+n-1} & \phi_{r+n-2} + w_{s+n} & \cdots & \phi_r + w_{s+2n-2} & \phi_{r-1} + w_{s+2n-1} \\ w_{k+s} & w_{k+s+1} & \cdots & w_{k+s+n-1} & w_{k+s+n} \end{pmatrix}, \tag{3.2.6}$$

and

$$\int_{\mathbb{T}} \mathcal{P}_n(z) z^{-k-r} \phi(z) \frac{dz}{2\pi i z} = \frac{1}{D_n} \det \begin{pmatrix} \phi_r + w_s & \phi_{r-1} + w_{s+1} & \cdots & \phi_{r-n+1} + w_{s+n-1} & \phi_{r-n} + w_{s+n} \\ \phi_{r+1} + w_{s+1} & \phi_r + w_{s+2} & \cdots & \phi_{r-n+2} + w_{s+n} & \phi_{r-n+1} + w_{s+n+1} \\ \vdots & \vdots & \ddots & \vdots & \\ \phi_{r+n-1} + w_{s+n-1} & \phi_{r+n-2} + w_{s+n} & \cdots & \phi_r + w_{s+2n-2} & \phi_{r-1} + w_{s+2n-1} \\ \phi_{k+r} & \phi_{k+r-1} & \cdots & \phi_{k+r-n+1} & \phi_{k+r-n} \end{pmatrix}. \tag{3.2.7}$$

hence

$$\int_{\mathbb{T}} \mathcal{P}_n(z) z^{-k-r} \phi(z) \frac{dz}{2\pi i z} + \int_{\mathbb{T}} \mathcal{P}_n(z) z^{k+s} \tilde{w}(z) \frac{dz}{2\pi i z} =$$

$$\frac{1}{D_n}\det\begin{pmatrix} \phi_r+w_s & \phi_{r-1}+w_{s+1} & \cdots & \phi_{r-n+1}+w_{s+n-1} & \phi_{r-n}+w_{s+n} \\ \phi_{r+1}+w_{s+1} & \phi_r+w_{s+2} & \cdots & \phi_{r-n+2}+w_{s+n} & \phi_{r-n+1}+w_{s+n+1} \\ \vdots & \vdots & \ddots & \vdots & \\ \phi_{r+n-1}+w_{s+n-1} & \phi_{r+n-2}+w_{s+n} & \cdots & \phi_r+w_{s+2n-2} & \phi_{r-1}+w_{s+2n-1} \\ \phi_{n+r}+w_{n+s} & \phi_{n+r-1}+w_{n+s+1} & \cdots & \phi_{r+1}+w_{s+2n-1} & \phi_r+w_{s+2n} \end{pmatrix} = \frac{D_{n+1}}{D_n}\delta_{n,k}.$$

So the polynomials defined by (3.2.5) are the uniqe polynomials satisfying (3.2.2), and

$$h_n = \frac{D_{n+1}}{D_n}. \tag{3.2.8}$$

Moreover, the polynomials $\{\mathcal{P}_n(z)\}$ are related to the following Riemann-Hilbert problem for the 2×2 matrix valued function $\mathcal{Y}$:

- **RH-$\mathcal{Y}$1** $\mathcal{Y}(z;n)$ is holomorphic in the complement of $\mathbb{T}$.
- **RH-$\mathcal{Y}$2** For $z\in\mathbb{T}$ we have

$$\mathcal{Y}^{(1)}_+(z) = \mathcal{Y}^{(1)}_-(z), \qquad z\in\mathbb{T}, \tag{3.2.9}$$

and

$$\mathcal{Y}^{(2)}_+(z) = \mathcal{Y}^{(2)}_-(z) + z^{-1+s}\tilde{w}(z)\mathcal{Y}^{(1)}_-(z) + z^{-1+r}\tilde{\phi}(z)\mathcal{Y}^{(1)}_-(z^{-1}), \qquad z\in\mathbb{T}, \tag{3.2.10}$$

- **RH-$\mathcal{Y}$3** As $z\to\infty$

$$\mathcal{Y}(z;n) = \left(I+\mathcal{O}(z^{-1})\right)z^{n\sigma_3} = \begin{pmatrix} z^n+O(z^{n-1}) & O(z^{-n-1}) \\ O(z^{n-1}) & z^{-n}+O(z^{-n-1}) \end{pmatrix}, \tag{3.2.11}$$

where $\mathcal{Y}^{(1)}$ and $\mathcal{Y}^{(2)}$ are the first and second columns of $\mathcal{Y}$ respectively. Let us see the relationship between this Riemann-Hilbert problem and the orthogonal polynomials satisfying the orthogonality relation (3.2.2). From (3.2.9) we see that $\mathcal{Y}_{11}$ and $\mathcal{Y}_{21}$ are entire functions, and from (3.2.11) we know that $\mathcal{Y}_{11}$ has to be a monic polynomial of degree n and $\mathcal{Y}_{21}$ has to be a polynomial of degree $n-1$. From (3.2.10) and what we just found about $\mathcal{Y}_{11}$ we would have

$$(\mathcal{Y}_{12}(z))_+ - (\mathcal{Y}_{12}(z))_- = z^{-1+s}\tilde{w}(z)\mathcal{Y}_{11}(z) + z^{-1+r}\tilde{\phi}(z)\widetilde{\mathcal{Y}}_{11}(z). \tag{3.2.12}$$

So by Plemelj-Sokhotskii formula we have

$$\mathcal{Y}_{12}(z) = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\xi^{-1+s}\tilde{w}(\xi)\mathcal{Y}_{11}(\xi) + \xi^{-1+r}\tilde{\phi}(\xi)\widetilde{\mathcal{Y}}_{11}(\xi)}{\xi - z} d\xi. \tag{3.2.13}$$

Using the identity

$$\frac{1}{\xi - z} = -\sum_{k=0}^{n} \frac{\xi^k}{z^{k+1}} + \frac{\xi^{n+1}}{(\xi - z)z^{n+1}}, \tag{3.2.14}$$

we get

$$\begin{aligned}\mathcal{Y}_{12}(z) = -\sum_{k=0}^{n}\frac{1}{z^{k+1}}\int_{\mathbb{T}}\left[\xi^{-1+s}\tilde{w}(\xi)\mathcal{Y}_{11}(\xi)\xi^k + \xi^{-1+r}\tilde{\phi}(\xi)\widetilde{\mathcal{Y}}_{11}(\xi)\xi^k\right]\frac{d\xi}{2\pi i} + \\ \frac{1}{z^{n+1}}\int_{\mathbb{T}}\frac{\xi^{n+1}}{(\xi - z)}\left[\xi^{-1+s}\tilde{w}(\xi)\mathcal{Y}_{11}(\xi) + \xi^{-1+r}\tilde{\phi}(\xi)\widetilde{\mathcal{Y}}_{11}(\xi)\right]\frac{d\xi}{2\pi i}.\end{aligned} \tag{3.2.15}$$

Note that due to $\mathcal{Y}_{12}(z) = O(z^{-n-1})$ we must have :

$$\int_{\mathbb{T}} \tilde{w}(\xi)\mathcal{Y}_{11}(\xi)\xi^{k+s}\frac{d\xi}{2\pi i\xi} + \int_{\mathbb{T}} \tilde{\phi}(\xi)\widetilde{\mathcal{Y}}_{11}(\xi)\xi^{k+r}\frac{d\xi}{2\pi i\xi} = 0, \quad 0 \le k \le n-1. \tag{3.2.16}$$

In the second integral we make the change of variable $\xi \mapsto \tau := \xi^{-1}$ and as a result we will arrive at

$$\int_{\mathbb{T}} \mathcal{Y}_{11}(\xi)\xi^{k+s}\tilde{w}(\xi)\frac{d\xi}{2\pi i\xi} + \int_{\mathbb{T}} \mathcal{Y}_{11}(\tau)\tau^{-k-r}\phi(\tau)\frac{d\tau}{2\pi i\tau} = 0, \quad 0 \le k \le n-1. \tag{3.2.17}$$

Since $\mathcal{Y}_{11}$ satisfies the orthogonality relations (3.2.2) we necessarily have

$$\mathcal{Y}_{11}(z) = \mathcal{P}_n(z). \tag{3.2.18}$$

In a similar fashion one can show that

$$\mathcal{Y}_{21}(z) = -\frac{1}{h_{n-1}}\mathcal{P}_{n-1}(z). \tag{3.2.19}$$

So we have shown that a representation of the solution to the $\mathcal{Y}$-RHP is given by

$$\mathcal{Y}(z;n) = \begin{pmatrix} \mathcal{P}_n(z) & \frac{1}{2\pi i}\int_{\mathbb{T}}\frac{\xi^{-1+s}\tilde{w}(\xi)\mathcal{P}_n(\xi) + \xi^{-1+r}\tilde{\phi}(\xi)\tilde{\mathcal{P}}_n(\xi)}{\xi - z}d\xi \\ -\frac{1}{h_{n-1}}\mathcal{P}_{n-1}(z) & -\frac{1}{2\pi i h_{n-1}}\int_{\mathbb{T}}\frac{\xi^{-1+s}\tilde{w}(\xi)\mathcal{P}_{n-1}(\xi) + \xi^{-1+r}\tilde{\phi}(\xi)\tilde{\mathcal{P}}_{n-1}(\xi)}{\xi - z}d\xi \end{pmatrix}. \tag{3.2.20}$$

It is important to notice that if the solution to the $\mathcal{Y}-$RHP exists, it is unique, because $\mathcal{Y}_{ij}$, $i,j=1,2$ are all uniquely identified with the *unique* orthogonal polynomials satisfying the orthogonality conditions (3.2.2). Also note that

$$h_n = -\lim_{z\to\infty} z^n/\mathcal{Y}_{21}(z;n+1). \tag{3.2.21}$$

This formula in conjunction with (3.2.8) will finally allow us to compute the asymptotics of the Toeplitz+Hankel determinants for specific choices of ϕ and w.

3.3 The steepest descent analysis for $r = s = 1$.

In this section we will develop a 4×4 analogue of the Deift/Zhou non-linear steepest descent method. For technical reasons that will be elaborated later, we will focus on the case where $r=s=1$. We are positive that our method has the capacity to allow for analyzing general values of r and s but the details of this generaliztion has not been fully worked out.[2] As the 2×2 $\mathcal{Y}-$RHP does not have jump conditions which could be written in the matrix form (see **RH-$\mathcal{Y}$2**), there is no prospect for developing a 2×2 Deift/Zhou non-linear steepest method for our particular Riemann-Hilbert problem. To this end, we will increase the size of the Riemann-Hilbert problem so that the jump conditions could be written in the matrix form. We first propose the associated 2×4 and then the associated 4×4 Riemann-Hilbert problem. Although more complicated, the analysis of the proposed 4×4 Riemann-Hilbert problem follows in the same spirit as the lower dimensional RHPs until we get to the model Riemann-Hilbert problem for Toeplitz+Hankel determinants introduced in section 3.3.6.

3.3.1 The associated 2×4 and 4×4 Riemann-Hilbert problems

Let us define the 2×4 matrix $\overset{\circ}{\mathcal{X}}$ out of the columns of $\mathcal{Y}$ as follows

$$\overset{\circ}{\mathcal{X}}(z;n) := \left(\mathcal{Y}^{(1)}(z;n), \mathcal{Y}^{(1)}(z^{-1};n), \mathcal{Y}^{(2)}(z;n), \mathcal{Y}^{(2)}(z^{-1};n)\right), \tag{3.3.1}$$

From (3.2.9), (3.2.10) and (3.2.11) we obtain the following Riemann-Hilbert problem for $\overset{\circ}{\mathcal{X}}$:

[2]We believe that it can be done in the same spirit as the arguments given in [9], see Lemma 2.4.

- **RH-$\mathring{\mathcal{X}}$1** $\mathring{\mathcal{X}}$ is holomorphic in the complement of $\mathbb{T}$.

- **RH-$\mathring{\mathcal{X}}$2** For $z \in \mathbb{T}$, $\mathring{\mathcal{X}}$ satisfies

$$\mathring{\mathcal{X}}(z;n)_+ = \mathring{\mathcal{X}}(z;n)_- \begin{pmatrix} 1 & 0 & \tilde{w}(z) & -\phi(z) \\ 0 & 1 & \tilde{\phi}(z) & -w(z) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.3.2}$$

- **RH-$\mathring{\mathcal{X}}$3** As $z \to \infty$ we have

$$\mathring{\mathcal{X}}(z;n) = \begin{pmatrix} 1+\mathcal{O}(z^{-1}) & C_1(n)+\mathcal{O}(z^{-1}) & \mathcal{O}(z^{-1}) & C_3(n)+\mathcal{O}(z^{-1}) \\ \mathcal{O}(z^{-1}) & C_2(n)+\mathcal{O}(z^{-1}) & 1+\mathcal{O}(z^{-1}) & C_4(n)+\mathcal{O}(z^{-1}) \end{pmatrix} \begin{pmatrix} z^n & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^{-n} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.3.3}$$

- **RH-$\mathring{\mathcal{X}}$4** As $z \to 0$ we have

$$\mathring{\mathcal{X}}(z) = \begin{pmatrix} C_1(n)+\mathcal{O}(z) & 1+\mathcal{O}(z) & C_3(n)+\mathcal{O}(z) & \mathcal{O}(z) \\ C_2(n)+\mathcal{O}(z) & \mathcal{O}(z) & C_4(n)+\mathcal{O}(z) & 1+\mathcal{O}(z) \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^{-n} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^n \end{pmatrix}, \tag{3.3.4}$$

where

$$C_1(n) = \mathcal{Y}_{11}(0), \quad C_3(n) = \mathcal{Y}_{12}(0), \quad C_2(n) = \mathcal{Y}_{21}(0), \quad C_4(n) = \mathcal{Y}_{22}(0). \tag{3.3.5}$$

In a natural way we will now consider the following 4×4 Riemann-Hilbert problem which we will refer to as the $\mathcal{X}$-RHP.

- **RH-$\mathcal{X}$1** $\mathcal{X}$ is holomorphic in the complement of $\mathbb{T} \cup \{0\}$.

- **RH-$\mathcal{X}$2** For $z \in \mathbb{T}$, $\mathcal{X}$ satisfies

$$\mathcal{X}_+(z;n) = \mathcal{X}_-(z;n) \begin{pmatrix} 1 & 0 & \tilde{w}(z) & -\phi(z) \\ 0 & 1 & \tilde{\phi}(z) & -w(z) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.3.6}$$

- **RH-$\mathcal{X}$3** As $z \to \infty$ we have

$$\mathcal{X}(z;n) = (I + \mathcal{O}(z^{-1})) \begin{pmatrix} z^n & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^{-n} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.3.7}$$

- **RH-$\mathcal{X}$4** As $z \to 0$ we have

$$\mathcal{X}(z;n) = P(n)(I + \mathcal{O}(z)) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^{-n} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^n \end{pmatrix}. \tag{3.3.8}$$

Remark 3.3.1 *Only in the case $r = s = 1$ we are certain that P in* (3.3.8) *is a constant matrix in z. This will be justified later and will have crucial significance in the analysis of the small-norm Riemann-Hilbert problem and also ensures that the solution of the $\mathcal{X}$-RHP is unique.*

3.3.2 Relation of the 2×4 and 4×4 Riemann-Hilbert problems

It is natural to consider

$$\mathfrak{R}(z;n) := \mathring{\mathcal{X}}(z;n)\mathcal{X}^{-1}(z;n). \tag{3.3.9}$$

From (3.3.2) and (3.3.6) it is clear that $\mathfrak{R}$ has no jumps. From (3.3.4) and (3.3.8) we can obtain the behavior of $\mathfrak{R}$ near zero :

$$\mathfrak{R}(z;n) = \begin{pmatrix} C_1(n) + \mathcal{O}(z) & 1 + \mathcal{O}(z) & C_3(n) + \mathcal{O}(z) & \mathcal{O}(z) \\ C_2(n) + \mathcal{O}(z) & \mathcal{O}(z) & C_4(n) + \mathcal{O}(z) & 1 + \mathcal{O}(z) \end{pmatrix} P^{-1}(n). \tag{3.3.10}$$

Therefore $\mathfrak{R}$ is an entire function. Also note that from (3.3.3) and (3.3.7) we have

$$\mathfrak{R}(z;n) = \begin{pmatrix} 1 + \mathcal{O}(z^{-1}) & C_1(n) + \mathcal{O}(z^{-1}) & \mathcal{O}(z^{-1}) & C_3(n) + \mathcal{O}(z^{-1}) \\ \mathcal{O}(z^{-1}) & C_2(n) + \mathcal{O}(z^{-1}) & 1 + \mathcal{O}(z^{-1}) & C_4(n) + \mathcal{O}(z^{-1}) \end{pmatrix}, \qquad z \to \infty. \tag{3.3.11}$$

Therefore by Liouville's theorem we conclude that

$$\mathfrak{R}(z;n) = \begin{pmatrix} 1 & C_1(n) & 0 & C_3(n) \\ 0 & C_2(n) & 1 & C_4(n) \end{pmatrix}. \tag{3.3.12}$$

And therefore we have

$$\begin{pmatrix} 1 & C_1(n) & 0 & C_3(n) \\ 0 & C_2(n) & 1 & C_4(n) \end{pmatrix} = \begin{pmatrix} C_1(n) & 1 & C_3(n) & 0 \\ C_2(n) & 0 & C_4(n) & 1 \end{pmatrix} P^{-1}(n). \tag{3.3.13}$$

Once we asymptotically solve the $\mathcal{X}$-RHP, A large-n asymptotic expression for P can be found from

$$P(n) = \mathcal{X}(z;n) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^n & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^{-n} \end{pmatrix} \Bigg|_{z=0}, \tag{3.3.14}$$

Which enables us to find asymptotic expressions for the constants C_i, $1 \le i \le 4$ via (3.3.13). This allows for construction of the asymptotic solution to the $\mathring{\mathcal{X}}$-RHP through (3.3.9).

3.3.3 Undressing of the $\mathcal{X}$-RHP

We observe the following factorization for the jump matrix of the $\mathcal{X}$-RHP, which we denote by $J_{\mathcal{X}}(z)$:

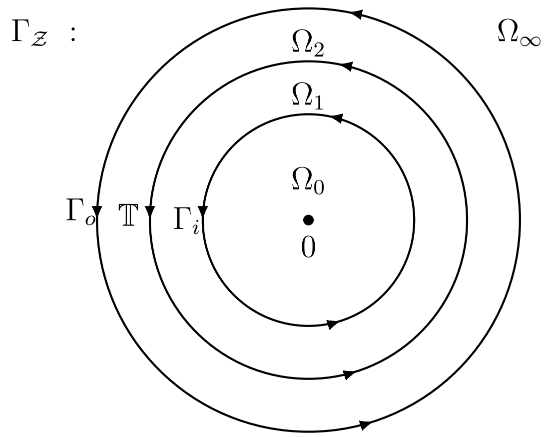

Figure 3.1. The jump contour for the $\mathcal{Z}$-RHP

$$J_{\mathcal{X}}(z) := \begin{pmatrix} 1 & 0 & \tilde{w}(z) & -\phi(z) \\ 0 & 1 & \tilde{\phi}(z) & -w(z) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -w(z) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -\phi(z) \\ 0 & 1 & \tilde{\phi}(z) & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \tilde{w}(z) & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$\equiv J_{\mathcal{X},o}(z) J_{\mathcal{X},\mathbb{T}}(z) J_{\mathcal{X},i}(z). \tag{3.3.15}$$

Let us define the function $\mathcal{Z}$ as

$$\mathcal{Z}(z;n) := \mathcal{X}(z;n) \begin{cases} J^{-1}_{\mathcal{X},i}(z), & z \in \Omega_1, \\ J_{\mathcal{X},o}(z), & z \in \Omega_2, \\ I, & z \in \Omega_0 \cup \Omega_\infty, \end{cases} \tag{3.3.16}$$

where $J_{\mathcal{X},i}$ and $J_{\mathcal{X},o}$ are defined in the factorization (3.3.15). the function $\mathcal{Z}$ satisfies the following RHP, which we will refer to as the $\mathcal{Z}-$RHP from now on:

- **RH-$\mathcal{Z}$1** $\mathcal{Z}$ is holomorphic in $\mathbb{C} \setminus (\mathbb{T} \cup \Gamma_i \cup \Gamma_o)$.

- **RH-$\mathcal{Z}$2** $\mathcal{Z}_+(z;n) = \mathcal{Z}_-(z;n)J_{\mathcal{Z}}(z)$, where

$$J_{\mathcal{Z}}(z) = \begin{cases} J_{\mathcal{X},\mathbb{T}}(z), & z \in \mathbb{T}, \\ J_{\mathcal{X},i}(z), & z \in \Gamma_i, \\ J_{\mathcal{X},o}(z), & z \in \Gamma_o. \end{cases} \tag{3.3.17}$$

- **RH-$\mathcal{Z}$3** As $z \to \infty$ we have

$$\mathcal{Z}(z;n) = (I + \mathcal{O}(z^{-1})) \begin{pmatrix} z^n & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^{-n} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.3.18}$$

- **RH-$\mathcal{Z}$4** As $z \to 0$ we have

$$\mathcal{Z}(z;n) = P(n)(I + \mathcal{O}(z)) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^{-n} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^n \end{pmatrix}. \tag{3.3.19}$$

Note that since we are considering Hankel weights w which are holomorphic in some neighborhood of the unit circle, $\mathcal{Z}$ does not have extra jumps in Ω_1 and Ω_2 (see (3.3.16) and (3.3.17)).

3.3.4 Normalization of behaviours at 0 and ∞

Following the natural steps of Riemann-Hilbert analysis, we will normalize the behavior of $\mathcal{Z}$ at 0 and ∞; to this end let us define

$$T(z;n) := \mathcal{Z}(z;n) \begin{cases} \begin{pmatrix} z^{-n} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^{n} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & |z| > 1, \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^{n} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^{-n} \end{pmatrix}, & |z| < 1. \end{cases} \tag{3.3.20}$$

It is very important to note that in order to have a suitable Riemann-Hilbert analysis, the normalization of behaviors at 0 and ∞ can only be carried out only after the undressing $\mathcal{X} \mapsto \mathcal{Z}$; this is due to technical reasons that will be further commented about at the end of this section. We have the following RHP for T :

- **RH-T1** T is holomorphic in $\mathbb{C} \setminus (\mathbb{T} \cup \Gamma_i \cup \Gamma_o)$.
- **RH-T2** $T_+(z;n) = T_-(z;n) J_T(z;n)$, where

$$J_T(z;n) = \begin{cases} \widehat{J}(z;n), & z \in \mathbb{T}, \\ J_{\mathcal{X},i}(z), & z \in \Gamma_i, \\ J_{\mathcal{X},o}(z), & z \in \Gamma_o, \end{cases} \qquad \text{where} \qquad \widehat{J}(z;n) = \begin{pmatrix} z^{n} & 0 & 0 & -\phi(z) \\ 0 & z^{n} & \tilde{\phi}(z) & 0 \\ 0 & 0 & z^{-n} & 0 \\ 0 & 0 & 0 & z^{-n} \end{pmatrix}, \tag{3.3.21}$$

  and the matrices $J_{\mathcal{X},i}(z)$ and $J_{\mathcal{X},o}(z)$ are defined by (3.3.15).

- **RH-T3** As $z \to \infty$, we have $T(z;n) = (I + \mathcal{O}(z^{-1}))$.
- **RH-T4** As $z \to 0$, we have $T(z;n) = P(n)(I + \mathcal{O}(z))$.

