On some Hamiltonian properties of the isomonodromic tau functions

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Abstract

We discuss some new aspects of the theory of the Jimbo-Miwa-Ueno tau function which have come to light within the recent developments in the global asymptotic analysis of the tau functions related to the Painlevé equations. Specifically, we show that up to the total differentials the logarithmic derivatives of the Painlevé tau functions coincide with the corresponding classical action differential. This fact simplifies considerably the evaluation of the constant factors in the asymptotics of tau-functions, which has been a long-standing problem of the asymptotic theory of Painlevé equations. Furthermore, we believe that this observation is yet another manifestation of L. D. Faddeev’s emphasis of the key role which the Hamiltonian aspects play in the theory of integrable system.

This article will appear in the WSPC memorial volume dedicated to Ludwig Faddeev.

1 Introduction

Consider a system of linear ordinary differential equations with rational coefficients,

$$\frac{d\Phi}{dz} = A(z)\Phi,$$

where \( A(z) \) is an \( N \times N, N > 1 \) matrix-valued rational function, The object of our study is the Jimbo-Miwa-Ueno tau function associated with the isomonodromic deformation of system \([1]\). Let us remind, following [JMU], the general set-up associated with this notion.

Denote the poles of the matrix valued rational function \( A(z) \) on \( \mathbb{C} P^1 \) by \( a_1, \ldots, a_n, \infty \) and by \( r_1, r_2, \ldots, r_n, r_\infty \) the corresponding Poincaré ranks. The matrix function \( A(z) \) can be then written as,

$$A(z) = \sum_{\nu=1}^n \sum_{k=0}^{r_\nu+1} A_{\nu,-k+1} \frac{z^k A_{\infty,-k-1}}{(z-a_\nu)^k} + \sum_{k=0}^{r_\nu} z^k A_{\infty,-k-1} \in \mathfrak{sl}_N(\mathbb{C}), \quad k = 1, \ldots, r_\nu + 1, \quad \nu = 1, \ldots, n.$$

We are going to make the standard assumption that all highest order matrix coefficients \( A_\nu \equiv A_{-r_\nu} \) are diagonalizable

$$A_{-r_\nu} = G_\nu \Theta_{-r_\nu} G_\nu^{-1}, \quad \Theta_{-r_\nu} = \text{diag}\{\theta_{\nu,1}, \ldots \theta_{\nu,N}\},$$

and that their eigenvalues are distinct and non-resonant:

$$\begin{cases} 
\theta_{\nu,a} \neq \theta_{\nu,b} & \text{if } r_\nu \geq 1, \quad a \neq b, \\
\theta_{\nu,a} \neq \theta_{\nu,b} \mod Z & \text{if } r_\nu = 0, \quad a \neq b.
\end{cases}$$

At each singular point, the system \([1]\) admits a unique formal solution,

$$\Phi_{\nu,\text{form}}(z) = G^{(\nu)}(z) e^{\Theta^{(\nu)}(z)}, \quad \nu = 1, \ldots, n, \infty, \quad (2)$$

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where $G^{(v)}(z)$ are formal series,
\[
G^{(v)}(z) = G_v \left[ I + \sum_{k=1}^{\infty} g_{v,k}(z-a_v)^k \right], \quad G^{(\infty)}(z) = G_\infty \left[ I + \sum_{k=1}^{\infty} g_{\infty,k}z^k \right],
\]
and $\Theta_v(z)$ are diagonal matrix-valued functions,
\[
\Theta_v(z) = \sum_{k=\nu}^{0} \frac{\Theta_{v,k}}{k}(z-a_v)^k + \Theta_{v,0} \ln(z-a_v), \quad \Theta_{\infty}(z) = -\sum_{k=1}^{\infty} \frac{\Theta_{\infty,k}}{k}z^k - \Theta_{\infty,0} \ln z.
\]
For every $v \in \{1, \ldots, n, \infty\}$, the matrix coefficients $g_{v,k}$ and $\Theta_{v,k}$ can be explicitly computed in terms of the coefficients of the matrix-valued rational function $G_v^{-1} A(z) G_v$, see [MU].

The non-formal global properties of solutions of the equation (1) are described by its monodromy data $M$ which include: i) formal monodromy exponents $\Theta_{v,0}$, ii) appropriate connection matrices between canonical solutions at different singular points, and iii) relevant Stokes matrices at irregular ($r_v \geq 1$) singular points. Let us denote the space of monodromy data of the system (1) by $\mathcal{M}$. It can be described in more details as follows.

Let $a_v$ be an irregular singular point of index $r_v$. For $j=1, \ldots, 2r_v + 1$, let also
\[
\Omega_{j,v} = \left\{ z : 0 < |z-a_v| < \epsilon, \quad \theta^{(1)}_j < \arg(z-a_v) < \theta^{(2)}_j, \quad \theta^{(1)}_j - \theta^{(2)}_j = \frac{\pi}{r_v} + \delta \right\}
\]
be the Stokes sectors around $a_v$ (see, e.g., [FIKN Chapter 1] or [Was] for more details). According to the general theory of linear systems, in each sector $\Omega_{j,v}$ there exists a unique canonical solution $\Phi_{j,v}^{(1)}(z)$ of (1) which satisfies the asymptotic condition
\[
\Phi_{j,v}^{(1)}(z) \approx \Phi_{v,\text{form}}^{(1)}(z) \quad \text{as} \quad z \to a_v, \quad z \in \Omega_{j,v}, \quad j = 1, \ldots, 2r_v + 1.
\]
Different canonical solutions are related by Stokes matrices, $S_{j,v}^{(v)}$, and connection matrices, $C_v$:
\[
\Phi_{j+1,v}^{(v)} = \Phi_{j,v}^{(v)} S_{j,v}^{(v)}, \quad j = 1, \ldots, 2r_v, \quad \Phi_1^{(v)} = \Phi_1^{(\infty)} C_v, \quad v = 1, \ldots, n.
\]
Let us assume that the irregular singular points are $\infty$ and the first $n_0 \leq n$ points among the singular points $a_1, \ldots, a_n$. Denote by $\mathcal{R}_v$ the collection of Stokes matrices at an irregular point $a_v$, i.e.
\[
\mathcal{R}_v = \left\{ S_1^{(v)}, \ldots, S_{2r_v}^{(v)} \right\}.
\]
The space $\mathcal{M}$ of monodromy data of the system (1) consists of formal monodromy exponents $\Theta_{v,0}$, connection matrices $C_v$ and Stokes matrices $S_{j,v}^{(v)}$, i.e.,
\[
\mathcal{M} = \left\{ M \equiv \left( \Theta_{v,0}, v = 1, \ldots, n, \infty; \quad C_v, v = 1, \ldots, n; \quad \mathcal{R}_v, v = 1, \ldots, n_0, \infty \right) \right\}.
\]
We shall use the notation,
\[
\tilde{m} = (m_1, \ldots, m_d), \quad d = N(n+1) + nN^2 + \left( \frac{N(N-1)}{2} \right) \left( \sum_{i=1}^{n_0} 2r_v + 2r_\infty \right),
\]
for the points $\tilde{m} \in \mathcal{M}$. In addition, we denote by $\mathcal{F}$ the set of times,
\[
a_1, \ldots, a_n, \quad \left( \Theta_{v,k} \right)_{l \nu}, \quad k = -r_v, \ldots, -1, \quad v = 1, \ldots, n_0, \infty, \quad l = 1, \ldots, N.
\]
We shall use the notation,
\[
\tilde{t} = (t_1, \ldots, t_L), \quad L = n + N \left( \sum_{v=1}^{n_0} r_v + r_\infty \right),
\]
for the points $\tilde{t} \in \mathcal{F}$. Let us also denote by $\mathcal{A}$ the variety of all rational matrix-valued functions $A(z)$ with a fixed number of poles of fixed orders. The so-called Riemann-Hilbert correspondence states that, up to submanifolds where the inverse monodromy problem for (1) is not solvable, the space $\mathcal{A}$ can be identified with the product $\mathcal{F} \times \mathcal{M}$, where $\mathcal{F}$ denotes the universal covering of $\mathcal{F}$. We shall loosely write
\[
\mathcal{A} \equiv \tilde{\mathcal{F}} \times \mathcal{M}.
\]
It should be mentioned that in each concrete case one has to specify the gauge normalization of the matrix $A(z)$ as well as the choice of the gauge matrices $G_r$ in order to make this identification well defined. In Section 3 we will demonstrate how these specifications can be done in the case of Painlevé equations.

The Jimbo-Miwa-Ueno 1-form is defined as the following differential form on $\mathcal{A}$:

$$\omega_{\text{JMU}} = - \sum_{v=1}^{n} \text{res}_{z=a_v} \text{Tr} \left( G^{(v)}(z)^{-1} \frac{dG^{(v)}}{dz}(z) \, d_{\mathcal{I}} \Theta_v(z) \right).$$

(3)

where $G^{(v)}(z)$ are the series from $\mathcal{A}$ and we put $a_{\infty} = \infty$. The notation $d_{\mathcal{I}} \Theta_v(z)$ stands for

$$d_{\mathcal{I}} \Theta_v(z) = \sum_{k=1}^{L} \frac{\partial \Theta_v(z)}{\partial t_k} \, dt_k, \quad L = n + N \left( \sum_{v=1}^{m} r_v + r_{\infty} \right).$$

The significance of this form is that, being restricted to any isomonodromic family in the space $\mathcal{A}$,

$$A(z) \equiv A(z; \tilde{t}, M), \quad \tilde{t} = (t_1, \ldots, t_L), \quad M \equiv \text{const}$$

it becomes closed with respect to times $\mathcal{I}$, i.e.

$$d_{\mathcal{I}} \left( \omega_{\text{JMU}} \big|_{A(z; \tilde{t}, M=\text{const})} \right) = 0.$$ 

The closedness of the 1-form $\omega_{\text{JMU}}$ with respect to $\mathcal{I}$ in turn implies that locally there is a function $\tau \equiv \tau (\tilde{t}; M)$ on $\mathcal{I} \times \mathcal{M}$ such that

$$d_{\mathcal{I}} \ln \tau = \omega_{\text{JMU}} \big|_{A(z; \tilde{t}, M)}.$$ 

(4)

A remarkable property of the tau function $\tau (\tilde{t}, M)$, which was established in [Mal] and [Miw], is that it admits analytic continuation as an entire function to the whole universal covering $\tilde{\mathcal{I}}$ of the parameter space $\mathcal{I}$. Furthermore, zeros of $\tau (\tilde{t}, M)$ correspond to the points in $\mathcal{I}$ where the inverse monodromy problem for $\mathcal{A}$ is not solvable for a given set $M$ of monodromy data (or, equivalently, where the holomorphic vector bundle over $\mathbb{C}P^1$ determined by $M$ becomes nontrivial). Hence a central role of the concept of tau function in the monodromy theory of systems of linear differential equations.