We observe that for $z \in \mathbb{T}$, G_T can be factorized as follows

$$\widehat{J}(z;n) = \begin{pmatrix} I_2 & 0_2 \\ z^{-n}\Phi^{-1}(z) & I_2 \end{pmatrix} \begin{pmatrix} 0_2 & \Phi(z) \\ -\Phi^{-1}(z) & 0_2 \end{pmatrix} \begin{pmatrix} I_2 & 0_2 \\ z^{n}\Phi^{-1}(z) & I_2 \end{pmatrix} \equiv J_{T,o}(z;n)\mathring{J}(z)J_{T,i}(z;n), \tag{3.3.22}$$

where 0_2 and I_2 are respectively 2×2 zero and identity matrices and

$$\Phi(z) = \begin{pmatrix} 0 & -\phi(z) \\ \tilde{\phi}(z) & 0 \end{pmatrix}. \tag{3.3.23}$$

Note that $J_{T,i}$ is exponentially close to the identity matrix for z inside of the unit circle and $J_{T,o}$ is exponentially close to the identity matrix for z outside of the unit circle.

It should be pointed out that if one normalizes the behaviors at 0 and ∞ without the un-dressing transformation $\mathcal{X} \mapsto \mathcal{Z}$; i.e. by directly defining the function $\mathcal{T}$ as

$$\mathcal{T}(z;n) := \mathcal{X}(z;n) \begin{cases} \begin{pmatrix} z^{-n} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^n & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & |z| > 1, \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^n & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^{-n} \end{pmatrix}, & |z| < 1. \end{cases} \tag{3.3.24}$$

then the jump matrix $J_{\mathcal{T}}(z) := \mathcal{T}_-^{-1}(z)\mathcal{T}_+(z)$ on the unit circle would be

$$J_{\mathcal{T}}(z) = \begin{pmatrix} z^n & 0 & z^n\tilde{w}(z) & -\phi(z) \\ 0 & z^n & \tilde{\phi}(z) & -z^{-n}w(z) \\ 0 & 0 & z^{-n} & 0 \\ 0 & 0 & 0 & z^{-n} \end{pmatrix}, \tag{3.3.25}$$

for which finding a factorization like (3.3.22) remains a challenge, mainly due to presence of the large parameter n in the 13 and 24 elements of $J_{\mathcal{T}}$. This fact justifies the necessity of the undressing step $\mathcal{X} \mapsto \mathcal{Z}$.

3.3.5 Opening of the lenses

The next Riemann-Hilbert transformation $T \mapsto S$, provides us with a problem with jump conditions on five contours where three jump matrices do not depend on n and the other two converge exponentially fast to the identity matrix as $n \to \infty$. Let us define the function S, suggested by (3.3.22), as

$$S(z;n) := T(z;n) \times \begin{cases} J_{T,i}^{-1}(z;n), & z \in \Omega_1', \\ J_{T,o}(z;n), & z \in \Omega_2', \\ I, & z \in \Omega_1'' \cup \Omega_2'' \cup \Omega_0 \cup \Omega_\infty, \end{cases} \tag{3.3.26}$$

where the regions Ω_1', Ω_2', Ω_1'' and Ω_2'' are shown in Figure 3.2. we have the following Riemann-Hilbert problem for S

- **RH-S1** S is holomorphic in $\mathbb{C} \setminus (\mathbb{T} \cup \Gamma_i \cup \Gamma_o \cup \Gamma_i' \cup \Gamma_o')$.
- **RH-S2** $S_+(z;n) = S_-(z;n) J_S(z;n)$, where

$$J_S(z;n) = \begin{cases} \overset{\circ}{J}(z), & z \in \mathbb{T}, \\ J_{T,i}(z;n), & z \in \Gamma_i', \\ J_{T,o}(z;n), & z \in \Gamma_o', \\ J_{\mathcal{X},i}(z), & z \in \Gamma_i, \\ J_{\mathcal{X},o}(z), & z \in \Gamma_o. \end{cases} \tag{3.3.27}$$

- **RH-S3** As $z \to \infty$, we have $S(z;n) = I + \mathcal{O}(z^{-1})$.
- **RH-S4** As $z \to 0$, we have $S(z;n) = P(n)(I + \mathcal{O}(z))$.

In the usual way, we will first try to solve this Riemann-Hilbert problem by ignoring the jump matrices which depend on n, this solution $\overset{\circ}{S}$ will be referred to as the global parametrix. Once we have constructed the global parametrix we will consider the small-norm Riemann-Hilbert problem for the ratio $R := S(\overset{\circ}{S})^{-1}$ and discuss its solvability in the forthcoming sections.

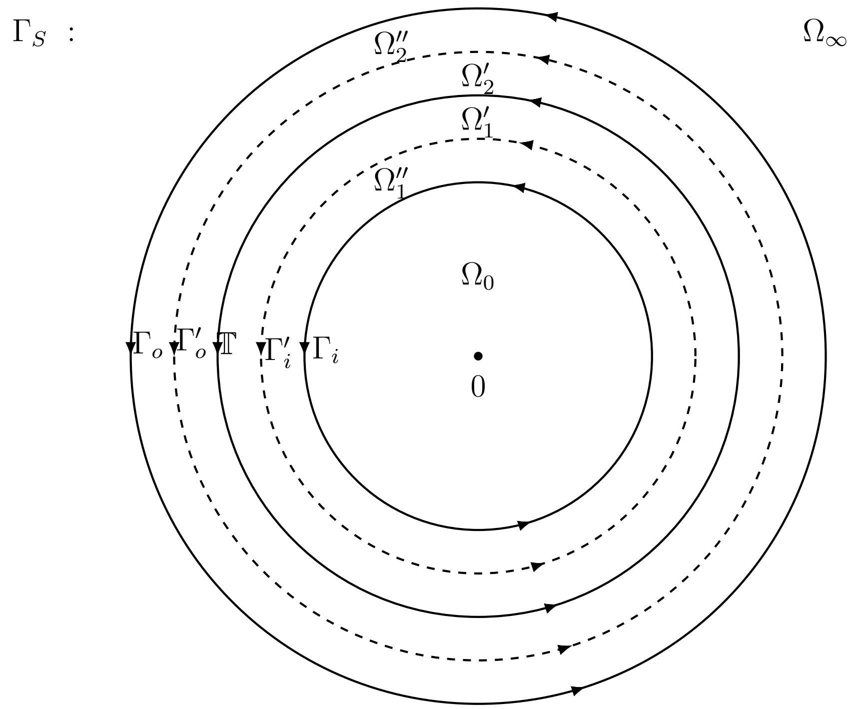

Figure 3.2. The jump contour for the S-RHP

3.3.6 The global parametrix and a model Riemann-Hilbert problem

In the same spirit as usual situations in nonlinear steepest-descent analysis, we will try to find a solution of S-RHP (the global parametrix) without regards to the jump matrices which are exponentially close to the identity matrix, indeed we consider the following RHP for $\mathring{S}$:

- **RH-$\mathring{S}$1** $\quad \mathring{S}$ is holomorphic in $\mathbb{C} \setminus (\mathbb{T} \cup \Gamma_i \cup \Gamma_o)$.
- **RH-$\mathring{S}$2** $\quad \mathring{S}_+(z) = \mathring{S}_-(z) J_{\mathring{S}}(z)$, where

$$J_{\mathring{S}}(z) = \begin{cases} \mathring{J}(z), & z \in \mathbb{T}, \\ J_{\mathcal{X},i}(z), & z \in \Gamma_i, \\ J_{\mathcal{X},o}(z), & z \in \Gamma_o. \end{cases} \tag{3.3.28}$$

- **RH-$\mathring{S}$3** $\quad$ As $z \to \infty$, we have $\mathring{S}(z) = I + \mathcal{O}(z^{-1})$.

And we finally *dress* the $\overset{\circ}{S}$-RHP to obtain a model problem for the global parametrix having jumps only on the unit circle. We define the function Λ as

$$\Lambda(z) := \overset{\circ}{S}(z) \times \begin{cases} J_{\mathcal{X},i}(z), & z \in \Omega_1, \\ J_{\mathcal{X},o}^{-1}(z), & z \in \Omega_2, \\ I, & z \in \Omega_0 \cup \Omega_\infty, \end{cases} \tag{3.3.29}$$

Now we arrive at the following Riemann-Hilbert problem for Λ that from now on we will refer to as *the model Riemann-Hilbert problem for Toeplitz+Hankel determinants*:

- **RH-Λ1** Λ is holomorphic in $\mathbb{C} \setminus \mathbb{T}$.
- **RH-Λ2** $\Lambda_+(z) = \Lambda_-(z) J_\Lambda(z)$, for $z \in \mathbb{T}$, where

$$J_\Lambda(z) = \begin{pmatrix} 0 & 0 & 0 & -\phi(z) \\ -\dfrac{w(z)}{\phi(z)} & 0 & \tilde{\phi}(z) - \dfrac{w(z)\tilde{w}(z)}{\phi(z)} & 0 \\ 0 & -\dfrac{1}{\tilde{\phi}(z)} & 0 & 0 \\ \dfrac{1}{\phi(z)} & 0 & \dfrac{\tilde{w}(z)}{\phi(z)} & 0 \end{pmatrix}. \tag{3.3.30}$$

- **RH-Λ3** As $z \to \infty$, we have $\Lambda(z) = I + \mathcal{O}(z^{-1})$.

The conditions on w and ϕ which ensure the solvability of this model problem are not completely known and categorized at this point. However, in the next section we will show a detailed analysis of this model problem for a family of Toeplitz and Hankel weights considered by E. Basor and T. Ehrhardt in their recent paper [37].

3.3.7 A solvable case

Using Operator-theoretic tools, in [37] the authors have obtained asymptotic formulas for the the Toeplitz+Hankel determinants $D_n(a, b; 0, 1) = \det(a_{j-k} + b_{j+k+1})_{j,k=0,\dots,n-1}$, where $a(z) = \phi(z)a_0(z)$, $b(z) = \phi(z)d(z)b_0(z)$. In their work a_0 and b_0 are any even pure Fisher-Hartwig symbol (see (1.1.4)) while ϕ and d are assumed to be smooth and nonvanishing with zero winding number. Neither ϕ nor d are assumed to be even functions but it is further required that d satisfies the conditions $d(z)\tilde{d}(z) = 1$ and $d(\pm 1) = 1$. Since in this work the symbols are not assumed to be of the Fisher-Hartwig type (which needs a more delicate treatment, e.g. see [9]), we should still expect that the model Riemann-Hilbert problem be solvable for the same choices of symbols in [37] when there is no Fisher-Hartwig singularity($a_0(z) = b_0(z) \equiv 0$). Indeed this is the case as will be elaborated in this section. As commented in the beginning of section 3.3, asymptotics of $D_n(\phi, d\phi; r, s)$, for general r and s requires a more delicate approach and we do not discuss the details here. So let us consider $D_n(\phi, d\phi; 1, 1)$, where d

a) is analytic in a neighborhood of the unit circle,

b) has zero winding number,

c) does not vanish on the unit circle, and

d) satisfies the condition $d(z)\tilde{d}(z) = 1$.

For instance, a class of functions satisfying these conditions is given by

$$d(z) = \prod_{i=1}^{m} d_i(z), \qquad d_i(z) = (z - a_i)^{\alpha_i}(z - b_i)^{\beta_i}(z - a_i^{-1})^{-\alpha_i}(z - b_i^{-1})^{-\beta_i}, \tag{3.3.31}$$

where $\alpha_i + \beta_i = 0$, $-\beta_i/\alpha_i > 1$, $0 < a_i = b_i^{-\beta_i/\alpha_i} < b_i < 1$, all factors are defined by their principal branch, and

$$0 < a_1 < b_1 < a_2 < b_2 < \cdots < a_m < b_m < 1.$$

Note that a similar construction can be found for $-1 < b_m < a_m < \cdots < b_1 < a_1 < 0$, and thus a larger class of functions can be found from multiplying functions of the first class with those of the second class. Although we have a class of functions satisfying the required properties, a complete categorization of functions satisfying the four required properties for d is yet to be found. We emphasize that the conditions $d(\pm 1) = 1$ required in [37] do not play a role in the Riemann-Hilbert analysis. However, for d defined as in (3.3.31) one can check that $d(1)=(-1)^{\sum_{i=1}^{m}(\alpha_i+\beta_i)}$ and $d(-1)=1$. So in this sense we are considering functions d which are slightly more general than those considered in [37].

Note that the condition $d(z)\tilde{d}(z)=1$ renders the 23-element of the jump matrix J_Λ zero; indeed

$$J_{\Lambda,23}(z)=\tilde{\phi}(z)-\frac{w(z)\tilde{w}(z)}{\phi(z)}=\tilde{\phi}(z)(1-d(z)\tilde{d}(z))=0.$$

Hence, for the particular choices made above, the jump matrix G_Λ reduces to

$$J_\Lambda(z)=\begin{pmatrix} 0 & 0 & 0 & -\phi(z) \\ -d(z) & 0 & 0 & 0 \\ 0 & -\dfrac{1}{\tilde{\phi}(z)} & 0 & 0 \\ \dfrac{1}{\phi(z)} & 0 & \dfrac{\tilde{w}(z)}{\phi(z)} & 0 \end{pmatrix}. \tag{3.3.32}$$

In order to factorize J_Λ, let us first consider the following Szegő functions

$$\alpha(z)=\exp\left[\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{\ln(\phi(\tau))}{\tau-z}d\tau\right],\qquad \beta(z)=\exp\left[\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{\ln(d(\tau))}{\tau-z}d\tau\right]. \tag{3.3.33}$$

By Plemelj-Sokhotskii formula α, β, $\tilde{\alpha}$ and $\tilde{\beta}$ satisfy the following jump conditions on the unit circle:

$$\begin{aligned} \alpha_+(z)&=\alpha_-(z)\phi(z), & \beta_+(z)&=\beta_-(z)d(z),\\ \tilde{\alpha}_-(z)&=\tilde{\alpha}_+(z)\tilde{\phi}(z), & \tilde{\beta}_-(z)&=\tilde{\beta}_+(z)\tilde{d}(z). \end{aligned} \tag{3.3.34}$$

It turns out that knowing the value of $\beta(0)$ is crucial for finding an asymptotic expression for h_n (see section 3.3.9) and the condition $d(z)\tilde{d}(z)=1$ on the unit circle allows us to evaluate $\beta(0)$ easily. Indeed

$$\int_{\mathbb{T}}\ln(d(\tau))\frac{d\tau}{\tau}=\int_{\mathbb{T}}\ln(\tilde{d}(\tau))\frac{d\tau}{\tau}=\int_{\mathbb{T}}\ln(d^{-1}(\tau))\frac{d\tau}{\tau}=-\int_{\mathbb{T}}\ln(d(\tau))\frac{d\tau}{\tau}.$$

Thus

$$\int_{\mathbb{T}}\ln(d(\tau))\frac{d\tau}{\tau}=0,\qquad \text{and therefore,}\qquad \beta(0)=1. \tag{3.3.35}$$

We also note that $\alpha(z), \beta(z) = 1 + \mathcal{O}(z^{-1})$, $\tilde{\alpha}(z) = \alpha(0)(1 + \mathcal{O}(z^{-1}))$ and $\tilde{\beta}(z) = 1 + \mathcal{O}(z^{-1})$ (by (3.3.35)) as $z \to \infty$. Now we can write the solution of the Λ-RHP (in the case $d\tilde{d} \equiv 1$ on $\mathbb{T}$) as

$$\Lambda(z) = \Lambda_\infty^{-1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ \mathcal{C}_\rho(z) & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \times \begin{cases} \begin{pmatrix} -\beta(z) & 0 & 0 & 0 \\ 0 & 0 & \dfrac{1}{\tilde{\alpha}(z)\tilde{\beta}(z)\alpha(z)} & 0 \\ 0 & -\tilde{\alpha}(z) & 0 & 0 \\ 0 & 0 & 0 & -\alpha(z) \end{pmatrix}, & |z| < 1, \\ \begin{pmatrix} 0 & \beta(z) & 0 & 0 \\ 0 & 0 & 0 & \dfrac{1}{\tilde{\beta}(z)\tilde{\alpha}(z)\alpha(z)} \\ 0 & 0 & \tilde{\alpha}(z) & 0 \\ \alpha(z) & 0 & 0 & 0 \end{pmatrix}, & |z| > 1. \end{cases} \tag{3.3.36}$$

where $\mathcal{C}_f(z)$ is the Cauchy-transform of $f(z)$:

$$\mathcal{C}_f(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(\tau)}{\tau - z} d\tau,$$

and

$$\Lambda_\infty^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{1}{\alpha(0)} & 0 \\ 0 & \alpha(0) & 0 & 0 \end{pmatrix}, \qquad \rho(z) = -\frac{1}{\tilde{\beta}_-(z)\beta_+(z)\tilde{\alpha}_-(z)\alpha_+(z)}. \tag{3.3.37}$$

Using (3.3.34), the Plemelj-Sokhotskii formula and general properties of the Cauchy integral, it can be checked that $\Lambda(z)$ given by (3.3.36) satisfies the Λ-RHP.

3.3.8 The small-norm Riemann-Hilbert problem associated to $D_n(\phi, d\phi, 1, 1)$

Let us consider

$$R(z;n) := S(z;n)\overset{\circ}{S}(z)^{-1}. \tag{3.3.38}$$

This function clearly has no jumps on Γ_i, Γ_o and $\mathbb{T}$, since S and $\overset{\circ}{S}$ have the same jumps on these contours. Thus, R satisfies the following small-norm Riemann-Hilbert problem

- **RH-R1** R is holomorphic in $\mathbb{C} \setminus \Sigma_R$.
- **RH-R2** $R_+(z;n) = R_-(z;n)J_R(z;n)$, for $z \in \Sigma_R$.
- **RH-R3** As $z \to \infty$, $R(z;n) = I + \mathcal{O}(z^{-1})$,

where $\Sigma_R := \Gamma_i' \cup \Gamma_o'$, and J_R is given by

$$J_R(z;n) = \overset{\circ}{S}(z)G_S(z;n)\overset{\circ}{S}(z)^{-1} = \begin{cases} \overset{\circ}{S}(z)G_{T,i}(z;n)\overset{\circ}{S}(z)^{-1}, & z \in \Gamma_i', \\ \overset{\circ}{S}(z)G_{T,o}(z;n)\overset{\circ}{S}(z)^{-1}, & z \in \Gamma_o'. \end{cases} \tag{3.3.39}$$

Using (3.3.36), (3.3.37) and the definitions of $G_{T,i}, G_{T,o}, G_{\mathcal{X},i}$ and $G_{\mathcal{X},i}$ given by (3.3.15) and (3.3.22) we find

$$J_R(z;n) - I = \begin{cases} z^n \cdot \begin{pmatrix} 0 & g_{12}(z) & 0 & g_{14}(z) \\ 0 & 0 & g_{23}(z) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & g_{43}(z) & 0 \end{pmatrix}, & z \in \Gamma_i', \\ z^{-n} \cdot \begin{pmatrix} 0 & 0 & 0 & 0 \\ g_{21}(z) & 0 & 0 & 0 \\ 0 & g_{32}(z) & 0 & g_{34}(z) \\ g_{41}(z) & 0 & 0 & 0 \end{pmatrix}, & z \in \Gamma_o', \end{cases} \tag{3.3.40}$$

where

$$\begin{aligned}
&g_{12}(z) = -\frac{\alpha(z)}{\phi(z)\beta(z)} - \frac{\tilde{w}(z)C_\rho(z)}{\phi(z)\tilde{\beta}(z)\tilde{\alpha}(z)}, \qquad g_{14}(z) = \frac{\tilde{w}(z)}{\phi(z)\tilde{\beta}(z)\tilde{\alpha}(z)\alpha(0)},\\
&g_{23}(z) = -\frac{\alpha(0)\tilde{w}(z)\beta(z)}{\tilde{\phi}(z)\tilde{\alpha}(z)}, \qquad g_{43}(z) = -\alpha^2(0)\left(\frac{\alpha(z)\tilde{\beta}(z)}{\tilde{\phi}(z)} + \frac{\beta(z)\tilde{w}(z)C_\rho(z)}{\tilde{\alpha}(z)\tilde{\phi}(z)}\right),\\
&g_{21}(z) = \frac{w(z)\beta(z)}{\phi(z)\alpha(z)}, \qquad g_{32}(z) = -\frac{1}{\alpha(0)\tilde{\phi}(z)}\left(\frac{\tilde{\alpha}(z)}{\beta(z)} - w(z)\tilde{\alpha}^2(z)\tilde{\beta}(z)\alpha(z)C_\rho(z)\right),\\
&g_{34}(z) = \frac{w(z)\tilde{\alpha}^2(z)\tilde{\beta}(z)\alpha(z)}{\tilde{\phi}(z)\alpha^2(0)}, \qquad g_{41}(z) = -\frac{\alpha(0)}{\phi(z)}\left(\frac{1}{\tilde{\alpha}(z)\tilde{\beta}(z)\alpha^2(z)} - \frac{w(z)\beta(z)C_\rho(z)}{\alpha(z)}\right).
\end{aligned} \tag{3.3.41}$$

Therefore by Lemma A.0.1 we have

$$R_1(z;n) = \frac{1}{2\pi i}\int_{\Sigma_R}\frac{J_R(\mu;n)-I}{\mu - z}d\mu = \begin{pmatrix} 0 & R_{1,12}(z;n) & 0 & R_{1,14}(z;n)\\ R_{1,21}(z;n) & 0 & R_{1,23}(z;n) & 0\\ 0 & R_{1,32}(z;n) & 0 & R_{1,34}(z;n)\\ R_{1,41}(z;n) & 0 & R_{1,43}(z;n) & 0\end{pmatrix}, \tag{3.3.42}$$

where

$$\begin{aligned}
R_{1,jk}(z;n) &= \frac{1}{2\pi i}\int_{\Gamma_i'}\frac{\mu^n g_{jk}(\mu)}{\mu - z}d\mu, \qquad jk = 12, 14, 23, 43,\\
R_{1,jk}(z;n) &= \frac{1}{2\pi i}\int_{\Gamma_o'}\frac{\mu^{-n} g_{jk}(\mu)}{\mu - z}d\mu, \qquad jk = 21, 32, 34, 41.
\end{aligned} \tag{3.3.43}$$

By Lemma A.0.1 we can also find $R_k(z)$, $k \geq 2$, recursively. For instance

$$R_2(z;n) = \frac{1}{2\pi i}\int_{\Sigma_R}\frac{[R_1(\mu;n)]_-\,(J_R(\mu;n)-I)}{\mu - z}d\mu = \begin{pmatrix} R_{2,11}(z;n) & 0 & R_{2,13}(z;n) & 0\\ 0 & R_{2,22}(z;n) & 0 & R_{2,24}(z;n)\\ R_{2,31}(z;n) & 0 & R_{2,33}(z;n) & 0\\ 0 & R_{2,42}(z;n) & 0 & R_{2,44}(z;n)\end{pmatrix}, \tag{3.3.44}$$

where

$$R_{2,kj}(z;n) = \begin{cases} \displaystyle\sum_{\ell\in\{2,4\}} \frac{1}{2\pi i}\int_{\Gamma'_o} \frac{\mu^{-n}\cdot[R_{1,k\ell}(\mu;n)]_-\, g_{\ell j}(\mu)}{\mu - z} d\mu, & j=1, k=1,3, \\ \displaystyle\sum_{\ell\in\{2,4\}} \frac{1}{2\pi i}\int_{\Gamma'_i} \frac{\mu^{n}\cdot[R_{1,k\ell}(\mu;n)]_-\, g_{\ell j}(\mu)}{\mu - z} d\mu, & j=3, k=1,3, \\ \displaystyle\frac{1}{2\pi i}\int_{\Gamma'_i} \frac{\mu^{n}\cdot[R_{1,k1}(\mu;n)]_-\, g_{1j}(\mu)}{\mu - z} d\mu + \frac{1}{2\pi i}\int_{\Gamma'_o} \frac{\mu^{-n}\cdot[R_{1,k3}(\mu;n)]_-\, g_{3j}(\mu)}{\mu - z} d\mu, & k,j=2,4. \end{cases} \tag{3.3.45}$$

Moreover, using (A.0.15) and a straightforward calculation one can justify that the matrix structure (i.e. the location of zero and nonzero elements) of R_{2k+1} and R_{2k}, $k \geq 1$, are similar to that of R_1 and R_2, respectively. It is also straightforward to show that

$$R_{k,ij}(z;n) = \mathcal{O}(e^{-kcn}), \qquad n \to \infty, \quad k \geq 1, \tag{3.3.46}$$

for some positive constant c.