The isomonodromicity of the family $A(z; \tilde{t}, M)$ means that all equations from it have the same set $M \in \mathcal{M}$ of monodromy data. This implies that the corresponding solution $\Phi(z) \equiv \Phi(z; \tilde{t})$ satisfies an overdetermined system

$$\begin{cases} 
\partial_z \Phi = A(z, \tilde{t}) \Phi(z, \tilde{t}), \\
\partial_{\mathcal{I}} \Phi = B(z, \tilde{t}) \Phi(z, \tilde{t}). 
\end{cases}$$

(5)

The coefficients of the matrix-valued differential form $B \equiv \sum_{k=1}^{L} B_k(z, \tilde{t}) \, dt_k$ are rational in $z$. Their explicit form may be algorithmically deduced from the expression for $A(z)$ (see again [IMU]). The compatibility of the system $\mathcal{A}$ yields the monodromy preserving deformation equation:

$$d_{\mathcal{I}} A = \partial_z B + [B, A].$$

(6)

Isomonodromy equation $\mathcal{A}$ is of great interest on its own. Indeed, it includes as special cases practically all known integrable differential equations. The first nontrivial cases of $\mathcal{A}$, where the set of isomonodromic times effectively reduces to a single variable $t$, cover all six classical Painlevé equations. Solutions of the latter are dubbed as nonlinear special functions, and they indeed play this role in many areas of modern nonlinear science (see [FKN], [BK], [DS], [GM], [TW1], [TW2]).

The principal analytic issue concerning the tau function, in particular from the point of view of applications, is its behavior near the critical hyperplanes, where either $a_\mu = a_\nu$ for some $\mu \neq \nu$, or $\theta_{\nu, \alpha} = \theta_{\nu, \beta}$ for some $\nu$ and some $\alpha \neq \beta$. In the case of Painlevé equations this is the behavior of respective tau functions near the $t = \infty$ (PI, II, IV), $t = \infty, 0$ (PIII, V), and $t = \infty, 0, 1$ (PVI). A special challenge in the asymptotic analysis of the tau functions is the evaluation of the constant pre-factors in their asymptotics. In fact, it is these pre-factors which usually contain the most important information about the physical properties of the model under investigation. At the same time, they can not be obtained directly via the Riemann-Hilbert approach. The latter method is one of the principal modern tools of the asymptotic analysis of Painlevé transcendents, and it is
based on the asymptotic evaluation of the above mentioned Riemann-Hilbert correspondence. In other words, the Riemann-Hilbert technique allows to evaluate the asymptotics of the matrix $A(z, \vec{t})$ and hence the asymptotics of the differential form $\omega_{JMU}$. In view of (4), this gives the asymptotics of the logarithmic derivatives of the tau function. In order to obtain the complete asymptotic description of the tau function itself, which would include the above mentioned pre-factors, one has to solve the "constant problem": to find the constant of integration arising from the formal integration of (4). More precisely, since the tau function is itself defined up to a multiplicative constant, we are actually talking about the evaluation, in terms of monodromy data, of the ratios of constant factors corresponding to different critical points (Painlevé III, V, VI) or to different critical directions (Painlevé I, II, IV).

The first rigorous solution of a constant problem for Painlevé equations (a special Painlevé III transcendent appearing in the Ising model) has been obtained in the work of C. Tracy [Tr]. After that, several other important special cases have been also solved. We refer the reader to [LLP] for a detailed history of the question. It is important to emphasize that all these works were concerned with the very special families of the Painlevé functions, and they used the techniques which could not be extended to the generic tau functions.

The means to solve the "constant problem" for tau functions corresponding to the generic solutions of Painlevé equations started to develop since the 2013-2014 works [ILT13], [ILST] of Iorgov, Lisovyy, Shchechkin, and Tykhyy where a very important discovery of the conformal block interpretation of tau functions was made. For the history of the question, we refer the reader to the paper [LP] where the heuristic though truly pioneering results of [ILT13], [ILST] have been rigorously proven. Another conjectural pre-factor formula, this time concerning the third Painlevé equation (work [ILT14]), was proven in [IP]. Later on, the method of [LP] and [IP] was successfully applied to the first Painlevé equation in [LR].

The method of [LP] and [IP] is inspired by the earlier works of B. Malgrange [Mal] and Bertola [Ber] and it is based on an extension of the Jimbo-Miwa-Ueno differential form $\omega_{JMU}$ to a 1-form, $\omega$ on the whole space $\mathcal{A} = \tilde{T} \times \mathcal{M}$,

$$\omega = \sum_{k=1}^{L} P_k(\vec{t}, M)d t_k + \sum_{j=1}^{d} Q_j(\vec{t}, M)d m_j,$$

such that

$$\omega(\partial_{t_k}) = \omega_{JMU}(\partial_{t_k}),$$

and the exterior differential of $\omega$, i.e., the form,

$$\Omega_0 := d\omega,$$

is a 2-form on $\mathcal{M}$ only. Furthermore, it is independent of isomonodromic times $\mathcal{T}$. The construction of the form $\omega$ will be described in detail in the next section.

The time-independence of the 2-form $\Omega_0$ in conjunction with the Riemann-Hilbert computability of the asymptotics of $\Phi(z)$ determines what should be added to the form $\omega$ to make it closed, i.e. to transform it into the form $\hat{\omega}$ which satisfies the two crucial properties:

$$d\hat{\omega} = d_{\mathcal{T}} \hat{\omega} + d_{\mathcal{M}} \hat{\omega} = 0, \quad \text{and} \quad \hat{\omega}(\partial_{t_k}) = \omega_{JMU}(\partial_{t_k}).$$

Having the form $\hat{\omega}$, the tau function can be represented as

$$\ln \tau = \int \hat{\omega}.$$  

Equation (8) allows one to use the asymptotic behavior of $\Phi(z)$ to evaluate the asymptotics of the associated tau function up to a numerical (i.e. independent of monodromy data) constant. The latter can be calculated by applying the final formulae to trivial solutions of deformation equations. This program has been first realized in [IP] for the sine-gordon reduction of the Painlevé III equation and later on in [LP] for Painlevé VI and II equations and in [LR] for Painlevé I equation.

We want also mention the most recent work [GL] where a general Fredholm determinant formula was found for the Painlevé VI tau function which allows to produce rigorously both the evaluation of the relevant asymptotic constants and the combinatorial series expansions of the tau function at the critical points.

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3 The exact relation of the form $\omega$ to the original Malgrange-Bertola 1-form is explained in detail in [LP] - see Remark 4.4 there.
In the course of the asymptotic analysis performed in [IP] and [ILP], an interesting observation has been made with respect to the form \( \omega \) in the Painlevé III and II cases. This observation is concerned with the Hamiltonian aspect of the theory of isomonodromic deformations which we have not yet discussed. As a matter of fact, the space \( \mathcal{A} \) can be equipped with a symplectic structure - see [H2], [Boal], [Kri], [B], so that the isomonodromic equation (6) induces \( L \) commuting Hamiltonian flows on \( \mathcal{A} \). A striking property of the tau function is that in many (though not all) known special cases its logarithm serves as the generating function of the Hamiltonians \( H_k \) of these flows:

\[
\frac{\partial \ln \tau(M)}{\partial t_k} = \gamma H_k|_{A(z,\tilde{t},M)},
\]

Here \( \gamma \) is numerical constant (in many cases, \( \gamma = 1 \)). This fact for the fourth, fifth and sixth Painlevé equations as well as for many higher rank isomonodromic systems was established in [Boal2] where also a generalization of the JMU form allowing repeated eigenvalues was worked out. The Hamiltonian formalism for all six Painlevé equations was first suggested by K. Okamoto [O].

The above mentioned observation of [IP] and [ILP] is that in the Painlevé II and III cases the 2-form \( \Omega_0 = d\omega \) is nothing else but, up to a numerical factor, the corresponding symplectic form. Hence, in these examples, the 1-form \( \omega \), up to a numerical factor and the addition of an explicit total differential, is an extension to the space \( \Sigma \times \mathcal{M} \) of the differential of classical action; moreover, in [IP] and [ILP] these total differentials have been explicitly found. Similar relation to the classical action in the case of the Painlevé I tau-function has been obtained in [LR], and in the case of the Schlesinger equations - the pure Fuchsian system (1), in [Mal2].

The goal of this paper is to show that the relation between the tau function and the classical action established in [IP], [ILP], and [LR] for the special cases of Painlevé III, II, and I is true for all Painlevé equations. We shall also present some arguments allowing one to expect that this relation is, most likely, a general fact of the monodromy theory of linear systems.

The detailed construction of the form \( \omega \) is given, following [ILP], in the next section. In this section we also provide the arguments in favor of the connection between the form \( \omega \) and the classical action in the general case of linear system (1) and formulate two conjectures concerning with this connection. In Section 3 these conjectures are justified for all six Painlevé equations and for an arbitrary Schlesinger system. In the cases of Painlevé I and Schlesinger equations we just reproduce the results of [LR] and [Mal2], respectively.

### 2 The extended Jimbo-Miwa-Ueno differential and the classical action functional.

As in introduction, we shall, unless the otherwise is explicitly indicated, be treating all the objects which are defined on \( \mathcal{A} = \Sigma \times \mathcal{M} \) as functions of \((\tilde{t}, M) \equiv (\tilde{t}, \tilde{m})\). In particular,

\[
\omega_{JMU} = \omega_{JMU}(\tilde{t}, M) = \omega_{JMU}|_{A(z,\tilde{t},M)},
\]

and for any function \( F \) on \( \mathcal{A} \), the partial derivatives with respect to \( t_k \) will mean the partial derivatives of \( F \) as a function of \((\tilde{t}, M)\), i.e.,

\[
\frac{\partial F}{\partial t_k} = \frac{\partial}{\partial t_k} F(\tilde{t}, M) = \frac{\partial}{\partial t_k} \left( F|_{A(z,\tilde{t},M)} \right).
\]

We will also use the notations

\[
dF \equiv dF(\tilde{t}, M) = \sum_{k=1}^{L} \frac{\partial F}{\partial t_k} dt_k + \sum_{k=1}^{d} \frac{\partial F}{\partial m_k} dm_k \equiv d_\Sigma F + d_\mathcal{M} F.
\]

Our starting point is the following Lemma.

**Lemma 1.** ([JMU]) The 1-form (3) (considered as a 1-form on \( \Sigma \times \mathcal{M} \)) can be alternatively written as

\[
\omega_{JMU} = \sum_{k=1}^{L} \sum_{v=1,\ldots,n,\infty} \text{res}_{z=a_v} \text{Tr} \left( A \frac{\partial G^{(v)}}{\partial t_k} (G^{(v)})^{-1} \right) dt_k \equiv \sum_{v=1,\ldots,n,\infty} \text{res}_{z=a_v} \text{Tr} \left( A d_\Sigma G^{(v)} (G^{(v)})^{-1} \right).
\]

4 The statement actually depends on the specific choice of the symplectic structure.
5 The authors are grateful to Marta Mazzocco for pointing out this result of B. Malgrange to us.
We give the version [ILP] of the proof of this Lemma in Section 4.1 of the Appendix. A direct corollary of Lemma [ILP] is the following integral formula for the tau function,

$$\ln \tau = \ln \tau (\tilde{t}_1, \tilde{t}_2, M) = \int \sum_{n, k} \text{res}_{z = a_n} \text{Tr} \left( A \frac{\partial G^{(v)}}{\partial t_k} (G^{(v)})^{-1} \right) dt_k. \quad (11)$$

Another consequence of the Lemma is the idea to take formula (10) as the motivation to introduce the following 1-form (cf. [ILP], [IP])

$$\omega = \sum_{k=1}^{L} \sum_{v=1,...,n,\infty} \text{res}_{z = a_v} \text{Tr} \left( A \frac{\partial G^{(v)}}{\partial t_k} (G^{(v)})^{-1} \right) dt_k + \sum_{k=1}^{d} \sum_{v=1,...,n,\infty} \text{res}_{z = a_v} \text{Tr} \left( A \frac{\partial G^{(v)}}{\partial m_k} (G^{(v)})^{-1} \right) dm_k \equiv \sum_{v=1,...,n,\infty} \text{res}_{z = a_v} \text{Tr} \left( A dG^{(v)} (G^{(v)})^{-1} \right) \equiv \sum_{v=1,...,n,\infty} \text{res}_{z = a_v} \text{Tr} \left( A dG^{(v)} (G^{(v)})^{-1} \right). \quad (12)$$

Now, the key observation.

**Lemma 2** [ILP]. The form $d\omega$ has no cross terms of the kind $dt_k \wedge dm_j$, \quad $k = 1, \ldots, L, \quad j = 1, \ldots, d$.

We present, following [ILP], the proof of this Lemma in Section 4.2 of the Appendix.

Lemma 2 plays a crucial role in the mentioned in the Introduction rigorous approach to the "constant problem". Indeed, a key issue in the determining of the monodromy dependence of the tau function is the possibility that $\omega$ is used in (11) with respect to the monodromy parameters $M_j$. Lemma 2 implies that

$$\frac{\partial}{\partial m_j} \sum_{v} \text{res}_{z = a_v} \text{Tr} \left( A \frac{\partial G^{(v)}}{\partial t_k} (G^{(v)})^{-1} \right) = \frac{\partial}{\partial t_k} \sum_{v} \text{res}_{z = a_v} \text{Tr} \left( A \frac{\partial G^{(v)}}{\partial m_j} (G^{(v)})^{-1} \right).$$

Therefore,

$$\frac{\partial \ln \tau}{\partial m_j} = \int \frac{\tilde{t}_2}{\tilde{t}_1} \sum_{k=1}^{L} \frac{\partial}{\partial m_j} \sum_{v} \text{res}_{z = a_v} \text{Tr} \left( A \frac{\partial G^{(v)}}{\partial t_k} (G^{(v)})^{-1} \right) dt_k$$

$$= \sum_{k=1}^{L} \int \frac{\tilde{t}_2}{\tilde{t}_1} \frac{\partial}{\partial t_k} \sum_{v} \text{res}_{z = a_v} \text{Tr} \left( A \frac{\partial G^{(v)}}{\partial m_j} (G^{(v)})^{-1} \right) dt_k = \sum_{v} \text{res}_{a_v} \text{Tr} \left( A \frac{\partial G^{(v)}}{\partial m_j} (G^{(v)})^{-1} \right) \int \frac{\tilde{t}_2}{\tilde{t}_1}. \quad (13)$$

In other words, we conclude that in addition to the differential relation (4), i.e.,

$$d_{\tilde{t}} \ln \tau = \sum_{v=1,...,n,\infty} \text{res}_{z = a_v} \text{Tr} \left( G^{(v)} (z) A(z) d_{\tilde{t}} G^{(v)} (z) \right),$$

the tau function satisfies the differential relation,

$$d_{\tilde{m}} \ln \tau = \sum_{v=1,...,n,\infty} \text{res}_{z = a_v} \text{Tr} \left( G^{(v)} (z) A(z) d_{\tilde{m}} G^{(v)} (z) \right).$$

These two differential identities allow to evaluate the asymptotic connection formulae up to the numerical constants and this is what is effectively done in [IP], [ILP].

The arguments which led to the representation (13) for the logarithmic derivative of the tau function with respect to $m_j$ are reminiscent to the variational equations for the classical action. Let us assume that we can...
identify the classical Darboux coordinates $p_j, q_j$ on the space $\mathcal{A}$ so that the isomonodromic deformation equations \( \text{[4]} \) can be written as the commuting system of Hamiltonian dynamical equations,\( \text{[3]} \)

\[
\frac{\partial q_j}{\partial t_k} = \frac{\partial H_k}{\partial p_j}, \quad \frac{\partial p_j}{\partial t_k} = -\frac{\partial H_k}{\partial q_j}.
\]

(14)

We remind that we are still identify $\mathcal{A} \cong \tilde{T} \times \mathcal{M}$, so that we consider $p_j$ and $q_j$ as the functions on $\tilde{T} \times \mathcal{M}$.

$q_j \equiv q_j(\vec{t}, M), \quad p_j \equiv p_j(\vec{t}, M), \quad H_k \equiv H_k(q_j(\vec{t}, M), p_j(\vec{t}, M), \vec{t})$.\( \text{[6]} \)

The compatibility of the system (14) means (see, e.g., \[B\]) that all

\[
c_kl := \{H_k, H_l\} + \frac{\partial H_k}{\partial t_l} - \frac{\partial H_l}{\partial t_k}
\]

are the Casimir functions\( \text{[8]} \)(maybe depending on the times $t_k$). We shall assume that

\[
c_kl = 0 \quad \forall \ k, l. \quad \text{(15)}
\]

This assumption works for all example of the isomonodromic deformations equations that we know. The classical action differential can be defined as the differential form on $\tilde{T} \times \mathcal{M}$,

\[
\omega_{\text{cla}} = \sum p_j dq_j - \sum H_k dt_k \equiv \sum_k \left( \sum_j p_j \frac{\partial q_j}{\partial t_k} - H_k \right) dt_k + \sum_k \left( \sum_j p_j \frac{\partial q_j}{\partial m_k} \right) dm_k
\]

and, using assumption (15), it is easy to check that it is closed on the trajectories of the dynamical system (14), i.e.,

\[
d_T (\omega_{\text{cla}} |_{M=\text{const}}) = 0.
\]

Note that in those cases when the logarithm of the tau function is the generating function for the Hamiltonians $H_k$, the Jimbo-Miwa-Ueno differential form is

\[
\omega_{\text{JMU}} = \sum H_k dt_k,
\]

so that the integral \( \text{[11]} \) is the truncated action integral,

\[
\ln \tau = \int_{\vec{t}_1}^{\vec{t}_2} \sum_k H_k dt_k.
\]

Suppose that instead of this integral we need to study the complete action, i.e. the integral,

\[
S \equiv S(\vec{t}_1, \vec{t}_2, M) = \int_{\vec{t}_1}^{\vec{t}_2} \omega_{\text{cla}}(M) \equiv \int_{\vec{t}_1}^{\vec{t}_2} \sum_k \left( \sum_j p_j \frac{\partial q_j}{\partial t_k} - H_k \right) dt_k.
\]

\( \text{[7]} \)

The Darboux coordinates on the phase spaces $\mathcal{A}$ corresponding to Painlevé equations are introduced in \[H1\], \[B\]; the Darboux coordinates for more general cases of the isomonodromic deformation equations are considered in \[H2\].