3.3.9 Asymptotics of h_n

From (3.2.21) we have

$$-\frac{1}{h_{n-1}} = \lim_{z\to 0} z^{n-1}\mathcal{Y}_{21}(z^{-1};n). \tag{3.3.47}$$

Let us denote

$$\mathcal{A}(z;n) := P^{-1}(n)\mathcal{X}(z;n)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^n & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^{-n} \end{pmatrix}, \tag{3.3.48}$$

and also let us define the matrix $\mathcal{B}(n)$ in the following expansion for $\mathcal{A}(z;n)$, which is equivalent to **RH-$\mathcal{X}$4**:

$$\mathcal{A}(z;n) = I + \mathcal{B}(n)z + \mathcal{O}(z^2), \qquad z \to 0. \tag{3.3.49}$$

Therefore by (3.3.9), (3.3.12), (3.3.13) and (3.3.48) we can write

$$\overset{\circ}{\mathcal{X}}(z,n) = \begin{pmatrix} C_1(n) & 1 & C_3(n) & 0 \\ C_2(n) & 0 & C_4(n) & 1 \end{pmatrix} \mathcal{A}(z;n) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^{-n} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^n \end{pmatrix}. \qquad (3.3.50)$$

Using (3.3.1) and (3.3.50) we can write

$$\mathcal{Y}_{21}(z^{-1};n) = \overset{\circ}{\mathcal{X}}_{22}(z;n) = C_2(n)\mathcal{A}_{12}(z;n)z^{-n} + C_4(n)\mathcal{A}_{32}(z;n)z^{-n} + \mathcal{A}_{42}(z;n)z^{-n}. \quad (3.3.51)$$

From (3.3.49) we have

$$z^{-n}\mathcal{A}(z;n) = z^{-n} \cdot I + z^{-n+1}\mathcal{B}(n) + \mathcal{O}(z^{-n+2}), \qquad z \to 0. \qquad (3.3.52)$$

Therefore, as $z \to 0$

$$z^{-n}\mathcal{A}_{ij}(z;n) = \begin{cases} z^{-n+1}\mathcal{B}_{ij}(n) + \mathcal{O}(z^{-n+2}), & i \neq j, \\ z^{-n} + z^{-n+1}\mathcal{B}_{ii}(n) + \mathcal{O}(z^{-n+2}), & i = j. \end{cases} \qquad (3.3.53)$$

Therefore by (3.3.47), (3.3.51) and (3.3.53) we have

$$-\frac{1}{h_{n-1}} = C_2(n)\mathcal{B}_{12}(n) + C_4(n)\mathcal{B}_{32}(n) + \mathcal{B}_{42}(n). \qquad (3.3.54)$$

Tracing back the Riemann-Hilbert transformations, we find that for $z \in \Omega_0$ we have

$$\mathcal{X}(z;n) = R(z;n)\Lambda(z) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^{-n} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^n \end{pmatrix}, \qquad \text{hence,} \qquad \mathcal{A}(z;n) = P^{-1}(n)R(z;n)\Lambda(z), \qquad (3.3.55)$$

by (3.3.48). Also from (3.3.14) and (3.3.55) we conclude that

$$P(n) = R(0;n)\Lambda(0). \qquad (3.3.56)$$

Let us denote the coefficients in the expansions of $R(z;n)$ and $\Lambda(z)$, as $z \to 0$, by

$$R(z;n) = R(0;n) + R^{(1)}(n)\cdot z + R^{(2)}(n)\cdot z^2 + \mathcal{O}(z^3), \qquad \Lambda(z) = \Lambda(0) + \Lambda^{(1)}\cdot z + \Lambda^{(2)}\cdot z^2 + \mathcal{O}(z^3). \tag{3.3.57}$$

Therefore from (3.3.49), (3.3.55), and (3.3.56) we have

$$\mathcal{B}(n) = \Lambda^{-1}(0)R^{-1}(0;n)R^{(1)}(n)\Lambda(0) + \Lambda^{-1}(0)\Lambda^{(1)}. \tag{3.3.58}$$

Note that

$$R^{(1)}(n) = \frac{1}{2\pi i}\int_{\Sigma_R}(J_R(\mu;n) - I)\frac{d\mu}{\mu^2} + \mathcal{O}(e^{-2cn}), \qquad R^{-1}(0;n) = I - R_1(0;n) + \mathcal{O}(e^{-2cn}), \tag{3.3.59}$$

as $n \to \infty$. More explicitly we have

$$R^{(1)}(n) = \begin{pmatrix} 0 & R^{(1)}_{12}(n) & 0 & R^{(1)}_{14}(n) \\ R^{(1)}_{21}(n) & 0 & R^{(1)}_{23}(n) & 0 \\ 0 & R^{(1)}_{32}(n) & 0 & R^{(1)}_{34}(n) \\ R^{(1)}_{41}(n) & 0 & R^{(1)}_{43}(n) & 0 \end{pmatrix}, \qquad n \to \infty, \tag{3.3.60}$$

where

$$\begin{aligned} R^{(1)}_{jk}(n) &= \frac{1}{2\pi i}\int_{\Gamma_i'} \mu^{n-2} g_{jk}(\mu)d\mu, \qquad jk = 12, 14, 23, 43, \\ R^{(1)}_{jk}(n) &= \frac{1}{2\pi i}\int_{\Gamma_o'} \mu^{-n-2} g_{jk}(\mu)d\mu, \qquad jk = 21, 32, 34, 41, \end{aligned} \tag{3.3.61}$$

and

$$R^{-1}(0;n) = \begin{pmatrix} 1 & -R_{1,12}(0;n) & 0 & -R_{1,14}(0;n) \\ -R_{1,21}(0;n) & 1 & -R_{1,23}(0;n) & 0 \\ 0 & -R_{1,32}(0;n) & 1 & -R_{1,34}(0;n) \\ -R_{1,41}(0;n) & 0 & -R_{1,43}(0;n) & 1 \end{pmatrix} + \mathcal{O}(e^{-2cn}), \qquad n \to \infty. \tag{3.3.62}$$

From (3.3.36) and (3.3.37) we have

$$\Lambda(0)=\begin{pmatrix}0 & 0 & 0 & -\alpha(0)\\ -1 & 0 & 0 & 0\\ 0 & -\dfrac{1}{\alpha(0)} & 0 & 0\\ -C_\rho(0)\alpha(0) & 0 & 1 & 0\end{pmatrix},\qquad \Lambda^{(1)}=\begin{pmatrix}0 & 0 & 0 & \Lambda^{(1)}_{14}\\ \Lambda^{(1)}_{21} & 0 & 0 & 0\\ 0 & \Lambda^{(1)}_{32} & 0 & 0\\ \Lambda^{(1)}_{41} & 0 & \Lambda^{(1)}_{43} & 0\end{pmatrix},\quad (3.3.63)$$

where

$$\begin{aligned}
\Lambda^{(1)}_{14} &= -\frac{\alpha(0)}{2\pi i}\int_{\mathbb{T}}\log\phi(\mu)\frac{d\mu}{\mu^2}, \qquad \Lambda^{(1)}_{21} = -\frac{1}{2\pi i}\int_{\mathbb{T}}\log d(\mu)\frac{d\mu}{\mu^2}, \qquad \Lambda^{(1)}_{32} = \frac{1}{2\pi i\alpha(0)}\int_{\mathbb{T}}\log\phi(\mu)d\mu,\\
\Lambda^{(1)}_{41} &= -\alpha(0)\left\{\frac{1}{2\pi i}\int_{\mathbb{T}}\rho(\mu)\frac{d\mu}{\mu^2}-\frac{1}{4\pi^2}\left(\int_{\mathbb{T}}\rho(\mu)\frac{d\mu}{\mu}\right)\left(\int_{\mathbb{T}}\log d(\mu)\frac{d\mu}{\mu^2}\right)\right\},\\
\Lambda^{(1)}_{43} &= \frac{1}{2\pi i}\left\{\int_{\mathbb{T}}\log w(\mu)d\mu-\int_{\mathbb{T}}\log\phi(\mu)\frac{d\mu}{\mu^2}\right\}.
\end{aligned}\quad (3.3.64)$$

From (3.3.58), (3.3.60), (3.3.62) and (3.3.63) we find that

$$\begin{aligned}
&\mathcal{B}_{12}(n)=\frac{R^{(1)}_{23}(n)}{\alpha(0)}, \qquad \mathcal{B}_{32}(n)=C_\rho(0)R^{(1)}_{23}(n)-\frac{R^{(1)}_{43}(n)}{\alpha(0)},\\
&\mathcal{B}_{42}(n)=-\frac{1}{\alpha^2(0)}\left(R_{1,12}(0;n)R^{(1)}_{23}(n)+R_{1,14}(0;n)R^{(1)}_{43}(n)\right).
\end{aligned}\quad (3.3.65)$$

Note that $\mathcal{B}_{12}(n),\mathcal{B}_{32}(n)$ are of order $\mathcal{O}(e^{-cn})$, while $\mathcal{B}_{42}(n)$ is of order $\mathcal{O}(e^{-2cn})$. From (3.3.56) we can write the asymptotic expansion for $P(n)$

$$P(n)=\begin{pmatrix}-C_\rho(0)\alpha(0)R_{1,14}(0;n)-R_{1,12}(0;n) & 0 & R_{1,14}(0;n) & -\alpha(0)\\ -1 & -\dfrac{R_{1,23}(0;n)}{\alpha(0)} & 0 & -\alpha(0)R_{1,21}(0;n)\\ -C_\rho(0)\alpha(0)R_{1,34}(0;n)-R_{1,32}(0;n) & -\dfrac{1}{\alpha(0)} & R_{1,34}(0;n) & 0\\ -C_\rho(0)\alpha(0) & -\dfrac{R_{1,43}(0;n)}{\alpha(0)} & 1 & -\alpha(0)R_{1,41}(0;n)\end{pmatrix}+\mathcal{O}(e^{-2cn}),\quad (3.3.66)$$

as $n\to\infty$. Revisiting (3.3.13) we have

$$\begin{pmatrix}1 & C_1(n) & 0 & C_3(n)\\ 0 & C_2(n) & 1 & C_4(n)\end{pmatrix}P(n)=\begin{pmatrix}C_1(n) & 1 & C_3(n) & 0\\ C_2(n) & 0 & C_4(n) & 1\end{pmatrix},\quad (3.3.67)$$

From this equation, in particular, we find the following four equations for the constants C_2 and C_4

$$C_2(n)P_{21}(n) + P_{31}(n) + C_4(n)P_{41}(n) = C_2(n), \qquad C_2(n)P_{22}(n) + P_{32}(n) + C_4(n)P_{42}(n) = 0, \tag{3.3.68}$$

Solving for C_2 and C_4 we find

$$C_2(n) = \frac{P_{42}(n)P_{31}(n) - P_{41}(n)P_{32}(n)}{(1-P_{21}(n))P_{42}(n) + P_{41}(n)P_{22}(n)}, \qquad C_4(n) = -\frac{P_{22}(n)P_{31}(n) + [1-P_{21}(n)]P_{32}(n)}{(1-P_{21}(n))P_{42}(n) + P_{41}(n)P_{22}(n)}. \tag{3.3.69}$$

From (3.3.66) we have

$$C_2(n) = \frac{C_\rho(0)}{\left(\frac{2}{\alpha(0)}\right) R_{1,43}(0;n) - C_\rho(0)R_{1,23}(0;n)} \left(1 + \mathcal{O}(e^{-2cn})\right), \tag{3.3.70}$$

and

$$C_4(n) = \frac{-\frac{2}{\alpha(0)}}{\left(\frac{2}{\alpha(0)}\right) R_{1,43}(0;n) - C_\rho(0)R_{1,23}(0;n)} \left(1 + \mathcal{O}(e^{-2cn})\right). \tag{3.3.71}$$

Combining (3.3.54), (3.3.65), (3.3.70) and (3.3.71) we obtain

$$h_{n-1} = -\alpha(0) \cdot \frac{\frac{2}{\alpha(0)} R_{1,43}(0;n) - C_\rho(0)R_{1,23}(0;n)}{\frac{2}{\alpha(0)} R_{43}^{(1)}(n) - C_\rho(0)R_{23}^{(1)}(n)} (1 + \mathcal{O}(e^{-2cn})), \qquad n \to \infty. \tag{3.3.72}$$

Note that from (3.3.43) and (3.3.61) we have

$$\begin{aligned} R_{1,jk}(0;n) &= R_{jk}^{(1)}(n+1), \qquad \text{for} \quad jk = 12, 14, 23, 43, \\ R_{1,jk}(0;n) &= R_{jk}^{(1)}(n-1), \qquad \text{for} \quad jk = 21, 32, 34, 41. \end{aligned} \tag{3.3.73}$$

Denoting

$$\mathcal{E}(n) := (-\alpha(0))^n \left(\frac{2}{\alpha(0)} R_{1,43}(0;n) - C_\rho(0)R_{1,23}(0;n) \right), \tag{3.3.74}$$

and using (3.3.73) we can write (3.3.72) as

$$h_{n-1} = \frac{\mathcal{E}(n)}{\mathcal{E}(n-1)} (1 + \mathcal{O}(e^{-2cn})), \qquad n \to \infty. \tag{3.3.75}$$

We have to mention that our main objective in this part of the thesis has been to develop a 4×4 steepest descent analysis for the Toeplitz+Hankel determinant and we have achieved that. However, to obtain the asymptotics of $D_n(\phi, d\phi, 1, 1)$ one has to derive suitable differential identities. We propose that the differential identity has to be with respect to the parameters α_i in the function d given by (3.3.31). Thus, one has to perform m integrations in the parameters α_i, $1 \leq i \leq m$. Note that for $\alpha_1 = \alpha_2 = \cdots = \alpha_m = 0$, we have $d \equiv 1$ and hence $\phi = w$. Thus the starting point of integration in α_1 is given by the result of E.Basor and T.Ehrhardt in [37]. Integration of the differential identity in α_1 will provide us with an asymptotic expression for $D_n(\phi, d_1\phi, 1, 1)$, which also serves as the starting point of integration in α_2. Thus we can find asymptotics of $D_n(\phi, d_1 d_2\phi, 1, 1)$ which also serves as the starting point of integration in α_3, and so on. Repeating this procedure will finally lead us to the asymptotics of $D_n(\phi, d\phi, 1, 1)$.

In order to derive the differential identities mentioned above, one has to find recurrence relations and prove a Christoffel-Darboux formula for the polynomials (3.2.2) and follow a path similar to that introduced by I.Krasovsky in [4]. Note that the recurrence relations can be found by analyzing the function $\mathcal{M}(z;n) := \mathcal{X}(z;n+1)\mathcal{X}^{-1}(z;n)$, which is holomorphic in $\mathbb{C} \setminus \{0\}$ and can be globally determined by its singular parts at zero and infinity.

3.4 Suggestions for future work

Through the proposed Riemann-Hilbert setting in this chapter, we think that a number of open problems could be approached, here we mention at least three of such possible avenues of research.

3.4.1 Ising model on different half-planes/Extension of the results to general offset values $r, s \neq 1$.

In an unpublished work, Dmitry Chelkak has been able to express the expectation $\mathbb{E}[\sigma_N]$ of the spin in the N-th column of the isotropic Ising model on the 45° rotated half plane and also for the Ising model on the half-plane. Below we give a short account of his formulation. First, let us denote

- $q := \sinh(2J/kT)$, so $q > 1$ in the sub-critical regime and $q < 1$ in the super-critical regime, as the critical temperature T_c in the isotropic homogeneous two-dimensional Ising model satisfies $\sinh(2J/kT_c) = 1$ (see [39] and [40], for example).

- $m := 2/(q + q^{-1}) \in [0, 1]$ and thus, the critical case corresponds to $m = 1$.

- $\mu := (1 + q^{-2})^{1/2}$.

D.Chelkak has considered Ising models on different half-planes, however, for the purposes of this thesis we only mention his findings on the 45° *rotated half plane* case. In this case one has different expressions for odd and even columns, indeed

$$\mathbb{E}[\sigma_{2n-1}] = \mu^{2n-3} \det\left(\phi_{i-j} + w_{i+j}\right)_{i,j=0}^{n-1}, \qquad \mathbb{E}[\sigma_{2n}] = \mu^{2n} \det\left(\phi_{i-j} + w_{i+j+1}\right)_{i,j=0}^{n-1}. \tag{3.4.1}$$

where

$$w_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{2ik\theta}\frac{q^2e^{2i\theta}-1}{e^{2i\theta}-q^2}\sqrt{1-m^2\cos^2\theta}d\theta, \tag{3.4.2}$$

and

$$\phi_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-2ik\theta}\sqrt{1-m^2\cos^2\theta}d\theta. \tag{3.4.3}$$

However we see that the matrix elements is not exactly given by the Fourier coefficients of a symbol, however, if we let $\alpha := 2\theta$, then we get

$$w_k = \frac{1}{4\pi}\int_{-2\pi}^{2\pi} e^{ik\alpha}\frac{q^2e^{i\alpha}-1}{e^{i\alpha}-q^2}\sqrt{1-m^2\cos^2(\alpha/2)}d\alpha =$$

$$\frac{1}{4\pi}\int_{-2\pi}^{0} e^{ik\alpha}\frac{q^2e^{i\alpha}-1}{e^{i\alpha}-q^2}\sqrt{1-m^2\cos^2(\alpha/2)}d\alpha + \frac{1}{4\pi}\int_{0}^{2\pi} e^{ik\alpha}\frac{q^2e^{i\alpha}-1}{e^{i\alpha}-q^2}\sqrt{1-m^2\cos^2(\alpha/2)}d\alpha.$$

In the first of the above integrals , let $\gamma = \alpha + 2\pi$, then we can write it as

$$\frac{1}{4\pi}\int_{0}^{2\pi} e^{ik\gamma}\frac{q^2e^{i\gamma}-1}{e^{i\gamma}-q^2}\sqrt{1-m^2\cos^2(\gamma/2)}d\gamma.$$

So we finally get

$$w_k = \frac{1}{2\pi}\int_{0}^{2\pi} e^{ik\alpha}\frac{q^2e^{i\alpha}-1}{e^{i\alpha}-q^2}\sqrt{1-m^2\cos^2(\alpha/2)}d\alpha = \frac{1}{2\pi}\int_{-2\pi}^{0} e^{-ik\alpha}\frac{q^2e^{-i\alpha}-1}{e^{-i\alpha}-q^2}\sqrt{1-m^2\cos^2(\alpha/2)}d\alpha.$$

Therefore the Hankel symbol with the above Fourier coefficients is given by

$$w(z) = \frac{q^2 - z}{1 - q^2 z} \sqrt{1 - m^2 \left(\frac{2 + z + z^{-1}}{4} \right)}, \tag{3.4.4}$$

where $z = e^{i\alpha}$. It is already obvious that the same calculation gives the Toeplitz symbol ϕ as

$$\phi(z) = \sqrt{1 - m^2 \left(\frac{2 + z + z^{-1}}{4} \right)}, \tag{3.4.5}$$

and therefore

$$w(z) = \frac{q^2 - z}{1 - q^2 z} \phi(z). \tag{3.4.6}$$

Now let us find the branch points of ϕ given by (3.4.5). Assume that $m \neq 0$, and thus $q \notin \{0, \infty\}$. Clearly $z = 0$ and $z = \infty$ are branch points of this square root. The other two branch points z_1 and z_2 are the roots of the following quadratic equation

$$z^2 + (2 - \frac{4}{m^2})z + 1 = 0, \qquad \text{and thus,} \qquad z_{1,2} = \frac{2}{m^2} \left[1 - \frac{m^2}{2} \mp \sqrt{1 - m^2} \right]. \tag{3.4.7}$$

Note that $z_1 z_2 = 1$, $0 < z_1 \leq 1$ and $z_2 = \dfrac{1}{z_1} \geq 1$. We can write ϕ as

$$\phi(z) = -\frac{im}{2} z^{-1/2} (z - z_1)^{1/2} (z - z_2)^{1/2}, \tag{3.4.8}$$

where the principal branches are chosen for $z^{-1/2}$ and $(z - z_1)^{1/2}$, while we pick the branch-cut $[z_2, +\infty)$ for $(z - z_2)^{1/2}$ and $0 < \arg(z - z_2) < 2\pi$. Equivalently, $\phi(z)$ has branch-cuts $[0, z_1] \cup [z_2, +\infty]$ and its branch is fixed by $\phi(1) = \sqrt{1 - m^2}$. Having fixed the branch of ϕ, the branch of w is automatically fixed as their ratio is a rational function. Also note that

$$m = \frac{2}{q + q^{-1}}, \quad \Rightarrow \quad q^2 - \frac{2}{m} q + 1 = 0, \quad \Rightarrow \quad q_{1,2} = \frac{1}{m} \mp \sqrt{\frac{1}{m^2} - 1},$$

where $q_1 q_2 = 1$, $0 < q_1 \leq 1$ and $q_2 = \dfrac{1}{q_1} \geq 1$. Note that

$$q_{1,2}^2 = \frac{2}{m^2} \left[1 - \frac{m^2}{2} \mp \sqrt{1 - m^2} \right]. \tag{3.4.9}$$

Therefore from (3.4.7) we have

$$q_1^2 = z_1, \qquad \text{and} \qquad q_2^2 = z_2. \tag{3.4.10}$$

Now let us consider the three Temperature regimes:

- At the critical temperature we have $q = 1 \iff m = 1 \iff z_1 = z_2 = 1$.
- In the sub-critical regime $T < T_c$ we have $q > 1$ and thus we can write w as

$$w(z) = \frac{1}{q^2} \cdot \frac{z - z_2}{z - z_1} \cdot \phi(z) = -\frac{im}{2q^2}(z - z_1)^{-1/2} z^{-1/2} (z - z_2)^{3/2}. \tag{3.4.11}$$

- In the super-critical regime $T > T_c$ we have $q < 1$ and thus we can write w as

$$w(z) = \frac{1}{q^2} \cdot \frac{z - z_1}{z - z_2} \cdot \phi(z) = -\frac{im}{2q^2}(z - z_1)^{3/2} z^{-1/2} (z - z_2)^{-1/2}. \tag{3.4.12}$$

It is important to note that the Toeplitz symbol $\phi(z) = \dfrac{im}{2}(z - z_1)^{1/2} z^{-1/2} (z - z_1^{-1})^{1/2}$ has no winding on the unit circle for $q \neq 1$, hence can be factorized using the Plemelj-Sokhotski formula. Also note that

$$d(z) \equiv \frac{w(z)}{\phi(z)} = \frac{1}{q^2} \begin{cases} \dfrac{z - z_2}{z - z_1}, & T < T_c, \\ \dfrac{z - z_1}{z - z_2}, & T > T_c, \end{cases} \qquad \text{and hence,} \qquad \tilde{d}(z) = \frac{1}{q^2} \begin{cases} \dfrac{z_2}{z_1} \cdot \dfrac{z - z_1}{z - z_2}, & T < T_c, \\ \dfrac{z_1}{z_2} \cdot \dfrac{z - z_2}{z - z_1}, & T > T_c. \end{cases} \tag{3.4.13}$$

Therefore for both $T < T_c$ and $T > T_c$ we have $d\tilde{d} \equiv 1$, recalling that $q^2 = z_2$, when $T < T_c$, and $q^2 = z_1$, when $T > T_c$. This ensures that $J_{\Lambda,23} = 0$, and hence (3.3.30) again reduces to (3.3.32). This fact highly increases the prospects of solvability of the associated Λ-model problem. However, in this case the function d does not have a zero winding number and hence does not fit the criteria of section 3.3.7. This is a concrete application of Toeplitz+Hankel determinants which motivates the study of large-n asymptotics of $D_n(\phi, w, r, s)$ for proper choices of w and $r, s \neq 1$.

3.4.2 Extension to Fisher-Hartwig symbols

One can study the large-n asymptotics of determinant $D_n(\phi, d\phi, 1, 1)$ (and with increasing effort $D_n(\phi, d\phi, r, s)$ for fixed $r, s \in \mathbb{Z}$) assuming that ϕ possesses Fisher-Hartwig singularities $\{z_i\}_{i=1}^m$ on the unit circle. It is in fact in this level of generality that E.Basor and T.Ehrhardt have been able to compute the asymptotics of $D_n(\phi, d\phi, 0, 1)$, $D_n(-\phi, d\phi, 0, 1)$,

$D_n(\phi, dz\phi, 0, 1)$, and $D_n(-z\phi, d\phi, 0, 1)$ via the operator-theoretic methods in [37]. However, the authors in [37] further require that the Fisher-Hartwig part of ϕ be *even*. In fact they used some results of the work [9] of P.Deift, A.Its and I.Krasovsky to prove their asymptotic formulas for Toeplitz+Hankel determinants, and for this reason they *inherited* the evenness assumption from the work [9] where the authors needed evenness of ϕ in their 2×2 setting to relate Hankel and Toeplitz+Hankel determinants to a Toeplitz determinant with symbol ϕ.
From a Riemann-Hilbert perspective, in the presence of Fisher-Hartwig singularities, one has to construct the 4×4 local parametrices near the points z_i. Expectedly, these local parametrices must be expressed in terms of confluent hypergeometric functions as suggested by [9]. We have not yet worked out the details of this construction but we believe that it should be well within reach. It would be methodologically important to achieve the results obtained from operator-theoretic tools via the Riemann-Hilbert approach as well. Moreover, we expect that the evenness of the Fisher-Hartwig part of ϕ would not play a role in our 4×4 setting, and in that sense there are reasonable prospects of generalizing the results of [37] to symbols ϕ with non-even Fisher-Hartwig part.

3.4.3 Characteristic polynomial of a Hankel matrix

Perhaps one of the most important motivations behind studying Toeplitz+Hankel determinants is to study the large n asymptotics of the eigenvalues of the matrix $H_n[w]$, as the characteristic polynomial $\det(H_n[w] - \lambda I)$ of the Hankel matrix $H_n[w]$ is indeed a particular *Toeplitz+Hankel* determinant, with $\phi(z) \equiv -\lambda$. Unlike the characteristic polynomial of a Hankel matrix, the key feature which allows an effective asymptotic spectral analysis of Toeplitz matrices and, in particular, the use of the Riemann-Hilbert method, is that the characteristic polynomial of a Toeplitz matrix is again a Toeplitz determinant with the symbol of the general Fisher-Hartwig type (see e.g. [38]).
In this case, i.e. $D_n(-\lambda, w, 0, 0)$, the associated Λ-model needs a special treatment, in a sense it is a simpler problem as the symbol ϕ is identically equal to a constant, but more complicated - compared to the situation in section 3.3.7 - as it does not enjoy $J_{\Lambda,23}(z) = 0$. In any case,

the solution to this model problem provides us with the constant term in the asymptotics of $D_n(-\lambda, w, 0, 0)$, and in the case of Fisher-Hartwig weight w, one can hope to obtain the leading terms of this asymptotic expansion (up to the constant term, viz. the solution of the Λ-model problem) from the local analysis near the Fisher-Hartwig singularities. This last point is yet another motivation to pursue the goals of section 3.4.2.

4. ASYMPTOTIC ANALYSIS OF A BORDERED-TOEPLITZ DETERMINANT AND THE NEXT-TO-DIAGONAL CORRELATIONS OF THE ANISOTROPIC SQUARE LATTICE ISING MODEL

Abstract. In 1987 Au-Yang and Perk expressed the spin-spin next-to-diagonal correlations of the anisotropic square lattice Ising model in terms of a bordered Toeplitz determinant [41], [42]. We will relate this bordered Toeplitz determinant to the 12-entry of the 2×2 matrix solution of the well known Riemann-Hilbert problem associated with Toeplitz determinants. We will use this connection to find the large N asymptotics of the next-to-diagonal correlations $\langle \sigma_{0,0}\sigma_{N,N-1} \rangle$. This is a joint work with A.Its.

4.1 Introduction and Background

The two-dimensional Ising model is defined on a $2M \times 2N$ rectangular lattice in $\mathbb{Z}^2$ with an associated spin variable $\sigma_{j,k}$ taking the values 1 and -1 at each vertex (j,k), $-M \leq j \leq M-1$ and $-N \leq k \leq N-1$. There are $4MN$ lattice points and hence 2^{4MN} possible spin configurations $\{\sigma\}$. By a spin configuration we mean a fixation of the values $\sigma_{j,k}$ for all vertices (j,k). To each spin configuration $\{\sigma\}$, we can associate its nearest-neighbor coupling energy (Hamiltonian) given by

$$E(\{\sigma\}) = -\sum_{j=-M}^{M-1} \sum_{k=-N}^{N-1} \left(J_h \sigma_{j,k}\sigma_{j,k+1} + J_v \sigma_{j,k}\sigma_{j+1,k} \right), \qquad J_h, J_v > 0. \tag{4.1.1}$$

The probability of a spin configuration $\{\sigma\}$ is given by

$$P_{\{\sigma\}} = \frac{1}{Z(T)} \exp\left(-\frac{E\left(\{\sigma\}\right)}{k_B T} \right), \tag{4.1.2}$$

where k_B is the Boltzmann's constant and $Z(T)$ denotes the partition function and is naturally defined as

$$Z(T) = \sum_{\{\sigma\}} \exp\left(-\frac{E(\{\sigma\})}{k_B T}\right), \tag{4.1.3}$$

where the sum is taken over all configurations. Also the spin-spin correlation function between the vertices (m', n') and (m, n) is defined as the following *thermodynamic limit*

$$\langle \sigma_{m',n'} \sigma_{m,n} \rangle = \lim_{M,N\to\infty} \frac{1}{Z(T)} \sum_{\{\sigma\}} \sigma_{m',n'} \sigma_{m,n} \exp\left(-\frac{E(\{\sigma\})}{k_B T}\right). \tag{4.1.4}$$

The quantity $\lim_{m^2+n^2\to\infty} \langle \sigma_{0,0} \sigma_{m,n} \rangle$ is referred to as the long-range order in the lattice at a temperature T. Indeed, the spontaneous magnetization M is defined as square of the large-n limit of *diagonal* correlations

$$M := \sqrt{\lim_{n\to\infty} \langle \sigma_{0,0} \sigma_{n,n} \rangle}. \tag{4.1.5}$$

It is famously known that, unlike the one-dimensional case, the two-dimensional Ising model exhibits a phase transition in the spontaneous magnetization at some temperature T_c (see [40, 43, 44]), characterized by

$$\sinh \frac{2J_h}{k_B T_c} \sinh \frac{2J_v}{k_B T_c} = 1. \tag{4.1.6}$$

Remarkably, for the diagonal correlations $\langle \sigma_{0,0} \sigma_{N,N} \rangle$ and the horizontal correlations $\langle \sigma_{0,0} \sigma_{0,N} \rangle$, one has Toeplitz determinant representations, indeed for the diagonal correlations we have

$$\langle \sigma_{0,0} \sigma_{N,N} \rangle = \det(\phi_{i-j})_{1\le i,j\le N} =: D_N[\phi], \tag{4.1.7}$$

where the symbol $\phi(z, k)$ is given by

$$\phi(z; k) = \sqrt{\frac{1 - k^{-1} z^{-1}}{1 - k^{-1} z}}, \qquad k = \sinh \frac{2J_h}{k_B T} \sinh \frac{2J_v}{k_B T}, \tag{4.1.8}$$

and ϕ_j are the Fourier coefficients of ϕ

$$\phi_j = \int_{\mathbb{T}} z^{-j} \phi(z; k) \frac{dz}{2\pi i z}, \qquad n \in \mathbb{Z}. \tag{4.1.9}$$

The Toeplitz determinant representation for the horizontal correlations is given by

$$\langle \sigma_{0,0}\sigma_{0,N}\rangle = \det(\psi_{i-j})_{1\le i,j\le N}, \qquad \psi(z) = \sqrt{\frac{(1-\alpha_1 z)(1-\alpha_2 z^{-1})}{(1-\alpha_1 z^{-1})(1-\alpha_2 z)}}, \tag{4.1.10}$$

where α_1 and α_2 are given by

$$\alpha_1 = \frac{z_h(1-z_v)}{1+z_v}, \quad \alpha_2 = \frac{1-z_v}{z_h(1+z_v)}, \qquad z_{h,v} = \tanh\frac{J_{h,v}}{k_B T}, \tag{4.1.11}$$

and ψ_j's are again Fourier coefficients of ψ.

The determinantal representation (4.1.7) allows one to compute the spontaneous magnetization M. In the low-temperature regime, this is achieved via the Strong Szegő limit theorem (SSLT) for Toeplitz determinants, which was originally proved by Szegő in 1952 for positive and $C^{1+\varepsilon}$ symbols, $\varepsilon > 0$, on the unit circle(see [45] and [46]). Following that achievement many mathematicians tried to prove SSLT for a more general class of symbols and we refer the interested reader to [40] for a comprehensive review of such developments.

Provided that $\phi(z)$ is non-zero and continuous on the unit circle and has zero winding number, then $\phi(z)$ can be written as $e^{V(z)}$ for the continuous and periodic function $V(z) = \log\phi(z)$. Here we state the strongest version of SSLT due to K. Johansson [47].

Theorem 4.1.1 (Johansson) *Let $V(e^{i\theta}) \in L^1(S^1)$ be a (possibly complex-valued) function on S^1 with Fourier coefficients $\{V_k\}_{k\in\mathbb{Z}}$ satisfying*

$$\sum_{k=-\infty}^{\infty} k|V_k|^2 < \infty, \tag{4.1.12}$$

then

$$\lim_{n\to\infty}\frac{D_n(e^{V(e^{i\theta})})}{e^{nV_0}} = e^{\sum_{k=-\infty}^{\infty} kV_kV_{-k}}. \tag{4.1.13}$$

In the temperature regime $T < T_c$, the symbols (4.1.8) and (4.1.10) enjoy the regularity properties required by the SSLT. Applying the theorem (see [48], chapter 10) to (4.1.8) and (4.1.10) separately one finds that the spontaneous magnetization can be found also from the horizontal correlations as well, and is given by

$$M = \langle \sigma_{0,0}\sigma_{N,N}\rangle = \langle \sigma_{0,0}\sigma_{0,N}\rangle = (1-k^{-2})^{1/8}, \qquad T < T_c. \tag{4.1.14}$$

For $T = T_c$ and $T > T_c$, the symbol (4.1.8) possesses Fisher-Hartwig singularities at $z = 1$ (see [49] and [40]), and the asymptotic analysis of the corresponding Toeplitz determinant can not be obtained via SSLT. We will discuss these temperature regimes later in this chapter. There have also been efforts to study correlation functions in the directions other that those discussed above. For instance, in the isotropic case ($J_h = J_v$), the expressions for the correlation functions $\langle \sigma_{0,0}\sigma_{m,n}\rangle$ with $(m,n) \neq (0,n), (m,0), (n,n)$, were explicitly derived by Shrock and Ghosh in [50]. In particular, via the Pfaffian method they found expressions in the cases $(m,n) = (2,1), (3,1), (3,2), (4,1), (4,2)$, and $(4,3)$, in terms of complete elliptic integrals K and E. Furthermore, They inferred a general structural formula for arbitrary $\langle \sigma_{0,0}\sigma_{m,n}\rangle$ in terms of these elliptic integrals K and E.

4.2 determinantal formula for next-to-diagonal correlations

In 1987 Au-Yang and Perk expressed the spin-spin next-to-diagonal correlations of the anisotropic square lattice Ising model in terms of the following *bordered* Toeplitz determinant (see [41], [42])

$$\langle \sigma_{0,0}\sigma_{N,N-1}\rangle = \det \begin{pmatrix} a_0 & \cdots & a_{N-2} & b_{N-1} \\ a_{-1} & \cdots & a_{N-3} & b_{N-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1-N} & \cdots & a_{-1} & b_0 \end{pmatrix}, \qquad N \geq 1, \tag{4.2.1}$$

where

$$a_j = \int_{\mathbb{T}} z^j \phi_{\mathtt{YP}}(z;k)\frac{dz}{2\pi i z}, \qquad \text{and,} \qquad b_j = C_h \int_{\mathbb{T}} \frac{z^j}{S_h + S_v z}\phi_{\mathtt{YP}}(z;k)\frac{dz}{2\pi i z}. \tag{4.2.2}$$

Here the symbol $\phi_{\mathtt{YP}}$ is defined as

$$\phi_{\mathtt{YP}}(z;k) = \sqrt{\frac{1-k^{-1}z}{1-k^{-1}z^{-1}}} = \phi(z^{-1};k), \tag{4.2.3}$$

and the parameters S_h, S_v and C_h are given by

$$S_h = \sinh\frac{2J_h}{k_B T}, \qquad S_v = \sinh\frac{2J_v}{k_B T}, \qquad C_h = \cosh\frac{2J_h}{k_B T}, \tag{4.2.4}$$

and thus $k = S_h S_v$. The determinantal representation (4.2.1) was later used by Witte [51] to express $\langle \sigma_{0,0}\sigma_{n,n-1}\rangle$ as a solution to an isomonodromic problem associated with the particular Painleve' VI system, which charactrises the diagonal correlation functions.

In [51], N.Witte expressed the bordered Toeplitz determinant (4.2.1) in terms of the function $\varepsilon_n^*(z)$ (see equations (34) and (59) in [51]) which is the Cauchy-Hilbert transform of the reciprocal polynomial associated to one of the bi-orthogonal polynomials (for example, see section 2 of [52]). More precisely, he found that

$$\langle \sigma_{0,0}\sigma_{N,N-1}\rangle = \frac{C_h}{2S_h\kappa_{N-1}} D_{N-1}[\phi]\varepsilon_{N-1}^*(z^*;k), \qquad N \geq 1, \tag{4.2.5}$$

where

$$z^* = -\frac{S_h}{S_v}, \tag{4.2.6}$$

and

$$\varepsilon_n^*(z) := \frac{1}{\kappa_n} - \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\phi(\zeta)Q_n^*(\zeta)\frac{d\zeta}{2\pi i\zeta}, \qquad Q_n^*(z) := z^n\widehat{Q}_n(\frac{1}{z}),$$

where Q and $\widehat{Q}$ are given by (1.2.2) and (1.2.3), respectively, and form the bi-orthogonal system of polynomials on the unit circle with respect to the weight ϕ given by (4.1.8). Prior to the latter work, In 2006, P.Forrester and N.Witte in [52] introduced a Riemann-Hilbert problem for the bi-orthogonal polynomials on the unit circle and were able write down the representation of a solution to that Riemann-Hilbert problem in terms of one of the bi-orthogonal polynomials, its reciprocal polynomial and their respective Cauchy- Hilbert transforms; However there was no attempt to asymptotically solve that Riemann-Hilbert problem in [52]. We observed that by an explicit transformation the Riemann-Hilbert problem in [52] can be mapped to the well-established Riemann-Hilbert problem found by J.Baik, P.Deift and K.Johansson in [8] for Toeplitz determinants which is studied in great detail in [9].[1] Using this explicit relationship one can readily find the asymptotics of $\varepsilon_n^*(z)$ and hence asymptotics of the bordered Toeplitz determianant by the formula given in [51]. However, a more convenient way seems to be relating the bordered Toeplitz determinant (4.2.1) directly to the

[1]This Riemann-Hilbert problem will be referred to as the Y-RHP in the rest of this chapter. For the convenience of the reader, We have provided its asymptotic solution (corresponding to the weight (4.1.8) when $T < T_c$) in appendix B.

12-element of the representation of the solution to the X-RHP (see section 1.2.1). The next proposition, due to N.Witte, is the needed first step in making the desired connection.

Proposition 4.2.1 *For all values of T, the next-to-diagonal correlations of the anisotropic Ising model on the square lattice is given by*

$$\langle\sigma_{0,0}\sigma_{N,N-1}\rangle = \frac{S_v}{S_h} D_{N-1}[\phi]\delta_{N-1}(z^*;k), \qquad N \geq 1, \tag{4.2.7}$$

where

$$\delta_n(z;k) := -\frac{C_h z}{S_v \kappa_n}\int_{\mathbb{T}} \frac{\zeta^n Q_n(\zeta^{-1})}{\zeta - z}\phi_{\text{YP}}(\zeta;k)\frac{d\zeta}{2\pi i\zeta}. \tag{4.2.8}$$

Proof Note that from (1.2.2) we can write $z^n Q_n(z^{-1})$ as

$$z^n Q_n(z^{-1}) = \frac{1}{\sqrt{D_n[\phi]D_{n+1}[\phi]}} \det \begin{pmatrix} \phi_0 & \phi_{-1} & \cdots & \phi_{-n} \\ \phi_1 & \phi_0 & \cdots & \phi_{-n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n-1} & \phi_{n-2} & \cdots & \phi_{-1} \\ z^n & z^{n-1} & \cdots & 1 \end{pmatrix}. \tag{4.2.9}$$

Now, recalling that $\kappa_n = \sqrt{D_n[\phi]/D_{n+1}[\phi]}$, we can write

$$\delta_n(z;k) = -\frac{z}{D_n[\phi]} \det \begin{pmatrix} \phi_0 & \phi_{-1} & \cdots & \phi_{-n} \\ \phi_1 & \phi_0 & \cdots & \phi_{-n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n-1} & \phi_{n-2} & \cdots & \phi_{-1} \\ \frac{C_h}{S_v}\int_{\mathbb{T}} \frac{\zeta^n \phi_{\text{YP}}(\zeta;k)}{\zeta - z}\frac{d\zeta}{2\pi i\zeta} & \frac{C_h}{S_v}\int_{\mathbb{T}} \frac{\zeta^{n-1} \phi_{\text{YP}}(\zeta;k)}{\zeta - z}\frac{d\zeta}{2\pi i\zeta} & \cdots & \frac{C_h}{S_v}\int_{\mathbb{T}} \frac{\phi_{\text{YP}}(\zeta;k)}{\zeta - z}\frac{d\zeta}{2\pi i\zeta} \end{pmatrix}. \tag{4.2.10}$$

Note that (4.2.3) implies that $a_j = \phi_j$. Thus we can express (4.2.1) as

$$\langle \sigma_{0,0}\sigma_{N,N-1}\rangle = \det \begin{pmatrix} \phi_0 & \phi_{-1} & \cdots & \phi_{-N+1} \\ \phi_1 & \phi_0 & \cdots & \phi_{-N+2} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{N-2} & \phi_{N-3} & \cdots & \phi_{-1} \\ b_{N-1} & b_{N-2} & \cdots & b_0 \end{pmatrix}, \qquad N \geq 1. \tag{4.2.11}$$

Now formula (4.2.7) follows from combining the formulas (4.2.2), (4.2.10) and (4.2.11). ■

The above proposition is basically an adaptation of proposition 3 of [51]. Our preference to express the next-to-diagonal correlations in terms of δ_n as opposed to ε_n^* is rooted in the fact that the former is, up to a constant, the evaluation at z^{-1} of the 12-entry of the representation for the solution to the X-RHP in terms of the associated orthogonal polynomials. The following lemma establishes this connection.

Lemma 4.2.1 *The function $\delta_n(z;k)$ is encoded in the (unique) solution of the Riemann-Hilbert problem associated with $D_n[\phi]$ through*

$$\delta_n(z;k) = \frac{C_h}{S_v} X_{12}(z^{-1};n). \tag{4.2.12}$$

Proof Indeed

$$\delta_n(z;k) = -\frac{C_h z}{S_v \kappa_n}\int_{\mathbb{T}} \frac{\zeta^{-n}Q_n(\zeta)}{\zeta^{-1}-z}\phi_{\text{YP}}(\zeta^{-1};k)\frac{d\zeta}{2\pi i\zeta} = -\frac{C_h z}{S_v \kappa_n}\int_{\mathbb{T}} \frac{\zeta^{-n+1}Q_n(\zeta)}{1-z\zeta}\phi(\zeta;k)\frac{d\zeta}{2\pi i\zeta}$$

$$= \frac{C_h}{S_v\kappa_n}\int_{\mathbb{T}}\frac{Q_n(\zeta)}{\zeta - z^{-1}}\phi(\zeta;k)\frac{d\zeta}{2\pi i \zeta^n} = \frac{C_h}{S_v}X_{12}(z^{-1};n),$$

where we have used (4.2.3) and (1.2.5).

■

Now we combine (4.2.7) with (4.2.12) to get

$$\langle \sigma_{0,0}\sigma_{N,N-1}\rangle = \frac{C_h}{S_h} D_{N-1}[\phi] X_{12}(\frac{1}{z^*};N-1), \qquad N \geq 1. \tag{4.2.13}$$

Note that in the anisotropic case for z^* given by (4.2.6), we have the following exclusive possibilities

$$\begin{cases} z^* < -1, & J_h > J_v, \\ -1 < z^* < 0, & J_h < J_v, \end{cases} \tag{4.2.14}$$

since for a fixed temperature, $J_h > J_v$ is equivalent to $S_h > S_v$ and vice versa. This has an important message, that the anisotropy plays the crucial role in the final asymptotic formula for $\langle \sigma_{0,0}\sigma_{N,N-1}\rangle$ as the expression for $X_{12}(z;n)$ is quite different when $|z| < 1$ and $|z| > 1$. Since we know the large-n asymptotics of $X(z;n)$ (see appendix B), we can derive an asymptotic expression for the next-to-diagonal correlations through the connection formula (4.2.13).

Corollary 4.2.1.1 *For $T < T_c$, we have the following asymptotics for $\langle \sigma_{0,0}\sigma_{N,N-1}\rangle$*

$$\langle \sigma_{0,0}\sigma_{N,N-1}\rangle = \begin{cases} \dfrac{C_h}{S_h}(1-k^{-2})^{1/4}\alpha(\dfrac{1}{z^*})(1+o(1)), & J_h > J_v, \\ \dfrac{C_h}{S_h\alpha(\frac{1}{z^*})}(1-k^{-2})^{1/4}R_{1,12}(\dfrac{1}{z^*};N-1)(z^*)^{N-1}(1+o(1)), & J_h < J_v, \end{cases} \quad N \to \infty, \tag{4.2.15}$$

where $\alpha(z)$ and $R_{1,12}(z;n)$ are given by (B.0.7) *and* (B.0.14)*, respectively.*

Proof Suggested by (4.2.13), to find the asymptotics of the next-to-diagonal correlations we need the asymptotics of both $D_n[\phi]$ and the solution of the X-RHP. When $T < T_c$ the weight (4.1.8) has no Fisher-Hartwig singularities and is analytic in a neighborhood of the unit circle. In appendix B we have outlined how the solution of the X-RHP can be found in this case. We will also use the well-known result (e.g. see [40, 48])

$$D_N[\phi] = (1-k^{-2})^{1/4}(1+o(1)), \qquad N \to \infty. \tag{4.2.16}$$

Recall that when $J_h > J_v$, we have $|z^*| > 1$ and hence from (B.0.16) we have

$$X_{12}(\frac{1}{z^*};N-1) = \alpha(\frac{1}{z^*})[1+\mathcal{O}(e^{-2c(N-1)})]. \tag{4.2.17}$$

Combining (4.2.13) and (4.2.16) with (4.2.17) yields the first member of (4.2.15). On the other hand if $J_h < J_v$, we have $|z^*| < 1$, hence from (B.0.16) we obtain

$$X_{12}(\frac{1}{z^*};N-1) = R_{1,12}(\frac{1}{z^*};N-1)\frac{(\frac{1}{z^*})^{-(N-1)}}{\alpha(\frac{1}{z^*})}[1+\mathcal{O}(e^{-2c(N-1)})]$$
$$= R_{1,12}(\frac{1}{z^*};N-1)\frac{(z^*)^{N-1}}{\alpha(\frac{1}{z^*})}[1+\mathcal{O}(e^{-2c(N-1)})]. \tag{4.2.18}$$

This equation combined with (4.2.13) and (4.2.16) gives the second member of (4.2.15). ■

So if the vertical coupling dominates the horizontal coupling, the next-to-diagonal correlations decay exponentially fast as $N \to \infty$ and conversely if the vertical coupling is dominated by the horizontal coupling, the next-to-diagonal correlations tend to a constant as $N \to \infty$.