\( \text{[6]} \)

In the special case of Painlevé and Schlesinger equations, which are our principal concern, their Hamiltonian representations \( \text{[13]} \) is described in all details in the main body of the paper. The interested reader can be referred to section 5 of \[Boal2\] for the general definition of a time-dependent Hamiltonian system in the context of isomonodromy setting. We notice that this general definition is a delicate issue since the original parametrization of the extended phase space $\mathcal{A}$ mixes the time and dynamical parameters.

\( \text{[8]} \)

Warning: here,

\[
\frac{\partial H_k}{\partial t_j} = \frac{\partial}{\partial t_j} |_{\vec{t} = \text{const}} \left( H_k(p, \vec{q}, \vec{t}) |_{\vec{t} = \text{const}} \right).
\]

\( \text{[7]} \)
Then, the usual variational calculus arguments show that, similar to (13), in any \( m_j \)-derivative of \( S \) the integral terms would disappear. In fact, we have,

\[
\frac{\partial S}{\partial m_{j_0}} = \int_{\tilde{t}_1}^{\tilde{t}_2} \sum_k \left( \sum_j \frac{\partial p_j}{\partial m_{j_0}} \frac{\partial q_j}{\partial t_k} + p_j \frac{\partial^2 q_j}{\partial t_k \partial m_{j_0}} - \frac{\partial H_k}{\partial p_j} \frac{\partial p_j}{\partial m_{j_0}} - \frac{\partial H_k}{\partial q_j} \frac{\partial q_j}{\partial m_{j_0}} \right) \, dt_k
\]

\[
= \sum_{j,k} p_j \frac{\partial q_j}{\partial m_{j_0}} \bigg|_{\tilde{t}_1}^{\tilde{t}_2} + \int_{\tilde{t}_1}^{\tilde{t}_2} \left( \sum_k \frac{\partial p_j}{\partial m_{j_0}} \frac{\partial q_j}{\partial t_k} - \frac{\partial H_k}{\partial p_j} \frac{\partial p_j}{\partial m_{j_0}} - \frac{\partial H_k}{\partial q_j} \frac{\partial q_j}{\partial m_{j_0}} \right) \, dt_k
\]

and the integral term vanishes because of the equations of motion (14). Comparison (13) and (16) makes one to suspect some deep connection between the tau function and the classical action. And, indeed, taken the full exterior derivation of \( \omega_{\text{cla}} \equiv \omega_{\text{cla}}(\vec{t}, M) \), one obtains,

\[
d\omega_{\text{cla}} = \sum_i d\omega p_j \wedge d\omega q_j \equiv \Omega.
\]

The form \( \Omega \) is the symplectic form associated with the dynamical system (14). Note, that both, the form \( \Omega \) and the form \( \Omega_0 \) from (7) are the closed 2-form on \( \mathcal{M} \) and they do not depend on the times \( \mathcal{T} \). This observation together with the similarities of the variational identities (19) and (16) allow us to formulate the following conjectures.

**Conjecture 1.** Suppose that the parameter space \( \mathcal{A} \) is equipped with the symplectic structure. Let \( \Omega \) be a corresponding symplectic form and \( \Omega_0 \) be the two-form defined in (7). Then, there exists a numerical constant \( \gamma \) such that,

\[
\Omega_0 = \gamma \Omega.
\]

If this conjecture is true then the two 1-forms, \( \omega \) and \( \omega_{\text{cla}} \), coincide up to the total differential. Hence our next conjecture

**Conjecture 2.** There exists a rational function \( G(\vec{p}, \vec{q}, \vec{t}) \) of \( \vec{p}, \vec{q}, \vec{t} \) such that,

\[
\omega = \gamma \omega_{\text{cla}} + dG(\vec{p}, \vec{q}, \vec{t}).
\]

Moreover, the function \( G(\vec{p}, \vec{q}, \vec{t}) \) is explicitly computable.

As it has already been indicated in the Introduction, the statement of Conjecture 2 has been proven to be true in the case of the Schlesinger equations [Mal2], in the case of the sine-gordon reduction of Painlevé III equation [OP], in the case of the (homogenous) Painlevé II equation [LP], and in the case of the Painlevé I equation [LE]. In the next section of this paper we demonstrate the validity of the both conjectures for the rest of the Painlevé equations, and we also present, for completeness, the result of [Mal2].

**Remark 3.** Restricting (17) to the isomonodromic family \( M \equiv \text{const} \), one arrives to the identity

\[
\sum_k \frac{\partial \ln r}{\partial t_k} \, dt_k = \sum_k \left( \sum_j p_j(\vec{t}, M) \frac{\partial q_j(\vec{t}, M)}{\partial t_k} - H_k(\vec{p}(\vec{t}, M), \vec{q}(\vec{t}, M), \vec{t}) \right) \, dt_k
\]

\[
+ \sum_k \frac{\partial}{\partial t_k} G(\vec{p}(\vec{t}, M), \vec{q}(\vec{t}, M), \vec{t}) \, dt_k.
\]

and hence,

\[
\ln r(\vec{t}_1, \vec{t}_2, M) = S(\vec{t}_1, \vec{t}_2, M) + G(\vec{p}(\vec{t}, M), \vec{q}(\vec{t}, M), \vec{t}) \bigg|_{\vec{t}_1}^{\vec{t}_2}.
\]
This, in turn, would produce, taking into account (16), the following, alternative to (13), formula for the \( m_j \)-derivative of \( \ln \tau \):

\[
\frac{\partial \ln \tau}{\partial m_{j0}} = \sum_{j,k} p_{j} \frac{\partial q_{j}}{\partial m_{j0}} \bigg|_{\vec{t}_1} + \frac{\partial G}{\partial m_{j0}} \bigg|_{\vec{t}_1}.
\]

This version of the variational logarithmic derivatives of the tau functions turns out even more efficient then (13) in the concrete examples related to the "constant problem". Indeed, the particular cases of (20) have been used in [BIP] in evaluation of the constant terms in the asymptotics of the several basic distribution functions of random matrix theory expressible in terms of the Painlevé transcendents.

Remark 4. In the pioneering papers [ILST] and [ILT], the evaluation of the constant pre-factors has been partially based on the conjectural interpretations of these constant factors as the generating functions of the canonical transformations between the Darboux asymptotic coordinates associated with the different critical points. This generating function interpretation of the constant pre-factors, which in the case of Painlevé II, III and I has been proven in [IP, ILP] and [LR], can be considered as a direct corollary of (19).

3 Painlevé equations

In this section we present the exact realization of the relations (17) – (18) between the tau-function and the classical action for all six Painlevé equations and also for the case of the general Schlesinger equation. We shall start with the second Painlevé equation which we take as a case study and illustrate in detail the general constructions of Section 2. In particular, we will describe exactly the both spaces, \( A \) and \( M \) corresponding to the Painlevé II equation. The other Painlevé equations will be treated with less details. We won’t describe the space \( M \) for other Painlevé equations explicitly, and instead we will refer the reader either to [JM] or to Chapter 5 of [FIKN].

3.1 Painlevé II

According to [JM], the second Painlevé equation describes the isomonodromic deformations of the \( 2 \times 2 \) linear system having only one irregular singular point at \( z = \infty \) of the Poincaré rank 3,

\[
\frac{d\Phi}{dz} = A(z) \Phi, \quad A(z) = A_2 z^2 + A_1 z + A_0.
\]

Following again [JM], we normalize the system by the conditions,

\[
\text{Tr } A(z) \equiv 0, \quad A_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_{11} = A_{12} = 0,
\]

so that the matrix coefficients \( A_1 \) and \( A_0 \) can be written in the form,

\[
A_1 = \begin{pmatrix} 0 & k \\ -2p & 0 \end{pmatrix}, \quad A_0 = \begin{pmatrix} p + \frac{1}{2} & -kq \\ -\frac{1}{2}(\theta + pq) & -p - \frac{1}{2} \end{pmatrix},
\]

and the complex parameters \( p, q, k, \theta \), and \( t \) can be taken as the original coordinates on the corresponding space \( D \),

\[
D = \{(p, q, k, \theta, t)\}.
\]

The formal solution of system (21) at its only (irregular) singular point, \( z = \infty \), has the structure (cf. 22),

\[
\Phi_{\text{form}}(z) = G(z)e^{\Theta(z)} = \left(1 + \frac{g_1}{z} + \frac{g_2}{z^2} + \frac{g_3}{z^3} + \ldots\right)e^{\Theta(z)}, \quad \Theta(z) = \sigma_3 \left(\frac{z^3}{3} + \frac{iz}{2} - \theta \ln z\right),
\]

with the first three matrix coefficients \( g_k, k = 1, 2, 3 \) given as functions on the space \( D \) by the explicit formulae,

\[
g_1 = \begin{pmatrix} -H & -\frac{k}{2} \\ -\frac{p}{x} & H \end{pmatrix},
\]

\[
g_2 = \begin{pmatrix} -\frac{p}{x} & -\frac{k}{2} \\ -H & H \end{pmatrix},
\]

\[
g_3 = \begin{pmatrix} -\frac{p}{x} & -\frac{k}{2} \\ -H & H \end{pmatrix}.
\]
Equation (23), also tells us that the parameter \( t \) where \( Lax \) pair (21) we have, \( \Phi \)