Also it is important to remember that the original bordered Toeplitz determinant representation (4.2.1) of Yang and Perk for the next-to-diagonal correlations is only valid for the anisotropic case ($J_h \neq J_v$) and consequently we are not making any claims about the asymptotics of the next-to-diagonal correlations in the isotropic case $J_h = J_v$.

Remark 4.2.2 *As it is clear from (B.0.16), it is immaterial whether $z^* \in \Omega_2$ or $z^* \in \Omega_\infty$ in the case $J_h > J_v$; and similarly in the case $J_h < J_v$ it is immaterial whether $z^* \in \Omega_1$ or $z^* \in \Omega_0$; to put it differently the freedom in opening of the lenses , i.e. the locations of Γ_0 and Γ_1, expectedly do not affect the asymptotics of the next-to-diagonal correlations.*

Remark 4.2.3 *One can also prove the analogue of the result in Corollary 4.2.1.1 for the cases $T = T_c$ and $T > T_c$. Indeed when $T = T_c$, there is a FH-singularity at $z = 1$ with parameters $\alpha = 0$ and $\beta = -\frac{1}{2}$. In this case both ingredients of the formula* (4.2.13)*, i.e. the asymptotics of the determinant and the asymptotic solution of the corresponding Riemann-Hilbert problem are known(see respectively [49] and [9]). However the analysis of the case $T > T_c$ will be more involved as the FH parameters in this case are $\alpha = 0$ and $\beta = -1$. This is an instance of degenerate FH singularity ($\alpha_j \pm \beta_j \in \mathbb{Z}_-$). Note that the X-RHP has not been solved in the degenerate case in the pioneering work [9]. Nevertheless, the asymptotics of the corresponding Toeplitz determinant can be found from lemma 2.4 of [9]*[2]*.*

[2]specifically equation 2.12 which relates $D_n(z^{-1}f(z))$ to $D_n(f(z))$.

5. EMPTINESS FORMATION PROBABILITY IN THE XXZ-SPIN 1/2 HEISENBERG CHAIN

Abstract. In this section we will derive an asymptotic formula for a Fredholm determinant of interest in studying the Emptiness formation probability in the XXZ-spin 1/2 Heisenberg chain. This Fredholm determinant corresponds to an integral operator with a *generalized* Sine kernel. Apart from deriving the relevant differential identities, it turns out that the Riemann-Hilbert approach to this generalized Sine kernel is very similar to the regular Sine kernel studied in [53]. This is a joint work with K.Kozlowski and A.Its.

5.1 Introduction

In this section we will consider one special example of the generalized sine kernel which has a particular importance in the theory of the XXZ spin-1/2 Heisinberg chain. Let $V \equiv V_m$ be the trace class integral operator acting on $L^2(\Gamma_\alpha)$, where Γ_α is the arc,

$$|\lambda| = 1, \quad -\alpha < \arg\lambda < \alpha \quad 0 < \alpha < \pi,$$

depicted in Figure 5.1. traversed counterclockwise, with the kernel is given by the equations,

$$V(\lambda,\mu) = \frac{1}{2\pi i(\lambda-\mu)}\left(\lambda^{m/2}\mu^{-m/2}e^{t(\frac{\psi(\lambda)-\psi(\mu)}{2})} - \lambda^{-m/2}\mu^{m/2}e^{t(\frac{\psi(\lambda)-\psi(\mu)}{2})}\right), \tag{5.1.1}$$

where, as before, m is a positive integer, t is the real parameter, and the function $\psi(\lambda)$ is assumed to be analytic in a neighborhood of the arc Γ_α. In the following we explain the connection of this kernel to the XXZ spin-1/2 Heisinberg chain. The XXZ spin-1/2 Heisenberg chain of size N is determined by the Hamiltonian,

$$H_{XXZ} = \sum_{n=1}^{N}\left(\sigma_n^x\sigma_{n+1}^x + \sigma_n^y\sigma_{n+1}^y + \Delta\left(\sigma_n^z\sigma_{n+1}^z - 1\right) - h\sigma_n^z\right), \tag{5.1.2}$$

where the periodic boundary conditions are assumed. In (5.1.2), σ^x, σ^y, σ^z are Pauli matrices, h, $0 \le h < 2$, is an external (moderate) magnetic field, and Δ is the anisotropy parameter, $-1 < \Delta < 1$. At the point $\Delta = 0$, the model becomes the free fermionoic XX0 spin chain.

One of the principal objects of the analysis of the XXZ model is the *emptiness formation probability* (EFP) which is defined at zero temperature as the correlation function,

$$P^{(N)}(m) = \langle 0| \prod_{j=1}^{m} \frac{\sigma_j^z + 1}{2} |0\rangle. \tag{5.1.3}$$

The physical meaning of $P^{(N)}(m)$ is the probability of finding a string of m adjacent parallel spins up (i.e., a piece of the *ferromagnetic* state) in the antiferromagnetic ground state $|0\rangle$ for a given value of the magnetic field h. We shall denote,

$$P(m) := \lim P^{(N)}(m), \quad N \to \infty, \tag{5.1.4}$$

the emptiness formation probability in the thermodynamical limit. The principal analytical question is the large m behavior of $P(m)$.

At the free fermonic case, when $\Delta = 0$, the EFP is given by the explicit determinant formula involving the integral operator (5.1.1). Indeed, one has that

$$P(m) = \det\Big(1 - V_m\Big)\Big|_{t=0}. \tag{5.1.5}$$

The Fredholm determinant in the right hand side of this formula is simultaneously the Toiplitz determinant whose symbol is the characteristic function of the complimentary arc, $C \setminus \Gamma_\alpha$. The large m asymptotic of this determinant was obtained in the classical work by Widom [54] and it reads (see also [55], [53] for the error estimate),

$$P(m) = m^2 \ln \cos \frac{\alpha}{2} - \frac{1}{4} \ln\left(m \sin \frac{\alpha}{2}\right) + c_0 + \mathcal{O}(m^{-1}), \quad m \to \infty \tag{5.1.6}$$

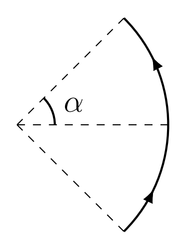

Figure 5.1. The contour Γ_α

where the constant c_0 is the famous Widom's constant,

$$c_0 = \frac{1}{12}\ln 2 + 3\zeta'(-1). \tag{5.1.7}$$

There exists the Fredholm determinant representations for the EFP in the general XXZ case as well. A remarkable fact is that this representation also involves the operator V_m but this time with $t \neq 0$. The exact formula relating $P(m)$ and V_m for $\Delta \neq 0$ was extracted by N. Slavnov from the analysis performed in the series of papers which he wrote together with Kitanine, Koslowski, Maillet and Terras devoted to the two-point correlation function in the XXZ model (see [KKMST] and references therein). The function $\psi(\lambda)$ in Slavnov's formula, however, is not a scalar function, but is in fact a *dual quantum field* acting in an auxiliary bosonic Fock space with vacuum $|0\rangle$. Indeed, Slavnov's representation is the following.

$$P(m) = \langle 0|C(\phi)\cdot\frac{\det\Big(1 - V_m\Big)}{\det(1 + \frac{1}{2\pi}K)}|0\rangle, \tag{5.1.8}$$

where the integral operator K and the quantity $C(\phi)$, which is also is expressed in terms of certain Fredholm determinants, do not depend on m. The constant $C(\phi)$ as well as the kernel $V_n(z, z')$ depend on the dual field $\phi(\lambda)$ and $\psi(\lambda)$. The dual fields commute for all values of spectral parameter z. Their contribution to the expectation value (5.1.8) is obtained through the averaging procedure which suggests the decomposition of the dual fields on the relevant creation and annihilation parts and then moving all exponentials of annihilation operators to the right, picking up contributions whenever passing by a creation operator.

The general strategy of using Slavnov's formula (5.1.8) can be formulated as the following two step procedure. First, treating $\psi(\lambda)$ as the usual function find the large m asymptotics of $\det\Big(1 - V_m\Big)$. The second step would be then an averaging of the asymptotic formulae obtained in the first step over the dual fields. [1]. In this work we will pass through the first step. Our main result is given in the following theorem:

[1]This strategy had already been used in the two point correlation function in the case of the 1D Bose gas at the final coupling [56] - another fundamental non-free fermion model .

Theorem 5.1.1 *The large-m asymptotics of the Fredholm determinant associated to the kernel* (5.1.1) *is given by*

$$\begin{aligned}\ln\det(1-V) &= m^2\ln\cos\frac{\alpha}{2} + m\frac{t}{2\pi i}\int_{\mathcal{L}}\psi(\lambda(z))\partial_z\ln g(z)dz\\ &-\frac{1}{4}\ln\left(m\sin\frac{\alpha}{2}\right) - \frac{t^2}{4\pi i}\int_{\mathcal{L}}\psi(\lambda(z))\partial_z\eta(z)dz + c_0 + \mathcal{O}(m^{-1}), \quad m\to\infty\end{aligned} \tag{5.1.9}$$

where $\mathcal{L}$ is a closed loop around the interval $[-1,1]$, c_0 is the Widom-Dyson constant given by (5.1.7)*, and the functions g and η are given by*

$$g(z) = \frac{1+i\sqrt{z^2-1}\sin(\alpha/2)}{1+iz\tan(\alpha/2)}, \qquad \eta(z) = -\frac{\sqrt{z^2-1}}{2\pi}\int_{-1}^{1}\frac{\psi(\lambda(\zeta))}{\sqrt{1-\zeta^2}(\zeta-z)}d\zeta. \tag{5.1.10}$$

Note that, the kernel (5.1.1) is very close to the integrable kernel studied in [53]. Indeed, the latter is the particular case of the former corresponding $\psi(\lambda)\equiv 0$. Moreover, as we will see below, most of the results and the constructions of [53], after some minimal modifications, can be used in the generalized case $\psi(\lambda)\not\equiv 0$. This observation allows us to simplify greatly the evaluation of the large m asymptotics of the $\det(1-V)$ with kernel (5.1.1). Basically the only analytical ingredient which is needed, in addition to a modification of the results in [53], is the relevant differential identity for $\det(1-V)$ in the generalized case.

5.1.1 The $\mathfrak{Y}$ - RH problem

The kernel (5.1.1) is of integrable type(see (1.1.7)). Precisely,

$$V(\lambda,\mu) = \frac{e_+(\lambda)e_-(\mu) - e_+(\mu)e_-(\lambda)}{2\pi i(\lambda-\mu)} \equiv \frac{f^T(\lambda)h(\mu)}{\lambda-\mu}, \tag{5.1.11}$$

where

$$e_\pm(\lambda) = \lambda^{\pm\frac{m}{2}}e^{\frac{\pm t\psi(\lambda)}{2}}, \qquad f(\lambda) = \sqrt{\frac{1}{2\pi i}}\begin{pmatrix} e_+(\lambda)\\ e_-(\lambda)\end{pmatrix}, \qquad \text{and} \qquad h(\lambda) = \sqrt{\frac{1}{2\pi i}}\begin{pmatrix} e_-(\lambda)\\ -e_+(\lambda)\end{pmatrix}. \tag{5.1.12}$$

Therefore the arguments of Section 1.2.3 are applicable and we can associate with this kernel the Riemann-Hilbert problem which consists in finding the 2×2 matrix valued function $\mathfrak{Y}$ satisfying the following properties.

- **RH-$\mathfrak{Y}$1** $\mathfrak{Y}$ is holomorphic in $\mathbb{C} \setminus \Gamma_\alpha$, and it has continuous boundary values $\mathfrak{Y}_\pm(\lambda)$ in $\Gamma_\alpha \setminus \{e^{\pm i\alpha}\}$.

- **RH-$\mathfrak{Y}$2** $\mathfrak{Y}_+(\lambda) = \mathfrak{Y}_-(\lambda) J_{\mathfrak{Y}}(\lambda)$, $\lambda \in \Gamma_\alpha \setminus \{e^{\pm i\alpha}\}$, where

$$J_{\mathfrak{Y}}(\lambda) = I - 2\pi i f(\lambda) h^T(\lambda) = \begin{pmatrix} 0 & \lambda^m e^{t\psi(\lambda)} \\ -\lambda^{-m} e^{-t\psi(\lambda)} & 2 \end{pmatrix}. \tag{5.1.13}$$

- **RH-$\mathfrak{Y}$3** $\mathfrak{Y}(\lambda) = O\Big(\log|\lambda - e^{\pm i\alpha}|\Big)$, as $\lambda \to e^{\pm i\alpha}$.

- **RH-$\mathfrak{Y}$4** $\mathfrak{Y}(\infty) = I$.

As shown in Section 1.2.3 the unique solution of this Riemann-Hilbert problem, which we will from now on call $\mathfrak{Y}$ - RH problem, admits the Cauchy representation (1.2.29), also

$$\mathfrak{Y}^{-1}(z) = I + \int_{\Gamma_\alpha} \frac{f(\mu) H^T(\mu)}{\mu - z} d\mu. \tag{5.1.14}$$

Conversely, the vector functions F and H from (1.2.28) are given in terms of $\mathfrak{Y}$ by the equations (1.2.35) and (1.2.36).

5.2 The differential identity for the Fredholm determinant

In this section we will prove the following proposition.

Proposition 5.2.1 *For the Fredholm determinant associated to* (5.1.1)*, we have the following differential identity*

$$\frac{d}{dt} \ln\det(1 - V) = -\frac{1}{4\pi i} \int_{\mathcal{C}} \psi(\lambda)\, \mathit{Trace}\left(\mathfrak{Y}(\lambda)\sigma_3 \partial_\lambda (\mathfrak{Y}^{-1}(\lambda))\right) d\lambda, \tag{5.2.1}$$

where, $\mathcal{C}$ is a small counterclockwise loop around the arc Γ_α.

Proof The starting point of the proof is the general formula $\ln\det(I - V) = \mathrm{Trace}(\ln(I - V))$, which yields

$$\frac{d}{dt} \ln\det(1 - V) = -\mathrm{Trace}(1 - V)^{-1} \frac{dV}{dt}. \tag{5.2.2}$$

In our case,

$$\frac{d}{dt}V(\lambda,\mu) = \frac{1}{4\pi i}\frac{\psi(\lambda)-\psi(\mu)}{\lambda-\mu}\left(\left(\frac{\lambda}{\mu}\right)^{m/2}\exp\left(t\frac{\psi(\lambda)-\psi(\mu}{2}\right)-\left(\frac{\mu}{\lambda}\right)^{m/2}\exp\left(t\frac{\psi(\mu)-\psi(\lambda)}{2}\right)\right),$$

which can be rewritten as,

$$\frac{dV}{dt}(\lambda,\mu) = \frac{1}{2}\frac{\psi(\lambda)-\psi(\mu)}{\lambda-\mu}\left(f_1(\lambda)h_1(\mu)-f_2(\lambda)h_2(\mu)\right). \tag{5.2.3}$$

Using the identity $(1-V)^{-1} = 1+R$, it can be shown that the kernel of the integral operator $(1-V)^{-1}\frac{dV}{dt}$ is equal to

$$\int_{\Gamma_\alpha}(1+R)(\lambda,\nu)\frac{dV}{dt}(\nu,\mu)d\nu.$$

Since the trace of an integral operator with kernel $K(\lambda,\mu)$ is, by definition, equal to $\int_\Gamma K(\lambda,\lambda)d\lambda$, we have

$$\begin{aligned}\mathrm{Trace}(1-V)^{-1}\frac{dV}{dt} &= \int_{\Gamma_\alpha}\int_{\Gamma_\alpha}(1+R)(\mu,\lambda)\frac{dV}{dt}(\lambda,\mu)d\lambda d\mu\\ &= \frac{1}{2}\int_{\Gamma_\alpha}\int_{\Gamma_\alpha}(I+R)(\mu,\lambda)\frac{\psi(\lambda)-\psi(\mu)}{\lambda-\mu}\left(f_1(\lambda)h_1(\mu)-f_2(\lambda)h_2(\mu)\right)d\lambda d\mu\\ &= I_0+\frac{1}{2}\int_{\Gamma_\alpha}\int_{\Gamma_\alpha}R(\mu,\lambda)\frac{\psi(\lambda)-\psi(\mu)}{\lambda-\mu}\left(f_1(\lambda)h_1(\mu)-f_2(\lambda)h_2(\mu)\right)d\lambda d\mu,\end{aligned} \tag{5.2.4}$$

where,

$$I_0 := \frac{1}{2}\int_{\Gamma_\alpha}\psi'(\lambda)\left(f_1(\lambda)h_1(\lambda)-f_2(\lambda)h_2(\lambda)\right)d\lambda. \tag{5.2.5}$$

Taking into account the identity,

$$\frac{\psi(\lambda)-\psi(\mu)}{\lambda-\mu} = \frac{1}{2\pi i}\int_{\mathcal{C}}\frac{\psi(s)}{(s-\lambda)(s-\mu)}ds,$$

where $\mathcal{C}$ is a small counterclockwise closed loop around Γ_α, we transform (5.2.4) into the following equation

$$\mathrm{Trace}(1-V)^{-1}\frac{dV}{dt} = I_0+\frac{1}{4\pi i}\int_{\mathcal{C}}\psi(s)\int_{\Gamma_\alpha}\int_{\Gamma_\alpha}R(\mu,\lambda)\frac{1}{(s-\lambda)(s-\mu)}\left(f_1(\lambda)h_1(\mu)-f_2(\lambda)h_2(\mu)\right)d\lambda d\mu ds$$

$$= I_0+\frac{1}{4\pi i}\int_{\mathcal{C}}\psi(s)\int_{\Gamma_\alpha}\int_{\Gamma_\alpha}\frac{F^T(\mu)H(\lambda)}{(s-\lambda)(s-\mu)(\mu-\lambda)}\left(f_1(\lambda)h_1(\mu)-f_2(\lambda)h_2(\mu)\right)d\lambda d\mu ds \tag{5.2.6}$$

By a direct calculation one can check that

$$F^T(\mu)H(\lambda)\left(f_1(\lambda)h_1(\mu)-f_2(\lambda)h_2(\mu)\right)=\text{Trace}\left(F(\mu)h^T(\mu)\sigma_3 f(\lambda)H^T(\lambda)\right). \tag{5.2.7}$$

Hence, (5.2.6) can be further transformed as follows

$$\begin{aligned}
\text{Trace}(1-V)^{-1}\frac{dV}{dt} &= I_0+\frac{1}{4\pi i}\int_{\mathcal{C}}\psi(s)\int_{\Gamma_\alpha}\int_{\Gamma_\alpha}\frac{\text{Trace}\left(F(\mu)h^T(\mu)\sigma_3 f(\lambda)H^T(\lambda)\right)}{(s-\lambda)(s-\mu)(\mu-\lambda)}d\lambda d\mu ds\\
&= I_0+\frac{1}{4\pi i}\int_{\mathcal{C}}\psi(s)\int_{\Gamma_\alpha}\int_{\Gamma_\alpha}\frac{\text{Trace}\left(F(\mu)h^T(\mu)\sigma_3 f(\lambda)H^T(\lambda)\right)}{(s-\lambda)^2}\left(\frac{1}{s-\mu}-\frac{1}{\lambda-\mu}\right)d\lambda d\mu ds\\
&= I_0+\frac{1}{4\pi i}\int_{\mathcal{C}}\psi(s)\int_{\Gamma_\alpha}\int_{\Gamma_\alpha}\frac{\text{Trace}\left(F(\mu)h^T(\mu)\sigma_3 f(\lambda)H^T(\lambda)\right)}{(s-\lambda)^2(s-\mu)}d\lambda d\mu ds\\
&\quad -\frac{1}{4\pi i}\int_{\mathcal{C}}\psi(s)\int_{\Gamma_\alpha}\int_{\Gamma_\alpha}\frac{\text{Trace}\left(F(\mu)h^T(\mu)\sigma_3 f(\lambda)H^T(\lambda)\right)}{(s-\lambda)^2(\lambda-\mu)}d\lambda d\mu ds.
\end{aligned} \tag{5.2.8}$$

Using (1.2.29) we can rewrite formula (5.2.8) in terms of double integrals,

$$\begin{aligned}
\text{Trace}(1-V)^{-1}\frac{dV}{dt} &= I_0+\frac{1}{4\pi i}\int_{\mathcal{C}}\psi(s)\int_{\Gamma_\alpha}\frac{\text{Trace}\left((\mathfrak{Y}(s)-I)\sigma_3 f(\lambda)H^T(\lambda)\right)}{(s-\lambda)^2}d\lambda ds\\
&\quad -\frac{1}{4\pi i}\int_{\mathcal{C}}\psi(s)\int_{\Gamma_\alpha}\frac{\text{Trace}\left(\mathfrak{Y}(\lambda)-I)\sigma_3 f(\lambda)H^T(\lambda)\right)}{(s-\lambda)^2}d\lambda ds.
\end{aligned} \tag{5.2.9}$$

Furthermore, from (5.1.14) it follows that

$$\partial_s(\mathfrak{Y}^{-1}(s))=\int_{\Gamma_\alpha}\frac{f(\lambda)H^T(\lambda)}{(\lambda-s)^2}dz. \tag{5.2.10}$$

Therefore, we can simplify (5.2.9) as

$$\begin{aligned}
&\text{Trace}(1-V)^{-1}\frac{dV}{dt} = I_0+\frac{1}{4\pi i}\int_{\mathcal{C}}\psi(s)\text{Trace}\left((\mathfrak{Y}(s)-I)\sigma_3\partial_s(\mathfrak{Y}^{-1}(s))\right)ds\\
&+\frac{1}{4\pi i}\int_{\mathcal{C}}\psi(s)\text{Trace}\left(\sigma_3\partial_s(\mathfrak{Y}^{-1}(s))\right)ds-\frac{1}{4\pi i}\int_{\mathcal{C}}\psi(s)\int_{\Gamma_\alpha}\frac{\text{Trace}\left(\mathfrak{Y}(\lambda)\sigma_3 f(\lambda)H^T(\lambda)\right)}{(s-\lambda)^2}d\lambda ds\\
&I_0+\frac{1}{4\pi i}\int_{\mathcal{C}}\psi(s)\text{Trace}\left(\mathfrak{Y}(s)\sigma_3\partial_s(\mathfrak{Y}^{-1}(s))\right)ds-\frac{1}{4\pi i}\int_{\mathcal{C}}\psi(s)\int_{\Gamma_\alpha}\frac{\text{Trace}\left(\sigma_3 f(\lambda)H^T(\lambda)\mathfrak{Y}(\lambda)\right)}{(s-\lambda)^2}d\lambda ds.
\end{aligned} \tag{5.2.11}$$

Now we observe that, for $\lambda\in\Gamma_\alpha$, from (1.2.36) we have

$$H^T(\lambda)\mathfrak{Y}(\lambda)=h^T(\lambda), \tag{5.2.12}$$

and hence

$$\text{Trace}\left(\sigma_3 f(\lambda)H^T(\lambda)\mathfrak{Y}(\lambda)\right) = \text{Trace}\left(\sigma_3 f(\lambda)h^T(\lambda)\right) = f_1(\lambda)h_1(\lambda) - f_2(\lambda)h_2(\lambda). \qquad (5.2.13)$$

Together with the identity,

$$\frac{1}{2\pi i}\int_{\mathcal{C}} \frac{\psi(s)}{(s-\lambda)^2} d\lambda = \psi'(\lambda),$$

relation (5.2.13) allows us to rewrite the last integral in (5.2.11) as

$$-\frac{1}{2}\int_{\Gamma_\alpha} \psi'(\lambda)\left(f_1(\lambda)h_1(\lambda) - f_2(\lambda)h_2(\lambda)\right) d\nu = -I_0, \qquad (5.2.14)$$

by (5.2.5). This means that the first and the last terms in (5.2.11) cancel each other and this concludes the proof of the proposition. ■

Formula (5.2.1) reduces the asymptotic evaluation of the $det(1-V)$ to the evaluation of the uniform in t asymptotics of the solution of the $\mathfrak{Y}$ - RH problem.

5.3 The Riemann-Hilbert analysis

The goal of this section is to produce the asymptotic solution of the $\mathfrak{Y}$ - RH problem. This Riemann-Hilbert problem is very close to the Riemann-Hilbert problem that was studied in [53]. In fact, if we put $\psi(\lambda) \equiv 0$, then $\mathfrak{Y}(\lambda)$ will be the solution of the Riemann-Hilbert problem whose asymptotics has been obtained in [53] (the m - RH problem of [53]). It turns out that the presence of the nontrivial phase function ψ does not affect the analysis of [53] much, so that we will be able to use most of the results obtained in the case $\psi(\lambda) \equiv 0$ and to shorten our analysis considerably. In the rest of this section we follow the steps used in [53].

5.3.1 Mapping onto a fixed interval

Define the linear-fractional transformation, $\lambda \mapsto z$, by the formulae,

$$z = -i\cot\frac{\alpha}{2}\frac{\lambda-1}{\lambda+1}, \qquad \lambda = \frac{1+iz\tan\frac{\alpha}{2}}{1-iz\tan\frac{\alpha}{2}}. \qquad (5.3.1)$$

This change of variable transforms the $\mathfrak{Y}$−RH problem to the following RHP which we call the Ψ-RH problem posed on the interval $(-1,1)$, traversed from -1 to 1 :

- **RH-Ψ1** Ψ is holomorphic in $\mathbb{C} \setminus [-1,1]$,
- **RH-Ψ2** $\Psi_+(z) = \Psi_-(z) J_\Psi(z), \qquad z \in (-1,1), \qquad$ where

$$J_\Psi(z) = \begin{pmatrix} 0 & \left(\frac{1+iz\tan(\frac{\alpha}{2})}{1-iz\tan(\frac{\alpha}{2})}\right)^m e^{t\psi(\lambda(z))} \\ -\left(\frac{1+iz\tan(\frac{\alpha}{2})}{1-iz\tan(\frac{\alpha}{2})}\right)^{-m} e^{-t\psi(\lambda(z))} & 2 \end{pmatrix}. \tag{5.3.2}$$

- **RH-Ψ3** $\Psi(\lambda) = \mathcal{O}\Big(\log|z \mp 1|\Big), \qquad$ as $z \to \pm 1$.
- **RH-Ψ4** $\Psi(\infty) = I$.

Once we have the solution $\Psi(z; m, t)$ of the $\Psi-$RH problem, we can find the solution $\mathfrak{Y}(\lambda; m, t)$ of the $\Psi-$RHP according to the equation

$$\mathfrak{Y}(\lambda; m, t) = \Big(\Psi(-i\cot\frac{\alpha}{2}; m, t)\Big)^{-1} \Psi(z(\lambda); m, t). \tag{5.3.3}$$

5.3.2 g - function transformation

Following again [53], we introduce the g-function

$$g(z) := \frac{1 + i\sqrt{z^2-1}\sin(\alpha/2)}{1 + iz\tan(\alpha/2)}. \tag{5.3.4}$$

The branch of the square root is fixed by the condition

$$\sqrt{z^2-1} \sim z, \qquad z \to \infty.$$

Let us list the key properties of the $g-$function (cf. Section 3.2 of [53]):

(i) g is holomorphic for all $z \notin [-1,1]$.

(ii) $g(z) \neq 0$ for all $z \notin [-1,1]$. At the points $z = -i\cot(\alpha/2)$ (or $z = \infty$) and $z = i\cot(\alpha/2)$ (or $z = 0$) the values of the function g are :

$$g(-i\cot(\alpha/2)) = 1, \qquad \text{and} \qquad g(i\cot(\alpha/2)) = \cos^2(\alpha/2) =: \kappa. \tag{5.3.5}$$

(iii) The boundary values $g_\pm(z), z \in [-1,1]$ satisfy the following equations :

$$g_+(z)g_-(z) = \kappa \frac{1 - iz\tan(\alpha/2)}{1 + iz\tan(\alpha/2)}, \tag{5.3.6}$$

and

$$\frac{g_+(z)}{g_-(z)} = \frac{1 - \sqrt{1-z^2}\sin(\alpha/2)}{1 + \sqrt{1-z^2}\sin(\alpha/2)}. \tag{5.3.7}$$

This means that for any fixed $0 < \delta < 1$, the following inequality holds

$$\left|\frac{g_+}{g_-}\right| \le \varepsilon_0 < 1, \qquad z \in [-1+\delta, 1-\delta], \tag{5.3.8}$$

for some $\varepsilon_0 = \varepsilon_0(\delta) > 0$.

(iv) The behavior of $g(z)$ as $z \to \infty$ is described by the asymptotic relation

$$g(z) = \cos(\alpha/2) + \mathcal{O}(\frac{1}{z}). \tag{5.3.9}$$

These properties suggest to transform the Riemann-Hilbert problem for Ψ by the formula,

$$\Phi(z) := \Psi(z) g^{-m\sigma_3} \kappa^{\frac{m}{2}\sigma_3} \tag{5.3.10}$$

The matrix valued function $\Phi(z) \equiv \Phi(z;m,t)$ is the solution of the following RHP, which we call the $\Phi-$ RH problem :

- **RH-Φ1** $\quad \Phi$ is holomorphic in the complement of the cut $[-1,1]$).
- **RH-Φ2** $\quad \Phi_+(z) = \Phi_-(z) J_\Phi(z), \quad z \in (-1,1), \qquad$ where

$$J_\Phi(z) = \begin{pmatrix} 0 & e^{t\psi(\lambda(z))} \\ -e^{-t\psi(\lambda(z))} & 2\left(g_+(z)/g_-(z)\right)^m \end{pmatrix}. \tag{5.3.11}$$

- **RH-Φ3** $\quad \Phi(z) = \mathcal{O}\Big(\log|z \mp 1|\Big), \qquad$ as $z \to \pm 1$.
- **RH-Φ4** $\quad \Phi(\infty) = I$.

Our original problem is now reduced to the asymptotic solution of the $\Phi-$ RH problem.

5.3.3 Global parametrix

By virtue of estimate (5.3.8), we have that for every $z \in (-1,1)$,

$$J_\Phi(z) \to \begin{pmatrix} 0 & e^{t\psi(\lambda(z))} \\ -e^{-t\psi(\lambda(z))} & 0 \end{pmatrix},$$

as $m \to \infty$. Hence, one expects that Φ be approximated by its *global parametrix* $P^{(\infty)}$ which is the solution of the following Riemann-Hilbert problem:

- **RH-Global1** $P^{(\infty)}$ is holomorphic in the complement of the cut $[-1,1]$,
- **RH-Global2** $P^{(\infty)}_+(z) = P^{(\infty)}_-(z) J^{(\infty)}(z), \qquad z \in (-1,1), \qquad$ where

$$J^{(\infty)}(z) = \begin{pmatrix} 0 & e^{t\psi(\lambda(z))} \\ -e^{-t\psi(\lambda(z))} & 0 \end{pmatrix}. \tag{5.3.12}$$

- **RH-Global3** $P^{(\infty)}(\infty) = I$.

The behavior of $P^{(\infty)}$ near the end points, $z = \pm 1$ is not specified. This problem admits an explicit solution. Indeed, put

$$\eta(z) := -\frac{\sqrt{z^2-1}}{2\pi} \int_{-1}^{1} \frac{\psi(\lambda(\zeta))}{\sqrt{1-\zeta^2}(\zeta - z)} d\zeta, \tag{5.3.13}$$

where

$$0 < \sqrt{1-z^2} \equiv -i \lim_{\epsilon \to +0} \sqrt{(z+i\epsilon)^2 - 1}, \qquad z \in (-1,1),$$

is the "plus" boundary value of the function$\sqrt{z^2-1}$ on the segment $(-1,1)$, oriented from the left to the right. The function η is analytic outside of the interval $[-1,1]$, and its boundary values satisfy the relation,

$$\eta_+(z) + \eta_-(z) = \psi(\lambda(z)), \qquad z \in (-1,1). \tag{5.3.14}$$

Observe also, that η is continuous at $z = \pm 1$, in fact,

$$\lim_{z \to \pm 1} \eta(z) = \frac{1}{2}\psi(\lambda(\pm 1)). \tag{5.3.15}$$

Therefore, if we make the transformation,

$$N(z) := e^{-t\eta_\infty \sigma_3} P^{(\infty)}(z) e^{t\eta(z)\sigma_3}, \tag{5.3.16}$$

where,

$$\eta_\infty := \lim_{z\to\infty} \eta(z) = \frac{1}{2\pi}\int_{-1}^{1} \frac{\psi(\lambda(z))}{\sqrt{1-z^2}} dz, \tag{5.3.17}$$

the Riemann-Hilbert problem for the global parametrix will be replaced by the following Riemann-Hilbert problem for the function N, enjoying constant jump on the segment $(-1, 1)$:

- **RH-N1** $\quad N$ is holomorphic in the complement of the cut $[-1, 1]$,
- **RH-N2** $\quad N_+(z) = N_-(z) J_N(z), \quad z \in (-1, 1) \qquad$ where,

$$J_N(z) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{5.3.18}$$

- **RH-N3** $\quad N(\infty) = I.$

The latter problem has already appeared numerous times in the context of the nonlinear steepest descent method, and its explicit solution is given by the formulae (see e.g. Section 3.3 of [53])

$$N(z) = \begin{pmatrix} \frac{\beta+\beta^{-1}}{2} & -\frac{\beta-\beta^{-1}}{2i} \\ \frac{\beta-\beta^{-1}}{2i} & \frac{\beta+\beta^{-1}}{2} \end{pmatrix} \equiv \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \beta^{-\sigma_3} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}, \tag{5.3.19}$$

where

$$\beta(z) = \left(\frac{z+1}{z-1}\right)^{1/4}. \tag{5.3.20}$$

This completes the construction of the global parametrix.

5.3.4 The local parametrix at $z = 1$

According to the standard nonlinear steepest descent approach, we shall construct the local parametrices in small enough neighborhoods of $z = 1$ and $z = -1$, respectively denoted by $U^{(1)}$ and $U^{(-1)}$, which are the solutions of the Φ-RH problem in these neighborhoods except for the condition at infinity gets replaced by the requirement that the local parametrices should match with $P^{(\infty)}(z)$ at the disks' boundaries to the leading order.

Consider first the neighborhood $U^{(1)}$. The parametrix we are looking for is the matrix valued function $P^{(1)}$ which solves the following local Riemann-Hilbert problem.

- **RH-P$^{(1)}$1** $\quad P^{(1)}$ is holomorphic in $U^{(1)} \setminus [-1,1]$.
- **RH-P$^{(1)}$2** $\quad P^{(1)}_+(z) = P^{(1)}_-(z) J_\Phi(z), \qquad z \in (-1,1) \cap U^{(1)}$.
- **RH-P$^{(1)}$3** $\quad P^{(1)}(z) = (I + \mathcal{O}(\frac{1}{m})) P^{(\infty)}(z), \qquad z \in \partial U^{(1)}$.

Using the assumed analyticity of $\psi(\lambda(z))$ around the interval$[-1,1]$, we can, in fact, get rid of $\psi(\lambda(z))$ in J_Φ with the help of the following simple transformation,

$$Q^{(1)}(z) := P^{(1)}(z) e^{\frac{t}{2}\psi(\lambda(z))\sigma_3}. \tag{5.3.21}$$

Note that we could not perform this transformation on the the full Φ-RH problem, since we shall not assume $\psi(\lambda(z))$ to be analytic everywhere in the z-plane. In terms of the function $Q^{(1)}(z)$, the parametrix RH problem reads,

- **RH-Q$^{(1)}$1** $\quad Q^{(1)}$ is holomorphic in $U^{(1)} \setminus [-1,1]$.
- **RH-Q$^{(1)}$2** $\quad Q^{(1)}_+(z) = Q^{(1)}_-(z) J_Q(z), \qquad z \in (-1,1) \cap U^{(1)}$, where

$$J_Q(z) = \begin{pmatrix} 0 & 1 \\ -1 & 2\left(g_+(z)/g_-(z)\right)^m \end{pmatrix}. \tag{5.3.22}$$

- **RH-Q$^{(1)}$3** $\quad Q^{(1)}(z) = (I + \mathcal{O}(\frac{1}{m})) P^{(\infty)}(z) e^{\frac{t}{2}\psi(\lambda(z))\sigma_3}, \qquad z \in \partial U^{(1)}$.

Up to the matching factors $e^{-t\eta(z)\sigma_3}$, $e^{t\eta_\infty\sigma_3}$, and $e^{\frac{t}{2}\psi(\lambda(z))\sigma_3}$, this is exactly the local problem which has been analyzed in [53]. Using the results obtained there, we arrive at the following formula for $Q^{(1)}(z)$ in terms of the Bessel functions $H_0^{(1)}$ and $H_0^{(2)}$ (cf. equation (64) of [53]),

$$Q^{(1)}(z) = E(z) \begin{pmatrix} H_0^{(1)}(\sqrt{\zeta(z)}) & H_0^{(2)}(\sqrt{\zeta(z)}) \\ \sqrt{\zeta(z)}(H_0^{(1)})'(\sqrt{\zeta(z)}) & \sqrt{\zeta(z)}(H_0^{(2)})'(\sqrt{\zeta(z)}) \end{pmatrix} f(z)^{-\frac{m}{2}\sigma_3}, \tag{5.3.23}$$

where, the function f and new local variable, $\zeta \equiv \zeta(z)$, are defined by the equations,

$$f(z) := \frac{1 + i(z^2-1)^{1/2}\sin(\alpha/2)}{1 - i(z^2-1)^{1/2}\sin(\alpha/2)}, \tag{5.3.24}$$

and

$$\zeta := \frac{1}{4}e^{-i\pi}m^2\Big(\ln f(z)\Big)^2. \tag{5.3.25}$$

The function $f(z)$ is analytic and has no zeros in $U^{(1)} \setminus (1-\delta, 1]$ (and, in fact, in $U^{(-1)} \setminus [-1, -1+\delta)$ as well), and the map $z \mapsto \zeta$ is a genuine local conformal mapping, such that (cf. (54) and (55) of [53]),

$$\zeta(z) = 2m^2(z-1)\sin^2\frac{\alpha}{2}\left(1 + (\frac{1}{2} - \frac{4}{3}\sin^2\frac{\alpha}{2})(z-1) + \mathcal{O}((z-1)^2)\right). \tag{5.3.26}$$

In particular, this implies that for sufficiently small δ, the following inequalities hold:

$$-\frac{3\pi}{4} \le \arg\sqrt{\zeta} \le \frac{3\pi}{4}, \tag{5.3.27}$$

and

$$|\sqrt{\zeta}| \ge m\sqrt{\delta}\left|\sin\frac{\alpha}{2}\right|, \tag{5.3.28}$$

for all $z \in \partial U^{(1)}$. The left matrix factor factor $E(z)$ is supposed to be analytic in $U^{(1)}$, and it should be chosen so that the matching condition on $\partial U^{(1)}$ be satisfied. Following again [53], we arrive at the following formula for $E(z)$ (cf. equation (63) of [53]),

$$E(z) = \sqrt{\frac{\pi}{8}}P^{(\infty)}(z)e^{i\frac{\pi}{4}\sigma_3}e^{\frac{t}{2}\psi(\lambda(z))\sigma_3}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\zeta^{\sigma_3/4}. \tag{5.3.29}$$

Inequality (5.3.28) implies that on the boundary of the disc $U^{(1)}$ the Hankel functions in (5.3.23) can be replaced by their known asymptotics and the verification of the matching condition is straightforward. It is also easy to verify that $E(z)$ has no jump across the segment $(1-\delta, 1]$, and hence $E(z)$ is analytic (and invertible) in $U^{(1)}$. Together with (5.3.21) equations (5.3.23), (5.3.29) yield the final formula for the parametrix of the solution of the Φ - RH problem at $z = 1$.

$$\begin{aligned} P^{(1)}(z) = {} & \sqrt{\frac{\pi}{8}}P^{(\infty)}(z)e^{i\frac{\pi}{4}\sigma_3}e^{\frac{t}{2}\psi(\lambda(z))\sigma_3}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\zeta^{\sigma_3/4} \\ & \times \begin{pmatrix} H_0^{(1)}(\sqrt{\zeta(z)}) & H_0^{(2)}(\sqrt{\zeta(z)}) \\ \sqrt{\zeta(z)}(H_0^{(1)})'(\sqrt{\zeta(z)}) & \sqrt{\zeta(z)}(H_0^{(2)})'(\sqrt{\zeta(z)}) \end{pmatrix} f(z)^{-\frac{m}{2}\sigma_3}e^{-\frac{t}{2}\psi(\lambda(z))\sigma_3}. \end{aligned} \tag{5.3.30}$$

The construction of the parametrix $P^{(-1)}$ at the point $z = -1$ is similar and again represents a very minor variation of the corresponding construction of [53]. The formula for $P^{(-1)}$ reads,

$$P^{(-1)}(z) = \sqrt{\frac{\pi}{8}} P^{(\infty)}(z) e^{i\frac{\pi}{4}\sigma_3} e^{\frac{t}{2}\psi(\lambda(z))\sigma_3} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \zeta^{\sigma_3/4}$$
$$\times\, \sigma_1 \begin{pmatrix} H_0^{(1)}(\sqrt{\zeta(z)}) & H_0^{(2)}(\sqrt{\zeta(z)}) \\ \sqrt{\zeta(z)}(H_0^{(1)})'(\sqrt{\zeta(z)}) & \sqrt{\zeta(z)}(H_0^{(2)})'(\sqrt{\zeta(z)}) \end{pmatrix} \sigma_1 f(z)^{-\frac{m}{2}\sigma_3} e^{-\frac{t}{2}\psi(\lambda(z))\sigma_3}, \tag{5.3.31}$$

where the local variable $\zeta(z)$ is given by the same formula (5.3.25), but is considered now in $U^{(-1)}$ and is fixed by the expansion,

$$\zeta(z) = -2m^2(z+1)\sin^2\frac{\alpha}{2}\left(1 + (\frac{4}{3}\sin^2\frac{\alpha}{2} - \frac{1}{2})(z+1) + \mathcal{O}((z+1)^2)\right). \tag{5.3.32}$$

The parametrix $P^{(-1)}$ solves the following local Riemann-Hilbert problem at the point $z = -1$.

- **RH-P$^{(-1)}$1** $\quad P^{(-1)}$ is holomorphic in $U^{(-1)} \setminus [-1, 1]$.
- **RH-P$^{(-1)}$2** $\quad P_+^{(-1)}(z) = P_-^{(-1)}(z) J_\Phi(z), \qquad z \in (-1, 1) \cap U^{(-1)}$.
- **RH-P$^{(-1)}$3** $\quad P^{(-1)}(z) = (I + \mathcal{O}(\frac{1}{m}))P^{(\infty)}(z), \qquad z \in \partial U^{(-1)}$.

Following the nonlinear steepest descent method as it is featured in [53], we introduce the function

$$R(z) := \begin{cases} \Phi(z)\left(P^{(\infty)}\right)^{-1}, & z \in \mathbb{C} \setminus (\overline{U^{(1)} \cup U^{(-1)}} \cup (-1,1)), \\ \Phi(z)P^{(1)}(z)^{-1}, & z \in U^{(1)} \setminus (1-\delta, 1], \\ \Phi(z)P^{(-1)}(z)^{-1}, & z \in U^{(-1)} \setminus [-1, -1+\delta). \end{cases} \tag{5.3.33}$$

By construction, the function R has no jumps across $(1-\delta, 1) \cup (-1, -1+\delta)$. Moreover, since *apriori* R can have no stronger than logarithmic singularities at the points ± 1, the function R is in fact analytic in the union of the discs $U^{(1)} \cup U^{(-1)}$. It solves the following RH-problem on the contour $\Sigma := \partial U^{(1)} \cup \partial U^{(-1)} \cup (-1+\delta, 1-\delta)$, where $\partial U^{(1)}$ and $\partial U^{(-1)}$ are oriented clockwise:

- **RH-R1** $\quad R(z)$ is holomorphic for all $z \notin \Sigma$,

- **RH-R2** $R_+(z) = R_-(z)J_R(z), \quad z \in \Sigma^{(0)} \equiv \Sigma \setminus \{1-\delta, -1+\delta\}$, where

$$J_R(x) = P_+^{(\infty)}(x) \begin{pmatrix} 1 & -2f_+^n(x) \\ 0 & 1 \end{pmatrix} P_+^{(\infty)}(x)^{-1}, \qquad x \in (-1+\delta, 1-\delta), \tag{5.3.34}$$

$$J_R(z) = P^{(1)}(z)P^{(\infty)}(z)^{-1}, \qquad z \in \partial U^{(1)} \setminus \{1-\delta\}, \tag{5.3.35}$$

$$J_R(z) = P^{(-1)}(z)P^{(\infty)}(z)^{-1}, \qquad z \in \partial U^{(-1)} \setminus \{-1+\delta\}, \tag{5.3.36}$$

- **RH-R3** $R(\infty) = I$.

This RH problem differs from the similar R - RH problem of [53] by the replacement,

$$N(z) \mapsto P^{(\infty)}(z) \equiv e^{t\eta_\infty \sigma_3} N(z) e^{-t\eta(z)\sigma_3},$$

only. This modification does not affect the principal arguments of [53]. In particular, we have that there exists a positive constant C_δ, depending on δ only, such that

$$|f_+(x)| \leq e^{-C_\delta},$$

for all $-1+\delta \leq x \leq 1-\delta$. Together with the matching conditions of the local parametrices $P^{(\pm 1)}$ with the global parametrix $P^{(\infty)}$ we arrive at the following uniform in t estimate for the jump matrix $J_R(z)$,

$$||I - J_R||_{L^\infty(\Sigma)\cap L^2(\Sigma)} < \frac{C_\delta}{m}. \tag{5.3.37}$$

By the arguments in the appendix A, this ensures that the R-RHP is solvable for $m \geq m_*$, for some $m^* \in \mathbb{N}$, and its solution can be written as

$$R(z) = I + R_1(z) + R_2(z) + \ldots + R_r(z) \tag{5.3.38}$$

where $R_j = \mathcal{O}(m^{-j})$, as $m \to \infty$, $1 \leq j \leq r$, uniformly for $0 < t \leq t_0$, for some $t_0 > 0$.