The system (21) has seven canonical solutions, characterized by the asymptotic condition,

\[ \Phi_j(z) = \Phi_{form}(z), \quad z \to \infty, \quad \frac{(2j-5)\pi}{6} < \arg z < \frac{(2j-1)\pi}{6}, \quad \Phi_T(z) = \Phi_1(z)e^{-2\pi i \theta_3}. \]

and hence it has six Stokes matrices, \( S_j = \Phi_j^{-1}(z)\Phi_{j+1}(z) \), which have the following triangular structure (for more detail see [JM] or Chapter 5 of [FIKN]),

\[ S_{2k+1} = \begin{pmatrix} 1 & 0 \\ s_{2k+1} & 1 \end{pmatrix}, \quad S_{2k} = \begin{pmatrix} 1 & s_{2k} \\ 0 & 1 \end{pmatrix}, \]

and satisfy one cyclic relation,

\[ S_1 S_2 ... S_6 = e^{-2\pi i \theta_3}. \]

Also, as it follows from (22), the parameter \( \theta \) determines the formal monodromy exponent, \( \Theta_0 = \theta \sigma_3. \) This means, that the space \( \mathcal{M} \) in the case of system (21) can be identified with the algebraic variety of dimension 4,

\[ \mathcal{M} = \{ \hat{\theta} = (s_1, s_2, ..., s_6, \theta) : 1 + s_1 s_2 = (1 + s_4 s_5) e^{2\pi i \theta}, 1 + s_2 s_3 = (1 + s_5 s_6) e^{-2\pi i \theta}, s_1 + s_3 + s_1 s_2 s_3 = -s_5 e^{2\pi i \theta} \}. \]

Equation (23), also tells us that the parameter \( t \) is the only isomonodromic time, so that, in the case of system (21) we have,

\[ \mathcal{T} = \{ t \}. \]

The isomonodromic deformations of system (21), i.e., the conditions,

\[ p = p(t, M), \quad q = q(t, M), \quad k = k(t, M), \quad M = \text{const}, \]

yield the second linear matrix differential equation, this time with respect to \( t \), for the function \( \Phi(z) \equiv \Phi(z; t, M) \),

\[ \frac{d\Phi}{dt} = B(z)\Phi, \quad B(z) = B_1 z + B_0, \]

where

\[ B_1 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B_0 = \frac{1}{2} \begin{pmatrix} 0 & k \\ -2p & 0 \end{pmatrix}. \]

Equations (21) and (22) form a Lax pair (cf. [3]),

\[ \begin{cases} \frac{d\Phi}{dz} = A(z)\Phi, \\ \frac{d\Phi}{dt} = B(z)\Phi, \end{cases} \]

whose compatibility condition [5], in terms of the functional parameters \( p, q, k \) and \( \theta \) reads

\[ \begin{align*}
\frac{dq}{dt} & = p + \frac{q^2 + t}{2}, \\
\frac{dp}{dt} & = -2pq - \theta, \\
\frac{dk}{dt} & = -kq, \\
\frac{d\theta}{dt} & = 0.
\end{align*} \]
The last equation of this system is just the statement that \( \theta \), as the part of the monodromy data, is constant. The third equation gives \( \ln k(t) \) as the antiderivative of \(-q(t)\). The first two first order differential equations are equivalent to one second order differential equation, indeed, the Painlevé II equation,

\[
q_{tt} = tq + 2q^3 + \alpha, \quad \alpha = \frac{1}{2} - \theta. \tag{30}
\]

Assuming that \( \theta \equiv \text{const} \), one can easily see that the function \( (27) \) is the Hamiltonian of the second Painlevé equation \( (30) \) with respect to the symplectic form,

\[
\Omega = dp \wedge dq,
\]

Indeed, the first and the second equations in \( (29) \) are just

\[
\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \text{and} \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q},
\]

respectively.

Let us now discuss the forms \( \omega_{\text{JMU}} \) and \( \omega \) corresponding to \( (21) \). The linear system \( (21) \) has only \( \infty \) as its singular point. Therefore, the general definition \( (3) \) of the form \( \omega_{\text{JMU}} \) transforms to the equation,

\[
\omega_{\text{JMU}} = -\text{res}_{z=\infty} \text{Tr} \left( G^{-1}(z) dG(z) \frac{d\Theta(z)}{dt} \right) dt.
\]

Plugging \( (23) \) at the right hand side we arrive at the formulae,

\[
\omega_{\text{JMU}} = -\text{Tr} \left( \frac{1}{2} g_1 \sigma_3 \right),
\]

or, taking into account \( (24) \) (cf. \( (9) \)),

\[
\omega_{\text{JMU}} = \frac{d \ln \tau}{dt} dt = H dt, \tag{31}
\]

Similarly, the general definition \( (12) \) of the form \( \omega \) transforms to the equation,

\[
\omega = \text{res}_{z=\infty} \text{Tr} \left( A(z) dG(z) G(z)^{-1} \right).
\]

Plugging \( (23) \) into \( (32) \) we arrive at the formula,

\[
\omega = \text{Tr} \left( A_2 d g_3 \right) A_2 d g_2 g_1 - A_2 d g_1 g_2 + A_2 d g_1 \sigma_3^2 + A_1 d g_2 - A_1 d g_1 g_1 + A_0 d g_1 \right).
\]

Now, it is more involved to plug \( (24) \) - \( (26) \) into the right hand side of the last equation. However, after performing some algebra, the final expression comes out rather simple,

\[
\omega = -\frac{1}{3} q dp + \frac{2}{3} p dq - \theta \frac{dk}{k} + \frac{2}{3} t dH - \frac{1}{3} H dt - \frac{2\theta - 1}{3} d\theta,
\]

and can be in turn easily transformed to the equation,

\[
\omega = p dq - H dt + d \left( \frac{2}{3} H t - \frac{1}{3} q p - \theta \ln k - \frac{\theta^2}{3} + \frac{\theta}{3} \right) + \ln k d\theta. \tag{33}
\]

If we again assume that \( \theta \equiv \text{const} \), relation \( (33) \) reduces to

\[
\omega = p dq - H dt + d \left( \frac{2}{3} H t - \frac{1}{3} q p - \theta \ln k \right). \tag{34}
\]
Equation (34) proves Conjecture 2 in the case of the $2 \times 2$ system (21) with the additional constraint, $\theta \equiv$ constant. Indeed, this is exactly the formula (17) with the specification $G(p, q, t) = \frac{2}{3} H t - \frac{1}{3} q p - \theta \ln k$.

The corresponding equation (18) is

$$d \ln \tau = p \frac{dq}{dt} - H + \frac{2}{3} H t - \frac{1}{3} q p - \theta \ln k$$

$$= p \frac{dq}{dt} - H + \frac{2}{3} H t - \frac{1}{3} q p + \theta q.$$  (35)

We also note that Conjecture 1 follows directly from (34).

**Remark 5.** Together with (31), equation (35) is just an identity which can be proven directly by substituting (31) into the left hand side of (35). However, it would be quite difficult to guess, without having the concept of the form $\omega$, the existence of such connection between the truncated and the full action differentials. We believe that the fact that in the case of the Painlevé dynamical systems the truncated action differs from the full action by a total differential is a manifestation of their Lax-pair integrability.

**Remark 6.** Let us denote

$p_1 = p, \quad q_1 = q, \quad p_2 = \ln k, \quad q_2 = \theta$,

then the whole system (29) becomes Hamiltonian with the same Hamiltonian (27),

$$H = \frac{p_1^2}{2} + p_1 q_1^2 + \frac{p_1 t}{3} + q_1 q_2,$$

and with respect to the symplectic form,

$$\Omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2.$$  

Moreover, (34) can be written as

$$\omega = \sum_{k=1}^{2} p_k dq_k - H dt + d \left( \frac{2}{3} H t - \frac{1}{3} q_1 p_1 - q_2 p_2 - \frac{q_2^2}{3} + \frac{q_2}{3} \right).$$

That is, if we associate with the linear system (21) not just the second Painlevé equation (30) but the full system (29) of the isomonodromic deformation equations of (21), then the $G$-function in the relation (17) will be totally local and, in fact, polynomial in the Darboux coordinates,

$$G(p_1, p_2, q_1, q_2, t) = \frac{2}{3} H t - \frac{1}{3} q_1 p_1 - q_2 p_2 - \frac{q_2^2}{3} + \frac{q_2}{3}.$$  

It is also worth noticing that since $H$ does not contain $p_2$, the variable $q_2$ is an action variable, i.e., it is constant, as it should be.

**Remark 7.** The Lax pair (21), (28) is not the only Lax pair for the Painlevé II equation. If we use another Lax pair, for instance the Lax pair of Flaschka and Newell [FN], we would get the another tau function, another Hamiltonian and another form $\omega$. However, the Conjectures 1 and 2 would still be true - see [ILP] and [BIP]. It is an interesting issue how much the Hamiltonian aspects we are promoting depend on the concrete Lax pair realization of the Painlevé equations.

---

9 From the point of view of the asymptotic analysis of the tau functions outlined in the Introduction, the appearance of the non-local term $\ln k = \int gdq$ is not an obstacle since the functional parameter $k$ equals $-2g_{1,12}$ (see (44)) where $g_1$ is the first coefficient of the series (24). This means that, similar to the functions $p$ and $q$, it can be recovered from the underlining Riemann-Hilbert problem, see also [BBD1].
3.2 Painlevé I

The results of this subsections belong to O. Lisovyy and J. Roussillon, and we follow here their paper [LR]. The linear system associated with the first Painlevé equation is the $2 \times 2$ matrix ODE with one irregular singular point of Poincaré rank 5 at $z = \infty$ and with one Fuchsian singular point at $z = 0$,

$$\frac{d\Phi}{dz} = A(z)\Phi, \quad A(z) = A_1 z^4 + A_2 z^2 + A_1 z + A_0 + \frac{A_{-1}}{z}. \tag{36}$$

The matrix coefficients are,

$$A_1 = \begin{pmatrix} 4 & 0 \\ 0 & -4 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & -4q \\ 4q & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & -2p \\ -2p & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 2q^2 + t - 2q^2 - t \\ 2q^2 + t - 2q^2 - t \end{pmatrix}, \quad A_{-1} = -\frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

and so that the space $\mathcal{A}$ is parametrized by $p, q,$ and $t$,

$$\mathcal{A} = \{(p, q, t)\}.$$

The formal solution at $z = \infty$ is given by the series,

$$\Phi_{\text{form}}(z) = (I + \frac{g_1}{z} + \frac{g_2}{z^2} + \frac{g_3}{z^3} + \frac{g_4}{z^4} + \frac{g_5}{z^5} + O(\frac{1}{z^6}))e^{\Theta(z)}, \quad \Theta(z) = \sigma_3 \left(\frac{4z^5}{5} + tz\right) \tag{37}$$

with the explicit formulae for the first five coefficient matrices $g_k$ given by the equations,

$$g_1 = \begin{pmatrix} -H & 0 \\ 0 & H \end{pmatrix}, \quad g_2 = \begin{pmatrix} \frac{H^2}{2} & \frac{q}{2} \\ \frac{q}{2} & \frac{p}{2} \end{pmatrix}, \quad g_3 = \begin{pmatrix} -\frac{H^3}{6} - \frac{2p-t^2}{24} & \frac{qH}{3} + \frac{p}{4} \\ -\frac{qH}{3} + \frac{p}{4} & \frac{H^2}{6} + \frac{2p-t^2}{24} \end{pmatrix}, \tag{38}$$

$$g_4 = \begin{pmatrix} \frac{H^4}{24} + \frac{2p-t^2}{24} & \frac{qH^2}{4} + \frac{pH}{4} + \frac{2q^2 + t}{8} \\ \frac{qH^2}{4} + \frac{pH}{4} + \frac{2q^2 + t}{8} & \frac{H^2}{24} + \frac{2p-t^2}{24} & \frac{q^2}{8} \end{pmatrix}, \tag{39}$$

$$g_5 = \begin{pmatrix} -\frac{H^5}{120} - \frac{2p-t^2}{48} & \frac{H^2}{24} - \frac{5q^2 - 2t}{48} & \frac{H}{48} + \frac{4pq+1}{160} \\ \frac{H^2}{120} - \frac{5q^2 - 2t}{48} & \frac{H^2}{24} - \frac{5q^2 - 2t}{48} & \frac{H}{48} + \frac{4pq-1}{160} \\ \frac{H}{120} - \frac{5q^2 - 2t}{48} & \frac{H}{48} + \frac{4pq+1}{160} & \frac{H^2}{24} - \frac{5q^2 - 2t}{48} \end{pmatrix}. \tag{40}$$

where

$$H = \frac{p^2}{2} - 2q^3 - tq. \tag{41}$$

The Fuchsian point $z = 0$ is a resonant point and hence the generic theory outlined in the Introduction is not applicable. In fact, the behavior of the solution $\Phi(z)$ at $z = 0$ is given by the formula [LR],

$$\Phi(z) = \left(1, \frac{1}{z}, \frac{1}{z^2}, \frac{1}{z^3}\right)z^{-\frac{1}{2}}\Phi(z),$$

where $\Phi(z)$ is holomorphic and invertible at $z = 0$.

The set of monodromy data $\mathcal{M}$ of system (36) consists of ten Stokes matrices associated with the irregular singularity at $z = \infty$ out of which, due to the symmetry $z \rightarrow -z$ of the system, only two are in fact free, i.e. dim $\mathcal{M} = 2$ (see [LR] for details). The structure of the essential singularity of $\Phi(z)$ at infinity described in [MW] indicates that the parameter $t$ is the only isomonodromic time. The corresponding Lax pair is formed by equation (36) and the following additional matrix equation,

$$\frac{d\Phi}{dt} = B(t)\Phi, \quad B(t) = B_1 z + \frac{B_{-1}}{z}, \tag{42}$$

$$B_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B_{-1} = \begin{pmatrix} q-q \\ q-q \end{pmatrix}.$$
The compatibility condition of (38) and (40) yields the system of ODEs on \( q \equiv q(t) \) and \( p \equiv p(t) \),

\[
\begin{align*}
\frac{dq}{dt} &= p, \\
\frac{dp}{dt} &= 6q^2 + t.
\end{align*}
\]  

which is equivalent to the Painlevé I equation

\[ q_{11t} = 6q^2 + t. \]

The system (29) is a Hamiltonian system with the Hamiltonian (41).

We have for the first four coefficients the relations,

\[
\begin{align*}
A_1 &= -g_1, & A_2 &= -g_2 + g_1^2, & A_3 &= -g_3 + g_2 g_1 + g_3 g_2 - g_1^3, \\
A_4 &= -g_4 + g_3 g_1 + g_1 g_3 + g_2^2 - g_2 g_1^2 - g_1 g_2 g_1 - g_1^2 g_2 + g_1^4.
\end{align*}
\]

Plugging (46) and (37) into (45) we arrive at the formula,

\[
\omega = \text{Tr} \left( -A_4(h_4 d g_1 + h_3 d g_2 + h_2 d g_3 + h_1 d g_4 + d g_5) - A_2(h_2 d g_1 + h_1 d g_2 + d g_3) - A_1(h_1 d g_1 + d g_2) - A_0 d g_1 \right).
\]

Using (39)–(40) we get after (quite a lot of) simplifications

\[
\omega = \frac{6}{5} p dq - \frac{4}{5} q dp - \frac{2}{5} H dt + \frac{8}{5} t dB,
\]

and properly combining the terms,

\[
\omega = 2 \left( p dq - H dt + d \left( \frac{4Ht}{5} - \frac{2pq}{5} \right) \right).
\]  

Equation (47) proves Conjectures 1 and 2, with \( \gamma = 2 \), in the case of the \( 2 \times 2 \) system (36) and gives the explicit formula for \( G(p, q, t) \),

\[
G(p, q, t) = \frac{2}{5} \left( 4Ht - 2pq \right).
\]

The corresponding equation (18) is

\[
\frac{d \ln \tau}{dt} = 2 \left( p \frac{dq}{dt} - H \right) + \frac{2}{5} \left( 4Ht - 2pq \right),
\]

and, of course, can be easily checked directly.
3.3 Painlevé III

The linear system associated with the third Painlevé equation we take again from [JM]. This is the system,

\[
\frac{d\Phi}{dz} = A(z)\Phi, \quad A(z) = \frac{A_{-2}}{z^2} + \frac{A_{-1}}{z} + A_0,
\]

with

\[
A_0 = \frac{1}{2} \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}, \quad A_{-1} = \begin{pmatrix} -\theta_{\infty} & -qk \theta_t \\ \frac{pq(t-p)}{kt} + \theta_0 + \theta_{\infty} & -\frac{2\theta_{\infty}}{k} \theta_t \end{pmatrix}, \quad A_{-2} = \begin{pmatrix} p - \frac{4}{t} & -kt \\ \frac{p(t-p)}{k} - p + \frac{2}{t} \end{pmatrix}.
\]

The system has two irregular singular points at \(z = \infty\) and \(z = 0\), both of the Poincaré rank 1. The corresponding formal solutions are:

\[
\Phi_{\text{form}}^{(z)}(z) = \left( I + \frac{g_{0,1}}{z} + O\left(\frac{1}{z^2}\right) \right) e^{\Theta_{\infty}(z)}, \quad \Theta_{\infty}(z) = \sigma_3 \left( \frac{iz}{2} - \theta_{\infty} \ln z \right),
\]

with

\[
g_{0,1} = \begin{pmatrix} -\frac{H}{2} - \frac{pq}{2t} + \frac{\theta_{\infty}^2 - \theta_0^2}{2t} + \frac{k}{2} & kq \\ \frac{pq(t-p)}{kt} + \theta_0 + \theta_{\infty} & -\frac{2\theta_{\infty}}{k} \theta_t + \frac{H}{2} + \frac{pq}{2t} - \frac{\theta_{\infty}^2 - \theta_0^2}{2t} - \frac{k}{2} \end{pmatrix},
\]

at \(z = \infty\), and

\[
\Phi_{\text{form}}^{(0)}(z) = G_0 \left( I + g_{0,1} z + O\left( z^2 \right) \right) e^{\Theta_0(z)}, \quad \Theta_0(z) = \sigma_3 \left( \frac{iz}{2z} + \theta_0 \ln z \right)
\]

with

\[
g_{0,1} = \begin{pmatrix} -\frac{H}{2} - \frac{pq}{2t} + \frac{\theta_{\infty}^2 - \theta_0^2}{2t} + \frac{k}{2} & \frac{k}{2} q \left( p - t \right) + \frac{k}{2} a \left( \theta_0 - \theta_{\infty} \right) \\ \frac{1}{k} \left( pq + \theta_0 + \theta_{\infty} \right) & \frac{H}{2} + \frac{pq}{2t} + \frac{\theta_{\infty}^2 - \theta_0^2}{2t} - \frac{k}{2} \end{pmatrix},
\]

at \(z = 0\). In (50) and (52),

\[
H = \frac{1}{t} \left( 2p^2 q^2 + p(2t - 2tq^2 + 4\theta_{\infty} - 1)q - 2tq(\theta_0 + \theta_{\infty}) + \theta_{\infty}^2 - \theta_0^2 \right),
\]

and \(G_0\) diagonalizes matrix \(A_{-2}\),

\[
G_0^{-1} A_{-2} G_0 = -\frac{1}{2} \sigma_3,
\]

and it is chosen in the form

\[
G_0 = \frac{1}{\sqrt{k}} \begin{pmatrix} k & -k \\ \frac{q}{t} & \frac{r}{t} \end{pmatrix} a^{\frac{\theta_{\infty}}{2}},
\]

with \(a\) being an extra gauge parameter, so that the full space \(\mathcal{A}\) is seven dimensional,

\[
\mathcal{A} = \{p, q, k, a, t, \theta_0, \theta_{\infty}\}.
\]