5.4 Asymptotics of the determinant

In this section we are going to compute the differential identity (5.2.1) in terms of the Riemann-Hilbert data and finally perform the integration in t. Tracing back the chain of RH transformations that led us from the original function $\mathfrak{Y}(\lambda)$ to the function $\Phi(z)$ we have that

$$\mathfrak{Y}(\lambda) = \kappa^{\frac{m}{2}\sigma_3}\Phi^{-1}\Big(-i\cot\frac{\alpha}{2}\Big)\Phi(z(\lambda))g^{m\sigma_3}(z(\lambda))\kappa^{-\frac{m}{2}\sigma_3}. \tag{5.4.1}$$

where we have used (5.3.5). This would yield the following expression for the product $\partial_\lambda(\mathfrak{Y}^{-1}(\lambda))\mathfrak{Y}(\lambda)$ which is involved in the integral in the right hand side of (5.2.1),

$$\partial_\lambda(\mathfrak{Y}^{-1}(\lambda))\mathfrak{Y}(\lambda) = -m\partial_\lambda g(z(\lambda))g^{-1}(z(\lambda))\sigma_3+\kappa^{\frac{m}{2}\sigma_3}g^{-m\sigma_3}(z(\lambda))\partial_\lambda\Big(\Phi^{-1}(z(\lambda))\Big)\Phi(z(\lambda))g^{m\sigma_3}(z(\lambda))\kappa^{-\frac{m}{2}\sigma_3}, \tag{5.4.2}$$

and, in turn, using the cyclic property of the trace we have

$$\text{Trace}\left(\mathfrak{Y}(\lambda)\sigma_3\partial_\lambda(\mathfrak{Y}^{-1}(\lambda))\right) = -2m\partial_\lambda g(z(\lambda))g^{-1}(z(\lambda)) + \text{Trace}\left(\sigma_3\partial_\lambda\Big(\Phi^{-1}(z(\lambda))\Big)\Phi(z(\lambda))\right), \tag{5.4.3}$$

On the loop $\mathcal{C}$ the function $\Phi(z(\lambda))$ can be approximated by the the global parametrix, $P^{(\infty)}$. Indeed, from (5.3.33) and (5.3.38) we have that

$$\Phi(z(\lambda)) = R(z(\lambda))P^{(\infty)}(z(\lambda)) = \left(I + \mathcal{O}(m^{-1})\right)P^{(\infty)}(z(\lambda)), \qquad m\to\infty, \tag{5.4.4}$$

where the estimate holds uniformly for $\lambda \in \mathcal{C}$, and it is differentiable with respect to λ. Combining (5.4.4) and (5.4.3) we conclude that

$$\text{Trace}\left(\mathfrak{Y}(\lambda)\sigma_3\partial_\lambda(\mathfrak{Y}^{-1}(\lambda))\right) = -2m\partial_\lambda g(z(\lambda))g^{-1}(z(\lambda))+\text{Trace}\left(\sigma_3\partial_\lambda\Big(P^{(\infty)-1}(z(\lambda))\Big)P^{(\infty)}(z(\lambda))\right)+\mathcal{O}(m^{-1}). \tag{5.4.5}$$

Using the definition of the global parametrix $P^{(\infty)}$, we derive from (5.4.5) the following asymptotic formula for the integrand (5.2.1) expressed in terms of the known objects,

$$\begin{aligned}\text{Trace}\left(\mathfrak{Y}(\lambda)\sigma_3\partial_\lambda(\mathfrak{Y}^{-1}(\lambda))\right) = &-2m\partial_\lambda g(z(\lambda))g^{-1}(z(\lambda)) + 2t\partial_\lambda\eta(z(\lambda))\\ &+\text{Trace}\left(\sigma_3\partial_\lambda\Big(N^{-1}(z(\lambda))\Big)N(z(\lambda))\right)+\mathcal{O}(m^{-1}),\end{aligned} \tag{5.4.6}$$

where the functions η and N are given by the equations (5.3.13) and (5.3.19), respectively. Observe that

$$\mathrm{Trace}\left(\sigma_3\partial_\lambda\left(N^{-1}(z(\lambda))\right)N(z(\lambda))\right) = \partial_\lambda \ln\beta(z(\lambda))\mathrm{Trace}\left(\sigma_3 B^{-1}\sigma_3 B\right) = 0, \tag{5.4.7}$$

where

$$B = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix},$$

and the last equation in (5.4.7) is just a simple direct calculation of the trace indicated. Therefore, the asymptotic formulae (5.4.6) reduces to the relation

$$\mathrm{Trace}\left(\mathfrak{Y}(\lambda)\sigma_3\partial_\lambda(\mathfrak{Y}^{-1}(\lambda))\right) = -2m\partial_\lambda g(z(\lambda))g^{-1}(z(\lambda)) + 2t\partial_\lambda\eta(z(\lambda)) + \mathcal{O}(m^{-1}), \tag{5.4.8}$$

as $m \to \infty$, uniformly for $\lambda \in \mathcal{C}$. Substituting the estimate (5.4.8) into the right hand side of (5.2.1) and changing the variable of integration, $\lambda \mapsto z$, we obtain

$$\frac{d}{dt}\ln\det(1-V) = \frac{m}{2\pi i}\int_{\mathcal{L}}\psi(\lambda(z))\partial_z \ln g(z)dz - \frac{t}{2\pi i}\int_{\mathcal{L}}\psi(\lambda(z))\partial_z\eta(z)dz + \mathcal{O}(m^{-1}), \quad m\to\infty, \tag{5.4.9}$$

where $\mathcal{L} \equiv z(\mathcal{C})$, is a small loop around the interval $[-1,1]$ and the estimate is uniform with respect to t. Integrating this estimate, we arrive at the following asymptotics for the determinant,

$$\ln\det(1-V) = \ln\det(1-V)\Big|_{t=0} + \frac{mt}{2\pi i}\int_{\mathcal{L}}\psi(\lambda(z))\partial_z\ln g(z)dz - \frac{t^2}{4\pi i}\int_{\mathcal{L}}\psi(\lambda(z))\partial_z\eta(z)dz + \mathcal{O}(m^{-1}), \tag{5.4.10}$$

as $m \to \infty$. Using the known [54] (see also [53]) large m asymptotics of the $\det(1-V)\Big|_{t=0}$, we transform (5.4.10) into our final asymptotic result,

$$\begin{aligned}\ln\det(1-V) &= m^2\ln\cos\frac{\alpha}{2} + m\frac{t}{2\pi i}\int_{\mathcal{L}}\psi(\lambda(z))\partial_z\ln g(z)dz \\ &\quad -\frac{1}{4}\ln\left(m\sin\frac{\alpha}{2}\right) - \frac{t^2}{4\pi i}\int_{\mathcal{L}}\psi(\lambda(z))\partial_z\eta(z)dz + c_0 + \mathcal{O}(m^{-1}), \quad m\to\infty\end{aligned} \tag{5.4.11}$$

where the constant c_0 is the famous Widom's constant

$$c_0 = \frac{1}{12} \ln 2 + 3\zeta'(-1). \tag{5.4.12}$$

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A. NORMALIZED SMALL-NORM RIEMANN-HILBERT PROBLEMS

In this appendix we include the basic facts from general Riemann-Hilbert theory regarding the solvability of normalized small-norm Riemann-Hilbert problems. Here we mainly follow the presentation given in chapter 8 of [57] which suffices for the purposes of this thesis. A normalized small-norm Riemann-Hilbert problem is the problem of finding a matrix-valued function $R : \mathbb{C}/\Sigma_R \to GL(k, \mathbb{C})$ such that

- **RH-R1** R is holomorphic in $\mathbb{C} \setminus \Sigma_R$.
- **RH-R2** $R_+(z) = R_-(z) G_R(z)$, for $z \in \Sigma_R$.
- **RH-R3** As $z \to \infty$, $R(z) = I + \mathcal{O}(z^{-1})$.

where G_R depends analytically on an extra parameter n such that

$$||G_R - I||_{L^2(\Sigma_R)} \leq \frac{C}{n^{\varepsilon}}, \qquad ||G_R - I||_{L^\infty(\Sigma_R)} \leq \frac{C}{n^{\varepsilon}}, \qquad n \geq n_*, \tag{A.0.1}$$

for some positive constants C and ε. First, note that the solution of this Riemann-Hilbert problem can be written as

$$R(z) = I + \frac{1}{2\pi i} \int_{\Sigma_R} \frac{\rho(\mu)(G_R(\mu) - I)}{\mu - z} d\mu, \qquad z \in \mathbb{C} \setminus \Sigma_R, \qquad n \geq n_*, \tag{A.0.2}$$

where $\rho(z)$ is the solution of the following singular integral equation

$$\rho(z) = I + \frac{1}{2\pi i} \int_{\Sigma_R} \frac{\rho(\mu)(G_R(\mu) - I)}{\mu - z_-} d\mu, \qquad z \in \Sigma_R. \tag{A.0.3}$$

This can be easily justified using the Plemelj-Sokhotskii formula and standard properties of the Cauchy operator (for a detailed justification see, e.g., chapter 3 of [57]). This integral equation can be equivalently written as

$$\rho_0(z) = F(z) + \mathcal{K}[\rho_0](z), \qquad z \in \Sigma_R, \qquad \rho_0 \in L^2(\Sigma_R), \tag{A.0.4}$$

where $\rho_0(z) := \rho(z) - I$,

$$F(z) = \frac{1}{2\pi i}\int_{\Sigma_R} \frac{G_R(\mu) - I}{\mu - z_-} d\mu \equiv C_-[G_R - I](z), \qquad z \in \Sigma_R, \tag{A.0.5}$$

and

$$\mathcal{K}[\rho_0](z) := \frac{1}{2\pi i}\int_{\Sigma_R} \frac{\rho_0(\mu)(G_R(\mu) - I)}{\mu - z_-} d\mu \equiv C_-[\rho_0(G_R - I)](z), \qquad z \in \Sigma_R. \tag{A.0.6}$$

Note that

$$||F||_{L^2(\Sigma_R)} \leq ||C_-||_{L^2(\Sigma_R)}||G_R - I||_{L^2(\Sigma_R)} \leq \frac{C}{n^\varepsilon}, \tag{A.0.7}$$

and

$$||\mathcal{K}||_{L^2(\Sigma_R)\to L^2(\Sigma_R)} = \sup_{\rho_0 \in L^2(\Sigma_R)} \frac{||\mathcal{K}[\rho_0]||_{L^2(\Sigma_R)}}{||\rho_0||_{L^2(\Sigma_R)}} \leq ||C_-||_{L^2(\Sigma_R)}||G_R - I||_{L^\infty(\Sigma_R)} \leq \frac{C}{n^\varepsilon}. \tag{A.0.8}$$

Let us define the operator $\mathcal{K}_0 : L^2(\Sigma_R) \to L^2(\Sigma_R)$ by $\mathcal{K}_0[f] := F(z)$; note that

$$||\mathcal{K}_0||_{L^2(\Sigma_R)\to L^2(\Sigma_R)} = \sup\{||\mathcal{K}_0[f]||_{L^2(\Sigma_R)}, ||f||_{L^2(\Sigma_R)} = 1\} = ||F||_{L^2(\Sigma_R)} \leq \frac{C}{n^\varepsilon}.$$

Now let $\mathcal{L} : L^2(\Sigma_R) \to L^2(\Sigma_R)$ be the operator $\mathcal{L} := \mathcal{K}_0 + \mathcal{K}$, then

$$||\mathcal{L}||_{L^2(\Sigma_R)\to L^2(\Sigma_R)} \leq ||\mathcal{K}_0||_{L^2(\Sigma_R)\to L^2(\Sigma_R)} + ||\mathcal{K}||_{L^2(\Sigma_R)\to L^2(\Sigma_R)} \leq \frac{C_1(n)}{n^\varepsilon}. \tag{A.0.9}$$

Therefore the operator $\mathcal{L}$ is a contraction and hence by the fixed point theorem $\mathcal{L}[\rho_0] = \rho_0$ has a solution in $L^2(\Sigma_R)$ with $||\rho_0||_{L^2(\Sigma_R)} \leq C_1(n)/n^\varepsilon$. We can express $\rho_0(z)$ as

$$\rho_0(z) = \sum_{k=0}^{\infty} \rho_{0,k}(z), \qquad \text{with} \qquad \rho_{0,0}(z) = F(z). \tag{A.0.10}$$

and $\rho_{0,k}(z)$, $k \geq 1$ can be recursively determined from the integral equation (A.0.4), they are given by

$$\rho_{0,k+1}(z) = \mathcal{K}[\rho_{0,k}](z). \tag{A.0.11}$$

Then from (A.0.2) we can write the solution of the R-RHP as

$$R(z) = I + \frac{1}{2\pi i}\int_{\Sigma_R} \frac{G_R(\mu) - I}{\mu - z} d\mu + \frac{1}{2\pi i}\sum_{k=0}^{\infty}\int_{\Sigma_R} \frac{\rho_{0,k}(\mu)(G_R(\mu) - I)}{\mu - z} d\mu, \quad z \in \mathbb{C}\setminus\Sigma_R, \quad n \geq n_*. \tag{A.0.12}$$

So we can write

$$R(z) = I + R_1(z) + R_2(z) + R_3(z) + \cdots, \qquad n \geq n_*, \tag{A.0.13}$$

where

$$R_1(z) = \frac{1}{2\pi i}\int_{\Sigma_R} \frac{G_R(\mu) - I}{\mu - z} d\mu, \qquad \text{and} \qquad R_k(z) = \frac{1}{2\pi i}\int_{\Sigma_R} \frac{\rho_{0,k-2}(\mu)\,(G_R(\mu) - I)}{\mu - z} d\mu, \qquad k > 1. \tag{A.0.14}$$

The following lemma provides a recursive description for R_k, $k \geq 1$.

Lemma A.0.1 *We have the following recursive relations for R_k*

$$R_k(z) = \frac{1}{2\pi i}\int_{\Sigma_R} \frac{[R_{k-1}(\mu)]_-\,(G_R(\mu) - I)}{\mu - z} d\mu, \qquad z \in \mathbb{C}\setminus\Sigma_R, \qquad k \geq 1. \tag{A.0.15}$$

Proof The identity (A.0.15) is obviously true for $k = 1$, since $R_0(z) \equiv I$, see (A.0.14). From (A.0.5),(A.0.10) and (A.0.14) we clearly have

$$(R_1(z))_- = \rho_{0,0}(z), \qquad z \in \Sigma_R. \tag{A.0.16}$$

Also note that from (A.0.14), for $j > 1$ we have

$$(R_j(z))_- = \frac{1}{2\pi i}\int_{\Sigma_R} \frac{\rho_{0,j-2}(\mu)\,(G_R(\mu) - I)}{\mu - z_-} d\mu = \mathcal{K}[\rho_{0,j-2}] = \rho_{0,j-1}(z), \qquad z \in \Sigma_R. \tag{A.0.17}$$

From (A.0.16) and (A.0.17) we have

$$\frac{1}{2\pi i}\int_{\Sigma_R} \frac{[R_{k-1}(\mu)]_-\,(G_R(\mu) - I)}{\mu - z} d\mu = \frac{1}{2\pi i}\int_{\Sigma_R} \frac{\rho_{0,k-2}(\mu)\,(G_R(\mu) - I)}{\mu - z} d\mu \equiv R_k(z), \qquad k \geq 1. \tag{A.0.18}$$

■

B. SOLUTION OF THE RIEMANN-HILBERT PROBLEM ASSOCIATED WITH THE ANISOTROPIC SQUARE LATTICE ISING MODEL IN THE LOW TEMPERATURE REGIME

As suggested by (4.2.13), we need the 12 entry of the solution X to the Riemann-Hilbert problem associated to the Toeplitz determinant with symbol ϕ given by (4.1.8). The Riemann-Hilbert problem associated with Toeplitz determinants is the X-RHP introduced in section 1.2.1 ([58], [9], [49]). Below we show the standard steepest descent analysis to asymptotically solve this problem, in the case where ϕ is a symbol analytic in a neighborhood of the unit circle and with zero winding number. Note that the symbol ϕ associated to the 2D Ising model in the low temperature regime enjoys these properties. We first normalize the behavior at ∞ by defining

$$T(z;n) := \begin{cases} Y(z;n)z^{-n\sigma_3}, & |z|>1, \\ Y(z;n), & |z|<1. \end{cases} \tag{B.0.1}$$

The function T defined above satisfies the following RH problem

- **RH-T1** $\quad T(\cdot;n) : \mathbb{C}\setminus\mathbb{T} \to \mathbb{C}^{2\times 2}$ is analytic,
- **RH-T2** $\quad T_+(z;n) = T_-(z;n)\begin{pmatrix} z^n & \phi(z) \\ 0 & z^{-n}\end{pmatrix}, \qquad z\in\mathbb{T},$
- **RH-T3** $\quad T(z;n) = I + \mathcal{O}(1/z), \qquad z\to\infty,$

So T has a highly-oscillatory jump matrix as $n\to\infty$. The next transformation yields a Riemann Hilbert problem, normalized at infinity, having an exponentially decaying jump matrix on the *lenses*. Note that we have the following factorization of the jump matrix of the T-RHP :

$$\begin{pmatrix} z^n & \phi(z) \\ 0 & z^{-n}\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ z^{-n}\phi(z)^{-1} & 1\end{pmatrix}\begin{pmatrix} 0 & \phi(z) \\ -\phi(z)^{-1} & 0\end{pmatrix}\begin{pmatrix} 1 & 0 \\ z^{n}\phi(z)^{-1} & 1\end{pmatrix} \equiv J_0(z;n)J^{(\infty)}(z)J_1(z;n). \tag{B.0.2}$$

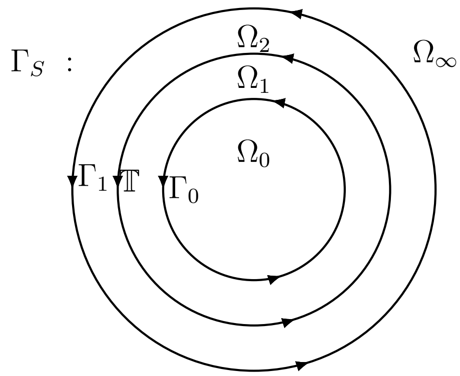

Figure B.1. Opening of lenses: the jump contour for the S-RHP.

Now, we define the following function :

$$S(z;n) := \begin{cases} T(z;n)J_1^{-1}(z;n), & z \in \Omega_1, \\ T(z;n)J_0(z;n), & z \in \Omega_2, \\ T(z;n), & z \in \Omega_0 \cup \Omega_\infty. \end{cases} \tag{B.0.3}$$

Also introduce the following function on $\Gamma_S := \Gamma_0 \cup \Gamma_1 \cup \mathbb{T}$

$$J_S(z;n) = \begin{cases} J_1(z;n), & z \in \Gamma_0, \\ J^{(\infty)}(z), & z \in \mathbb{T}, \\ J_0(z;n), & z \in \Gamma_1. \end{cases} \tag{B.0.4}$$

We have the following Riemann-Hilbert problem for $S(z;n)$

- **RH-S1** $\quad S(\cdot;n) : \mathbb{C} \setminus \Gamma_S \to \mathbb{C}^{2\times 2}$ is analytic.
- **RH-S2** $\quad S_+(z;n) = S_-(z;n)J_S(z;n), \qquad z \in \Gamma_S.$
- **RH-S3** $\quad S(z;n) = I + \mathcal{O}(1/z), \qquad$ as $z \to \infty$.

Note that the matrices $J_0(z;n)$ and $J_1(z;n)$ tend to the identity matrix uniformly on their respective contours, exponentially fast as $n \to \infty$.

Global parametrix RHP

We are looking for a piecewise analytic function $P^{(\infty)}(z) : \mathbb{C} \setminus \mathbb{T} :\to \mathbb{C}^{2\times 2}$ such that

- **RH-Global1** $P^{(\infty)}$ is holomorphic in $\mathbb{C} \setminus \mathbb{T}$.
- **RH-Global2** for $z \in \mathbb{T}$ we have

$$P_+^{(\infty)}(z) = P_-^{(\infty)}(z) \begin{pmatrix} 0 & \phi(z) \\ -\phi^{-1}(z) & 0 \end{pmatrix}. \tag{B.0.5}$$

- **RH-Global3** $P^{(\infty)}(z) = I + O(1/z), \qquad \text{as } z \to \infty.$

We can find a piecewise analytic function α which solves the following scalar multiplicative Riemann-Hilbert problem

$$\alpha_+(z) = \alpha_-(z)\phi(z) \qquad z \in \mathbb{T}. \tag{B.0.6}$$

By Plemelj-Sokhotski formula we have

$$\alpha(z) = \exp\left[\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\ln(\phi(\tau))}{\tau - z} d\tau\right], \tag{B.0.7}$$

Now, using (B.0.6) we have the following factorization

$$\begin{pmatrix} 0 & \phi(z) \\ -\phi^{-1}(z) & 0 \end{pmatrix} = \begin{pmatrix} 0\alpha_-^{-1}(z) & 0 \\ 0 & \alpha_-(z) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_+^{-1}(z) & 0 \\ 0 & \alpha_+(z) \end{pmatrix}. \tag{B.0.8}$$

So, the following function satisfies (B.0.5)

$$P^{(\infty)}(z) := \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \alpha^{-1}(z) & 0 \\ 0 & \alpha(z) \end{pmatrix}, & |z| < 1, \\ \begin{pmatrix} \alpha(z) & 0 \\ 0 & \alpha^{-1}(z) \end{pmatrix}, & |z| > 1. \end{cases} \tag{B.0.9}$$

Also, by the properties of the Cauchy integral, $P^{(\infty)}(z)$ is holomorphic in $\mathbb{C} \setminus \mathbb{T}$. Moreover, $\alpha(z) = 1 + \mathcal{O}(z^{-1})$, as $z \to \infty$ and hence

$$P^{(\infty)}(z) = I + O(1/z), \qquad z \to \infty. \tag{B.0.10}$$

Therefore $P^{(\infty)}$ given by (B.0.9) is the unique solution of the Global parametrix Riemann-Hilbert problem.