From the series (49) and (51) it follows that \(\theta_{\infty}\) and \(\theta_0\) are the formal monodromy exponents and the parameter \(t\) is the isomonodromic time. The isomonodromic with respect to \(t\) yields the second differential equation for \(\Phi(z) = \Phi(z, t)\),

\[
\frac{d\Phi}{dt} = B(z)\Phi, \quad B(z) = B_1 z + B_0 + \frac{B_{-1}}{z},
\]

where,

\[
B_1 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B_0 = \frac{1}{t} \begin{pmatrix} 0 & 0 & -qk \theta_t \\ \frac{pq(t-p)}{kt} + \theta_0 + \theta_{\infty} & -\frac{2\theta_{\infty}}{k} \theta_t & 0 \end{pmatrix}, \quad B_{-1} = \begin{pmatrix} \frac{1-5p}{2t} & k \\ \frac{p(t-p)}{k} & \frac{2p-1}{2t} \end{pmatrix}.
\]
The compatibility condition of the matrix equations (49) and (56) implies the following dynamical system on (59),

\[
\begin{align*}
\frac{dq}{dt} &= 4pq^2 - 2q^2 + \frac{q(4\theta_\infty - 1)}{t} + 2, \\
\frac{dp}{dt} &= -\frac{4p^2 q}{t} + \frac{p(4t q - 4\theta_\infty + 1)}{t} + 2\theta_0 + 2\theta_\infty, \\
\frac{dk}{dt} &= -\frac{4pqk}{t} + 2qk - \frac{2\theta_\infty k}{t}, \\
\frac{d\theta_\infty}{dt} &= 0, \quad \frac{d\theta_0}{dt} = 0.
\end{align*}
\]

(57)

It should be also mentioned, that the fourth equation, i.e. the equation for the function \(a(t)\), follows from plugging (51) into equation (56) – the second equation of the Lax pair, and equating the terms of zero order with respect to \(z\).

The last two equations of system (57) just state that \(\theta_\infty\) and \(\theta_0\) as the part of the monodromy data, are constant. The third and the fourth equations give \(\ln k(t)\) and \(\ln a(t)\) as the antiderivatives of the simple combinations of \(p\) and \(q\). The first two equations are equivalent to the third Painlevé equation,

\[
q_{tt} = \frac{(q_t)^2}{q} - \frac{q_{tt}}{2} \frac{t}{t} + \frac{1}{t}(a q^2 + \beta) + \gamma q^3 + \delta,
\]

(58)

where

\[
\alpha = 8\theta_0, \quad \beta = 4 - 8\theta_\infty, \quad \gamma = 4, \quad \delta = 4.
\]

Assuming that \(\theta_\infty\) and \(\theta_0\) are numerical constants, the function (59) becomes the Hamiltonian of (58) with \(p, q\) being the canonical variables. Also, if we denote

\[
p_1 = p, \quad q_1 = q, \quad p_2 = \ln k, \quad q_2 = \theta_\infty, \quad p_3 = \ln a, \quad q_3 = \theta_0,
\]

(59)

then the whole system (57) becomes Hamiltonian with the same Hamiltonian (53), i.e. with

\[
H = \frac{1}{t} \left(2p_1^2 q_1^2 + p_1(2t - 2t q_1^2 + (4q_2 - 1)q_1) - 2t q_1(q_2 + q_3) + q_2^2 - q_3^2\right),
\]

and with respect to the symplectic form,

\[
\Omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2 + dp_3 \wedge dq_3.
\]

The general formulae (3) and (12) transform, in the case of system (49), into the equations,

\[
\omega_{\text{IMU}} = -\operatorname{res}_{z=\infty} \operatorname{Tr} \left( G^{(\infty)}(z) \frac{1}{dG^{(\infty)}(z) \frac{d\Theta_{\infty}(z)}{dt}} d\frac{dz}{dt} - \operatorname{res}_{z=0} \operatorname{Tr} \left( G^{(0)}(z) \frac{1}{dG^{(0)}(z) \frac{d\Theta_0(z)}{dt}} d\frac{dz}{dt} \right) \right)
\]

(60)

and

\[
\omega = \operatorname{res}_{z=\infty} \operatorname{Tr} \left( A(z) dG^{(\infty)}(z) G^{(\infty)}(z)^{-1} \right) + \operatorname{res}_{z=0} \operatorname{Tr} \left( A(z) dG^{(0)}(z) G^{(0)}(z)^{-1} \right),
\]

(61)

respectively. Substituting the series \(G^{(\infty,0)}(z)\) and the exponentials \(\Theta_{\infty,0}(z)\) from (49) and (51) into (60), we obtain that

\[
\omega_{\text{IMU}} = -\frac{1}{2} \operatorname{Tr} \left( g_{\infty,1} \sigma_3 \right) dt - \frac{1}{2} \operatorname{Tr} \left( g_0 \sigma_3 \right) dt
\]

and using (50) and (52) we arrive at the final formula for \(\omega_{\text{IMU}}\),

\[
\omega_{\text{IMU}} = \frac{d\ln r}{dt} dt = H dt + \frac{pq}{t} dt - td t
\]

(62)

Note the additional to \(H dt\) terms in the right hand side of (62). Similar substitution of \(G^{(\infty,0)}(z)\) from (49) and (51) into (61) leads us to the formula,

\[
\omega = \operatorname{Tr} \left( A_{-1} dG_0 G_0^{-1} + G_0^{-1} A_{-2} G_0 dG_{0,1} - A_0 dG_{\infty,1} \right).
\]
and using again (50), (52) and (54) we arrive at the equation,
\[ \omega = pdq + tdH - \theta_\infty \frac{dk}{k} - \theta_0 \frac{da}{a} - tdt. \]

After regrouping the terms we obtain that
\[ \omega = pdq - Hdt + d\left( Ht - \theta_\infty \ln k - \theta_0 \ln a - \frac{r^2}{2} \right) + \ln k d\theta_\infty + \ln a d\theta_0, \]

or, using the definition (59) of the canonical coordinates,
\[ \omega = p_1 dq_1 + p_2 dq_2 + p_3 dq_3 - Hdt + d\left( Ht - q_2 p_2 - q_3 p_3 - \frac{r^2}{2} \right). \]  

Equation (63) proves Conjectures 1 and 2, with \( \gamma = 1 \), in the case of the 2 \times 2 system (48) and gives the explicit formula for \( G(p_j, q_j, t) \),
\[ G(p_1, p_2, p_3, q_1, q_2, q_3, t) = Ht - q_2 p_2 - q_3 p_3 - \frac{r^2}{2}. \]

The corresponding equation (19) is
\[ \frac{d\ln r}{dt} = p \frac{dq}{dt} - H + d\left( Ht - \theta_\infty \ln k - \theta_0 \ln a - \frac{r^2}{2} \right). \] (64)

Remark. One can deduce from (57) that
\[ \frac{pq}{t} = \frac{1}{4} \frac{a}{k} \frac{\partial}{ \partial \theta} + \frac{\theta_0 + \theta_\infty}{2t}. \]

Combining this with (62) and (64), we arrive at the equation,
\[ Hdt = pdq - Hdt + d\left( Ht + \frac{1 - 4\theta_0}{4} \ln k - \frac{1 + 4\theta_0}{4} \ln a + \frac{\theta_0 + \theta_\infty}{2t} \right). \]

where \( df \equiv d_1 f = \frac{df}{dt} \). In other words, although the truncated action, \( Hdt \), is not in this case exactly the Jimbo-Miwa-Ueno form \( \omega_{JMU} \), it still coincides with the full classical action, up to a total differential. As we will see, this is true in all other examples when accidentally \( Hdt \neq \omega_{JMU} \).

### 3.4 Painlevé IV

This time (see again [1]), the linear system is the 2 \times 2 system with one irregular singular point at \( z = \infty \) with the Poincaré rank 2 and one Fuchsian point at \( z = 0 \):
\[ \frac{d\Phi}{dz} = A(z) \Phi, \quad A(z) = \frac{A_{-1}}{z} + A_0 + A_1 z, \] (65)

where
\[ A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_0 = \begin{pmatrix} \frac{q}{4} & \frac{k}{2} \\ -\frac{k}{2} & -t \end{pmatrix}, \quad A_{-1} = \frac{1}{2} \left( \frac{g(4p - q - 2t)}{2 - k} \right) \left( \frac{\theta_{\infty}}{2} \right) \left( \frac{\theta_{\infty}}{2} \right). \]