Small-norm RHP

Let us consider the ratio

$$R(z;n) := S(z;n)\left[P^{(\infty)}(z)\right]^{-1}. \tag{B.0.11}$$

We have the following Riemann-Hilbert problem for $R(z;n)$

- **RH-R1** R is holomorphic in $\mathbb{C}\setminus(\Gamma_0\cup\Gamma_1)$.
- **RH-R2** $R_+(z;n) = R_-(z;n)J_R(z;n), \qquad z\in\Gamma_0\cup\Gamma_1,$
- **RH-R3** $R(z;n) = I + \mathcal{O}(1/z) \qquad \text{as } z\to\infty.$

This Riemann Hilbert problem is solvable for large n(see appendix A) and $R(z;n)$ can be written as

$$R(z;n) = I + R_1(z;n) + R_2(z;n) + R_3(z;n) + \cdots, \qquad n\geq n_0 \tag{B.0.12}$$

where R_k is given by the formula (A.0.15). It is easy to check that $R_{2\ell}(z;n)$ is diagonal and $R_{2\ell+1}(z;n)$ is off-diagonal; $\ell\in\mathbb{N}\cup\{0\}$. Let us compute $R_1(z;n)$; we have

$$J_R(z)-I = \begin{cases} P^{(\infty)}(z)\begin{pmatrix} 0 & 0 \\ z^n\phi^{-1}(z) & 0\end{pmatrix}\left[P^{(\infty)}(z)\right]^{-1}, & z\in\Gamma_0, \\ P^{(\infty)}(z)\begin{pmatrix} 0 & 0 \\ z^{-n}\phi^{-1}(z) & 0\end{pmatrix}\left[P^{(\infty)}(z)\right]^{-1}, & z\in\Gamma_1, \end{cases} = \begin{cases} \begin{pmatrix} 0 & -z^n\phi^{-1}(z)\alpha^2(z) \\ 0 & 0\end{pmatrix}, & z\in\Gamma_0, \\ \begin{pmatrix} 0 & 0 \\ z^{-n}\phi^{-1}(z)\alpha^{-2}(z) & 0\end{pmatrix}, & z\in\Gamma_1. \end{cases} \tag{B.0.13}$$

Therefore

$$R_1(z;n) = \begin{pmatrix} 0 & -\dfrac{1}{2\pi i}\displaystyle\int_{\Gamma_0}\frac{\tau^n\phi^{-1}(\tau)\alpha^2(\tau)}{\tau-z}d\tau \\ \dfrac{1}{2\pi i}\displaystyle\int_{\Gamma_1}\frac{\tau^{-n}\phi^{-1}(\tau)\alpha^{-2}(\tau)}{\tau-z}d\tau & 0 \end{pmatrix}. \tag{B.0.14}$$

Tracing back Riemann-Hilbert problems

If we trace back the Riemann-Hilbert problems $R \mapsto S \mapsto T \mapsto Y$ we will obtain

$$X(z;n) = R(z;n) \begin{cases} \begin{pmatrix} \alpha(z) & 0 \\ 0 & \alpha^{-1}(z) \end{pmatrix} z^{n\sigma_3}, & z \in \Omega_\infty, \\ \begin{pmatrix} \alpha(z) & 0 \\ -z^{-n}\alpha^{-1}(z)\phi^{-1}(z) & \alpha^{-1}(z) \end{pmatrix} z^{n\sigma_3}, & z \in \Omega_2, \\ \begin{pmatrix} z^n\alpha(z)\phi^{-1}(z) & \alpha(z) \\ -\alpha^{-1}(z) & 0 \end{pmatrix}, & z \in \Omega_1, \\ \begin{pmatrix} 0 & \alpha(z) \\ -\alpha^{-1}(z) & 0 \end{pmatrix}, & z \in \Omega_0. \end{cases} \tag{B.0.15}$$

Note that $R(z;n) = \begin{pmatrix} 1+\mathcal{O}(e^{-2cn}) & R_{1,12}(z;n)(1+\mathcal{O}(e^{-2cn})) \\ R_{1,21}(z;n)(1+\mathcal{O}(e^{-2cn})) & 1+\mathcal{O}(e^{-2cn}) \end{pmatrix}$, hence

$$X(z;n) = (1+\mathcal{O}(e^{-2cn})) \begin{cases} \begin{pmatrix} \alpha(z)z^n & R_{1,12}(z;n)\alpha^{-1}(z)z^{-n} \\ R_{1,21}(z;n)\alpha(z)z^n & \alpha^{-1}(z)z^{-n} \end{pmatrix}, & z \in \Omega_\infty, \\ \begin{pmatrix} \alpha(z)z^n - \alpha^{-1}(z)\phi^{-1}(z)R_{1,12}(z;n) & R_{1,12}(z;n)\alpha^{-1}(z)z^{-n} \\ R_{1,21}(z;n)\alpha(z)z^n - \alpha^{-1}(z)\phi^{-1}(z) & \alpha^{-1}(z)z^{-n} \end{pmatrix}, & z \in \Omega_2, \\ \begin{pmatrix} z^n\alpha(z)\phi^{-1}(z) - R_{1,12}(z;n)\alpha^{-1}(z) & \alpha(z) \\ -\alpha^{-1}(z) + z^n\alpha(z)\phi^{-1}(z)R_{1,21}(z;n) & R_{1,21}(z;n)\alpha(z) \end{pmatrix}, & z \in \Omega_1, \\ \begin{pmatrix} -R_{1,12}(z;n)\alpha^{-1}(z) & \alpha(z) \\ -\alpha^{-1}(z) & R_{1,21}(z;n)\alpha(z) \end{pmatrix}, & z \in \Omega_0. \end{cases} \tag{B.0.16}$$

We can now read the asymptotic expressions for $X_{12}(\frac{1}{z^*}; N-1)$, for the cases $|z^*| < 1$ and $|z^*| > 1$, see (4.2.13).

C. AIRY, BESSEL AND CONFLUENT HYPERGEOMETRIC MODEL RIEMANN-HILBERT PROBLEMS

We recall here some well-known model RH problems: the Airy model RH problem, whose solution is denoted Φ_{Ai} and the Bessel model RH problem, whose solution is denoted $\Phi_{\mathrm{Be}}(\cdot) = \Phi_{\mathrm{Be}}(\cdot;\alpha)$, where the parameter α is such that $\Re\alpha > -1$.

C.1 Airy model RH problem

(a) $\Phi_{\mathrm{Ai}} : \mathbb{C} \setminus \Sigma_{\mathrm{Ai}} \to \mathbb{C}^{2\times 2}$ is analytic, and Σ_{Ai} is shown in Figure C.1.

(b) Φ_{Ai} has the jump relations

$$\begin{aligned} \Phi_{\mathrm{Ai},+}(z) &= \Phi_{\mathrm{Ai},-}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, && \text{on } \mathbb{R}^-, \\ \Phi_{\mathrm{Ai},+}(z) &= \Phi_{\mathrm{Ai},-}(z) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, && \text{on } \mathbb{R}^+, \\ \Phi_{\mathrm{Ai},+}(z) &= \Phi_{\mathrm{Ai},-}(z) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, && \text{on } e^{\frac{2\pi i}{3}}\mathbb{R}^+, \\ \Phi_{\mathrm{Ai},+}(z) &= \Phi_{\mathrm{Ai},-}(z) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, && \text{on } e^{-\frac{2\pi i}{3}}\mathbb{R}^+. \end{aligned} \tag{C.1.1}$$

(c) As $z \to \infty$, $z \notin \Sigma_{\mathrm{Ai}}$, we have

$$\Phi_{\mathrm{Ai}}(z) = z^{-\frac{\sigma_3}{4}} N \left(I + \sum_{k=1}^{\infty} \frac{\Phi_{\mathrm{Ai,k}}}{z^{3k/2}} \right) e^{-\frac{2}{3} z^{3/2}\sigma_3}, \tag{C.1.2}$$

where $N = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ and $\Phi_{\mathrm{Ai},1} = \frac{1}{8} \begin{pmatrix} \frac{1}{6} & i \\ i & -\frac{1}{6} \end{pmatrix}$.

As $z \to 0$, we have

$$\Phi_{\mathrm{Ai}}(z) = \mathcal{O}(1). \tag{C.1.3}$$

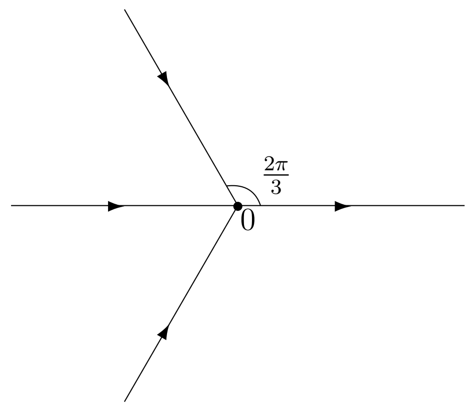

Figure C.1. The jump contour Σ_{Ai} for Φ_{Ai}.

The Airy model RH problem was introduced and solved in [34]. We have

$$\Phi_{\text{Ai}}(z) := M_A \times \begin{cases} \begin{pmatrix} \text{Ai}(z) & \text{Ai}(\omega^2 z) \\ \text{Ai}'(z) & \omega^2 \text{Ai}'(\omega^2 z) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3}, & \text{for } 0 < \arg z < \frac{2\pi}{3}, \\ \begin{pmatrix} \text{Ai}(z) & \text{Ai}(\omega^2 z) \\ \text{Ai}'(z) & \omega^2 \text{Ai}'(\omega^2 z) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, & \text{for } \frac{2\pi}{3} < \arg z < \pi, \\ \begin{pmatrix} \text{Ai}(z) & -\omega^2 \text{Ai}(\omega z) \\ \text{Ai}'(z) & -\text{Ai}'(\omega z) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & \text{for } -\pi < \arg z < -\frac{2\pi}{3}, \\ \begin{pmatrix} \text{Ai}(z) & -\omega^2 \text{Ai}(\omega z) \\ \text{Ai}'(z) & -\text{Ai}'(\omega z) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3}, & \text{for } -\frac{2\pi}{3} < \arg z < 0, \end{cases} \tag{C.1.4}$$

with $\omega = e^{\frac{2\pi i}{3}}$, Ai the Airy function and

$$M_A = \sqrt{2\pi} e^{\frac{\pi i}{6}} \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}. \tag{C.1.5}$$

C.2 Bessel model RH problem

(a) $\Phi_{\text{Be}} : \mathbb{C} \setminus \Sigma_{\text{Be}} \to \mathbb{C}^{2\times 2}$ is analytic, where Σ_{Be} is shown in Figure C.2.

(b) Φ_{Be} satisfies the jump conditions

$$\begin{aligned}
\Phi_{\mathrm{Be},+}(z) &= \Phi_{\mathrm{Be},-}(z)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad z \in \mathbb{R}^-, \\
\Phi_{\mathrm{Be},+}(z) &= \Phi_{\mathrm{Be},-}(z)\begin{pmatrix} 1 & 0 \\ e^{\pi i\alpha} & 1 \end{pmatrix}, \qquad z \in e^{\frac{2\pi i}{3}}\mathbb{R}^+, \\
\Phi_{\mathrm{Be},+}(z) &= \Phi_{\mathrm{Be},-}(z)\begin{pmatrix} 1 & 0 \\ e^{-\pi i\alpha} & 1 \end{pmatrix}, \quad z \in e^{-\frac{2\pi i}{3}}\mathbb{R}^+.
\end{aligned} \tag{C.2.1}$$

(c) As $z \to \infty$, $z \notin \Sigma_{\mathrm{Be}}$, we have

$$\Phi_{\mathrm{Be}}(z) = (2\pi z^{\frac{1}{2}})^{-\frac{\sigma_3}{2}} N \left(I + \sum_{k=1}^{\infty} \Phi_{\mathrm{Be},k} z^{-k/2} \right) e^{2z^{\frac{1}{2}}\sigma_3}, \tag{C.2.2}$$

where $\Phi_{\mathrm{Be},1} = \frac{1}{16}\begin{pmatrix} -(1+4\alpha^2) & -2i \\ -2i & 1+4\alpha^2 \end{pmatrix}$.

(d) As z tends to 0, the behaviour of $\Phi_{\mathrm{Be}}(z)$ is

$$\begin{aligned}
\Phi_{\mathrm{Be}}(z) &= \begin{cases} \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\log z) \\ \mathcal{O}(1) & \mathcal{O}(\log z) \end{pmatrix}, & |\arg z| < \frac{2\pi}{3}, \\ \begin{pmatrix} \mathcal{O}(\log z) & \mathcal{O}(\log z) \\ \mathcal{O}(\log z) & \mathcal{O}(\log z) \end{pmatrix}, & \frac{2\pi}{3} < |\arg z| < \pi, \end{cases} \quad \text{if } \Re\alpha = 0, \\
\Phi_{\mathrm{Be}}(z) &= \begin{cases} \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix} z^{\frac{\alpha}{2}\sigma_3}, & |\arg z| < \frac{2\pi}{3}, \\ \begin{pmatrix} \mathcal{O}(z^{-\frac{\alpha}{2}}) & \mathcal{O}(z^{-\frac{\alpha}{2}}) \\ \mathcal{O}(z^{-\frac{\alpha}{2}}) & \mathcal{O}(z^{-\frac{\alpha}{2}}) \end{pmatrix}, & \frac{2\pi}{3} < |\arg z| < \pi, \end{cases} \quad \text{if } \Re\alpha > 0, \\
\Phi_{\mathrm{Be}}(z) &= \begin{pmatrix} \mathcal{O}(z^{\frac{\alpha}{2}}) & \mathcal{O}(z^{\frac{\alpha}{2}}) \\ \mathcal{O}(z^{\frac{\alpha}{2}}) & \mathcal{O}(z^{\frac{\alpha}{2}}) \end{pmatrix}, \qquad \text{if } \Re\alpha < 0.
\end{aligned} \tag{C.2.3}$$

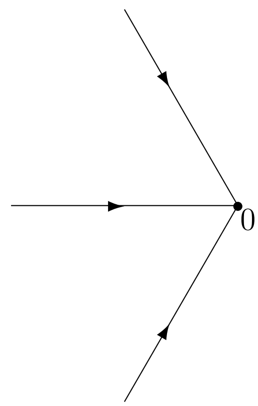

Figure C.2. The jump contour Σ_{Be} for $\Phi_{\mathrm{Be}}(\zeta)$.

This RH problem was introduced and solved in [10]. Its unique solution is given by

$$\Phi_{\mathrm{Be}}(z)=\begin{cases}\begin{pmatrix} I_\alpha(2z^{\frac{1}{2}}) & \frac{i}{\pi}K_\alpha(2z^{\frac{1}{2}}) \\ 2\pi i z^{\frac{1}{2}} I_\alpha'(2z^{\frac{1}{2}}) & -2z^{\frac{1}{2}}K_\alpha'(2z^{\frac{1}{2}}) \end{pmatrix}, & |\arg z|<\frac{2\pi}{3}, \\ \begin{pmatrix} \frac{1}{2}H_\alpha^{(1)}(2(-z)^{\frac{1}{2}}) & \frac{1}{2}H_\alpha^{(2)}(2(-z)^{\frac{1}{2}}) \\ \pi z^{\frac{1}{2}}\left(H_\alpha^{(1)}\right)'(2(-z)^{\frac{1}{2}}) & \pi z^{\frac{1}{2}}\left(H_\alpha^{(2)}\right)'(2(-z)^{\frac{1}{2}}) \end{pmatrix} e^{\frac{\pi i\alpha}{2}\sigma_3}, & \frac{2\pi}{3}<\arg z<\pi, \\ \begin{pmatrix} \frac{1}{2}H_\alpha^{(2)}(2(-z)^{\frac{1}{2}}) & -\frac{1}{2}H_\alpha^{(1)}(2(-z)^{\frac{1}{2}}) \\ -\pi z^{\frac{1}{2}}\left(H_\alpha^{(2)}\right)'(2(-z)^{\frac{1}{2}}) & \pi z^{\frac{1}{2}}\left(H_\alpha^{(1)}\right)'(2(-z)^{\frac{1}{2}}) \end{pmatrix} e^{-\frac{\pi i\alpha}{2}\sigma_3}, & -\pi<\arg z<-\frac{2\pi}{3}, \end{cases} \tag{C.2.4}$$

where $H_\alpha^{(1)}$ and $H_\alpha^{(2)}$ are the Hankel functions of the first and second kind, and I_α and K_α are the modified Bessel functions of the first and second kind.

C.3 Confluent hypergeometric model RH problem

(a) $\Phi_{\mathrm{HG}}:\mathbb{C}\setminus\Sigma_{\mathrm{HG}}\to\mathbb{C}^{2\times 2}$ is analytic, and Σ_{HG} is shown in Figure C.3.

(b) Φ_{HG} has the jump relations

$$\Phi_{\mathrm{HG},+}(z)=\Phi_{\mathrm{HG},-}(z)J_k, \qquad z\in\Gamma_k, \tag{C.3.1}$$

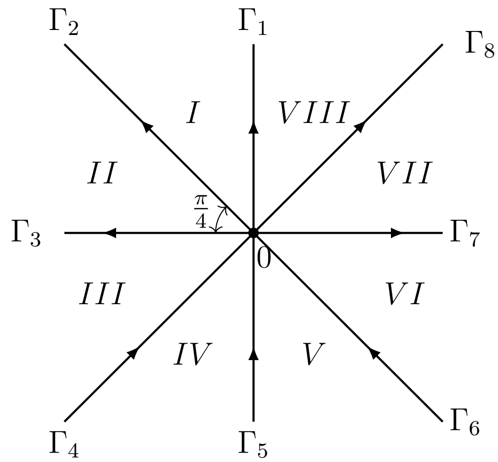

Figure C.3. The contour $\Sigma_{\mathrm{HG}} = \cup_{k=1}^{8}\Gamma_k$. Each Γ_k extends to ∞ and forms an angle $\frac{\pi}{4}$ with its adjacent ray.

where

$$J_1 = \begin{pmatrix} 0 & e^{-i\pi\beta} \\ -e^{i\pi\beta} & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 1 & 0 \\ e^{-i\pi\alpha}e^{i\pi\beta} & 1 \end{pmatrix}, \quad J_3 = J_7 = \begin{pmatrix} e^{\frac{i\pi\alpha}{2}} & 0 \\ 0 & e^{-\frac{i\pi\alpha}{2}} \end{pmatrix},$$

$$J_4 = \begin{pmatrix} 1 & 0 \\ e^{i\pi\alpha}e^{-i\pi\beta} & 1 \end{pmatrix}, \quad J_5 = \begin{pmatrix} 0 & e^{i\pi\beta} \\ -e^{-i\pi\beta} & 0 \end{pmatrix}, \quad J_6 = \begin{pmatrix} 1 & 0 \\ e^{-i\pi\alpha}e^{-i\pi\beta} & 1 \end{pmatrix}, \quad J_8 = \begin{pmatrix} 1 & 0 \\ e^{i\pi\alpha}e^{i\pi\beta} & 1 \end{pmatrix}.$$

(c) As $z \to \infty$, $z \notin \Sigma_{HG}$, we have

$$\Phi_{\mathrm{HG}}(z) = \left(I + \sum_{k=1}^{\infty} \frac{\Phi_{\mathrm{HG,k}}}{z^k} \right) z^{-\beta\sigma_3} e^{-\frac{z}{2}\sigma_3} M^{-1}(z), \tag{C.3.2}$$

where

$$\Phi_{\mathrm{HG},1} = (\beta^2 - \frac{\alpha^2}{4}) \begin{pmatrix} -1 & \tau(\alpha,\beta) \\ -\tau(\alpha,-\beta) & 1 \end{pmatrix}, \qquad \tau(\alpha,\beta) = \frac{-\Gamma(\frac{\alpha}{2} - \beta)}{\Gamma(\frac{\alpha}{2} + \beta + 1)}, \tag{C.3.3}$$

and

$$M(z)=\begin{cases} e^{\frac{i\pi\alpha}{4}\sigma_3}e^{-i\pi\beta\sigma_3}, & \frac{\pi}{2}<\arg z<\pi,\\ e^{-\frac{i\pi\alpha}{4}\sigma_3}e^{-i\pi\beta\sigma_3}, & \pi<\arg z<\frac{3\pi}{2},\\ e^{\frac{i\pi\alpha}{4}\sigma_3}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, & -\frac{\pi}{2}<\arg z<0,\\ e^{-\frac{i\pi\alpha}{4}\sigma_3}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, & 0<\arg z<\frac{\pi}{2}.\end{cases} \tag{C.3.4}$$

The factor $z^{-\beta}$ in (C.3.2), has a cut along $i\mathbb{R}^-$, such that $z^{-\beta}\in\mathbb{R}$ when $z\in\mathbb{R}^+$. As $z\to 0$, we have

$$\Phi_{\mathrm{HG}}(z)=\begin{cases}\begin{pmatrix}\mathcal{O}(1) & \mathcal{O}(\log z)\\ \mathcal{O}(1) & \mathcal{O}(\log z)\end{pmatrix}, & z\in II\cup III\cup VI\cup VII,\\ \begin{pmatrix}\mathcal{O}(\log z) & \mathcal{O}(\log z)\\ \mathcal{O}(\log z) & \mathcal{O}(\log z)\end{pmatrix}, & z\in I\cup IV\cup V\cup VIII,\end{cases}, \qquad \text{if } \Re\alpha=0,$$

$$\Phi_{\mathrm{HG}}(z)=\begin{cases}\begin{pmatrix}\mathcal{O}(z^{\frac{\Re\alpha}{2}}) & \mathcal{O}(z^{-\frac{\Re\alpha}{2}})\\ \mathcal{O}(z^{\frac{\Re\alpha}{2}}) & \mathcal{O}(z^{-\frac{\Re\alpha}{2}})\end{pmatrix}, & z\in II\cup III\cup VI\cup VII,\\ \begin{pmatrix}\mathcal{O}(z^{-\frac{\Re\alpha}{2}}) & \mathcal{O}(z^{-\frac{\Re\alpha}{2}})\\ \mathcal{O}(z^{-\frac{\Re\alpha}{2}}) & \mathcal{O}(z^{-\frac{\Re\alpha}{2}})\end{pmatrix}, & z\in I\cup IV\cup V\cup VIII,\end{cases}, \qquad \text{if } \Re\alpha>0,$$

$$\Phi_{\mathrm{HG}}(z)=\begin{pmatrix}\mathcal{O}(z^{\frac{\Re\alpha}{2}}) & \mathcal{O}(z^{\frac{\Re\alpha}{2}})\\ \mathcal{O}(z^{\frac{\Re\alpha}{2}}) & \mathcal{O}(z^{\frac{\Re\alpha}{2}})\end{pmatrix}, \qquad \text{if } \Re\alpha<0. \tag{C.3.5}$$

This model problem was introduced and solved in [5] for $\alpha=0$ and was later solved in the general case in [9] and [33]. We consider the function

$$\Psi_{\mathrm{HG}}(z;\alpha,\beta):=\begin{pmatrix}\frac{\Gamma(1+\frac{\alpha}{2}-\beta)}{\Gamma(1+\alpha)}G(\frac{\alpha}{2}+\beta,\alpha;z)e^{-\frac{\pi i\alpha}{2}} & -\frac{\Gamma(1+\frac{\alpha}{2}-\beta)}{\Gamma(\frac{\alpha}{2}+\beta)}H(1+\frac{\alpha}{2}-\beta,\alpha;ze^{-\pi i})\\ \frac{\Gamma(1+\frac{\alpha}{2}+\beta)}{\Gamma(1+\alpha)}G(1+\frac{\alpha}{2}+\beta,\alpha;z)e^{-\frac{\pi i\alpha}{2}} & H(\frac{\alpha}{2}-\beta,\alpha;ze^{-\pi i})\end{pmatrix}e^{-\frac{\pi i\alpha}{4}\sigma_3}, \tag{C.3.6}$$

where G and H are related to Whittaker functions :

$$G(a,\alpha;z) = \frac{M_{\kappa,\mu}(z)}{\sqrt{z}}, \qquad H(a,\alpha;z) = \frac{W_{\kappa,\mu}(z)}{\sqrt{z}}, \quad \mu = \frac{\alpha}{2}, \quad \kappa = \frac{1}{2} + \frac{\alpha}{2} - a. \tag{C.3.7}$$

Now, the solution to the confluent hypergeometric model Riemann-Hilbert problem is given by

$$\Phi_{\mathrm{HG}}(z) = \begin{cases} \Psi_{\mathrm{HG}} J_2^{-1}, & z \in I, \\ \Psi_{\mathrm{HG}}, & z \in II, \\ \Psi_{\mathrm{HG}} J_3, & z \in III, \\ \Psi_{\mathrm{HG}} J_3 J_4^{-1}, & z \in IV, \\ \Psi_{\mathrm{HG}} J_2^{-1} J_1^{-1} J_8^{-1} J_7^{-1} J_6, & z \in V, \\ \Psi_{\mathrm{HG}} J_2^{-1} J_1^{-1} J_8^{-1} J_7^{-1}, & z \in VI, \\ \Psi_{\mathrm{HG}} J_2^{-1} J_1^{-1} J_8^{-1}, & z \in VII, \\ \Psi_{\mathrm{HG}} J_2^{-1} J_1^{-1}, & z \in VIII, \end{cases} \tag{C.3.8}$$

where $\Psi_{\mathrm{HG}}(z) \equiv \Psi_{\mathrm{HG}}(z;\alpha,\beta)$.

VITA

Roozbeh Gharakhloo was born on June 9th, 1989 in Shiraz, Iran. He pursued professional chess during middle school, high school and college years when he received gold medals in several state and national tournaments. He studied Mechanical Engineering (Solid Mechanics) in his undergraduate studies at Shiraz University, Iran. Towards the end of his undergraduate studies, he found a deeper passion for mathematics which lead him to apply to the department of Mathematical Sciences at IUPUI. His first intention was to do research in dynamical systems, but he later changed his mind and considered working in the area of mathematical physics. His mathematical research, Under guidance of professor Alexander Its, has been focused on the asymptotic analysis of structured determinants arising in random matrix theory, statistical mechanics, theory of integrable operators and theory of orthogonal polynomials, where he primarily employs the Riemann-Hilbert techniques.