The corresponding formal solution at \( z = \infty \) is
\[ \Phi_{\text{form}}(z) = \left( I + \frac{g_1}{z} + \frac{g_2}{z^2} + O\left( \frac{1}{z^3} \right) \right) e^{\Theta(z)}, \quad \Theta(z) = \sigma_3 \left( \frac{z^2}{2} + tz - \theta_\infty \ln z \right) \] (66)

with
\[ g_1 = \frac{1}{2} \left( \frac{-2H + q}{\theta_{\infty}} \right) \left( \frac{-2H + q}{2k} \right) \left( \frac{-k}{\theta_{\infty}} \right), \]
\[ g_2 = \frac{1}{8} \left( \frac{1}{2} \right) \left( \frac{(2H + q + 2t)^2 - 4t^2 - 8\theta_{\infty}^2 + 8\theta_{\infty}^2}{2k} \right) \left( \frac{-k(2H - q - 4t)}{\theta_{\infty}^2} \right) \left( \frac{2H + q + 2t)^2 - 4t^2 + 8\theta_{\infty}^2 - 8\theta_{\infty}^2}{2k} \right), \] (67)
and
\[ H = 2p^2 q - \frac{1}{8} q^3 - \frac{1}{2} t q^2 + \frac{1}{2} (2 \theta_\infty - 1 - t^2) q + 2 \theta_\infty t - \frac{2 \theta_0^2}{q}. \] (68)
The behavior of the solution of (65) at the (non-resonant, this time) Fuchsian point \( z = 0 \) is described by the equation,
\[ \Phi^{(0)}(z) = G_0(I + O(z)) z^{\theta_0 \sigma_3}, \quad z \to 0, \] (69)
where \( G_0 \) diagonalizes the matrix \( A_{-1} \),
\[ G_0^{-1} A_{-1} G_0 = \theta_0 \sigma_3, \]
and it is chosen in the form,
\[ G_0 = \frac{1}{2 \sqrt{kq} q_0} \left( \begin{array}{cc} -kq & -kq \\ a(q(4p-q-2t)-4t_0) & a(q(4p-q-2t)+4t_0) \end{array} \right) \alpha^{-\frac{q_0^2}{q}}. \] (70)
The full parameter space,
\[ \mathcal{A} = \{ p, q, k, a, t, \theta_0, \theta_\infty, \}, \]
is again seven dimensional with \( t \) being the isomonodromic time and \( \theta_\infty \) and \( \theta_0 \) serving as the formal monodromy exponents at the respective singular points. The isomonodromicity with respect to \( t \) yields the second differential equation for \( \Phi(z) \),
\[ \frac{d\Phi}{dt} = B(z) \Phi, \quad B(z) = B_1 z + B_0, \] (71)
where
\[ B_1 = A_1, \quad B_0 = \begin{pmatrix} 0 & k \\ \frac{q(4p-q-2t)+4t_0}{2k} & 0 \end{pmatrix}. \]
and the compatibility of (71) and (65) implies,
\[ \frac{dq}{dt} = 4pq, \]
\[ \frac{dp}{dt} = -2q^2 + \frac{3}{2} q^2 + qt + \frac{1}{2} t^2 - \theta_\infty + \frac{1}{2} + \frac{2 \theta_0^2}{q^2}, \]
\[ \frac{dk}{dt} = -(q+2t)k, \quad \frac{da}{dt} = \frac{4 \theta_0}{q}, \]
\[ \frac{d \theta_\infty}{dt} = 0, \quad \frac{d \theta_0}{dt} = 0. \] (72)
As in the previous section, the fourth equation follows from the substitution of (65) into (71).
Similar to the previous cases, the last equations of (72) manifest the time-independence of the formal monodromy exponents, the third and the fourth equations express \( k \) and \( a \) in terms of \( p \) and \( q \), while the first and the second equations are equivalent to a Painlevé equation, this time to the fourth Painlevé equation,
\[ q_{tt} = \frac{(q_t)^2}{2q} + \frac{3}{2} q^2 + 4 t_q q + 2 t^2 - a q + \frac{\beta}{q}, \]
(73)
where
\[ a = 2 \theta_\infty - 1, \quad \beta = -8 \theta_0^2. \]
Assuming that \( \theta_\infty \) and \( \theta_0 \) are numerical constants, the function (68) becomes the Hamiltonian of (73) with \( p, q \) being the canonical variables. Also, if we again denote
\[ p_1 = p, \quad q_1 = q, \quad p_2 = \ln k, \quad q_2 = \theta_\infty, \quad p_3 = \ln a, \quad q_3 = \theta_0, \]
(74)
then the whole system (72) becomes Hamiltonian with the same Hamiltonian (68), i.e. with
\[ H = 2p_1^2 q_1 - \frac{1}{15} q_1^3 - \frac{1}{2} t q_1^2 + \frac{1}{2} (2q_2 - 1 - t^2) q_1 + 2 q_2 t - \frac{2 \theta_0^2}{q_1}, \]
(75)
and with respect to the symplectic form, \[ \Omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2 + dp_3 \wedge dq_3. \]

The general formulae \[ 5 \] and \[ 12 \] transform, in the case of system \[ 65 \], into the equations,

\[
\omega_{\text{JMU}} = -\operatorname{res}_{z=\infty} \text{Tr} \left( G^{(\infty)}(z) \frac{dG^{(\infty)}(z)}{dz} \frac{d\Theta_{\infty}(z)}{dt} \right) dt
\]

and

\[
\omega = \operatorname{res}_{z=\infty} \text{Tr} \left( A(z) dG^{(\infty)}(z) G^{(\infty)}(z)^{-1} \right) + \operatorname{res}_{z=0} \text{Tr} \left( A(z) dG^{(0)}(z) G^{(0)}(z)^{-1} \right),
\]

respectively. Substituting the series \( G^{(\infty)}(z) \) and the exponentials \( \Theta_{\infty}(z) \) from \[ 65 \] into \[ 75 \], and using \[ 3.4 \] we obtain that

\[
\omega_{\text{JMU}} \equiv \frac{d\ln r}{dt} dt = H dt + \frac{1}{2} qdt.
\]

Note again the additional to \( Hdt \) term in the right hand side of \[ 77 \]. Similar substitution of \( G^{(\infty,0)}(z) \) from \[ 65 \] and \[ 69 \] into \[ 70 \] followed by the use of \[ 3.4 \] and \[ 70 \] leads us to the formulae,

\[
\omega = \text{Tr}(A_1 dG_0 G_0^{-1} - A_1 dq_2 + A_1 dG_1 q_1 - A_0 dq_1)
\]

\[
= -\frac{1}{2} qdq + \frac{1}{2} pdq + \frac{1}{2} tH - \frac{1}{2} Hdt - \frac{\theta_0}{2} \frac{d\alpha}{\alpha} + \theta_0 dt - \frac{2\theta_0 - 1}{2} d\theta_0.
\]

Regrouping the last equation, we arrive at the final answer for the form \( \omega \),

\[
\omega = pdq - Hdt + d \left( \frac{Ht}{2} - \frac{pq}{2} - \frac{\theta_0}{2} \ln k - \frac{\theta_0 - 1}{2} a + \frac{\theta_0}{2} a + \frac{\theta_0}{2} a - \frac{\theta_0}{2} a \right) + \ln k d\theta_0 + \ln a d\theta_0
\]

or, using the definition \[ 73 \] of the canonical coordinates,

\[
\omega = p_1 dq_1 + p_2 dq_2 + p_3 dq_3 - Hdt + d \left( \frac{Ht}{2} - \frac{p_1 q_1}{2} - \frac{q_2 p_2}{2} - \frac{q_3 p_3}{2} + \frac{q_2}{2} p_2 - \frac{q_3}{2} p_3 + \frac{q_1}{2} p_1 \right).
\]

Equation \[ 78 \] proves Conjectures 1 and 2, with \( \gamma = 1 \), in the case of the \( 2 \times 2 \) system \[ 65 \] and gives the explicit formula for \( G(p_1, q_1, t) \),

\[
G(p_1, p_2, p_3, q_1, q_2, q_3, t) = \frac{Ht}{2} - \frac{p_1 q_1}{2} - \frac{q_2 p_2}{2} - \frac{q_3 p_3}{2} + \frac{q_1}{2} p_1 + \frac{q_2}{2} p_2 + \frac{q_3}{2} p_3.
\]

The corresponding equation \[ 13 \] and the formula for the truncated action are

\[
\frac{d\ln r}{dt} = pq + \frac{1}{2} \ln k - \frac{\theta_0}{2} \ln a + \frac{\theta_0}{2} a + \frac{\theta_0}{2} a,
\]

\[
Hdt = pdq - Hdt + d \left( \frac{Ht}{2} - \frac{pq}{2} - \frac{1}{2} \ln k - \frac{\theta_0}{2} \ln a + \frac{\theta_0}{2} a \right),
\]

respectively.

### 3.5 Painlevé V

In \[ 1 \], the following linear system is associated with the fifth Painlevé equation,

\[
\frac{d\Phi}{dz} = A(z) \Phi, \quad A(z) = A_2 + \frac{A_0}{z} + \frac{A_1}{z-1},
\]

where

\[
A_2 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_0 = \begin{pmatrix} -pq - \theta_0 - \theta_1 & k(pq + \theta_0 + \theta_1 - \theta_0) \\ -\frac{1}{2}(pq + \theta_0 + \theta_1 + \theta_0) & pq + \theta_0 + \theta_1 \end{pmatrix}.
\]
This system has one irregular singular point of Poincaré rank 1 at $z = \infty$ and two Fuchsian singular points $z = 0$ and $z = 1$. The corresponding formal solution at $z = \infty$ is given by the formulae,

$$
\Phi_{\text{form}}(z) = \left( I + \frac{g_1}{z} + O\left( \frac{1}{z^2} \right) \right) \epsilon^{\Theta(z)}, \quad z \to \infty, \quad \Theta(z) = \sigma_3 \left( \frac{t z}{2} - \theta_1 \ln z \right)
$$

with

$$
g_1 = \begin{pmatrix}
-H & \frac{k(p q^2 - p q + 2 \theta_1 q - \theta_\infty - \theta_1 + \theta_0)}{t} \\
\frac{-2 p q - 2 p + 2 \theta_\infty + 2 \theta_1 + 2 \theta_0}{t k} & \frac{t}{H}
\end{pmatrix}
$$

and

$$
H = \frac{p^2 (q - 1)^2 q}{t} + p \left( \frac{q^2}{t} (\theta_0 + 3 \theta_1 + \theta_\infty) + \frac{d}{t} (t - 2 \theta_\infty - 4 \theta_1) + \frac{1}{t} (\theta_\infty + \theta_1 - \theta_0) \right) + \frac{q^2 \theta_1}{t} (\theta_\infty + \theta_1 + \theta_0)
\quad + \frac{\theta_0^2 - \theta_1^2 - \theta_\infty^2 + \theta_1 t - 2 \theta_1 \theta_\infty}{t}
$$

The behavior of the solutions of (79) at the (non-resonant) Fuchsian points $z = 0$ and $z = 1$ are described by the equations,

$$
\Phi^{(0)}(z) = G_0 (I + O(z)) z^{\theta_0 \sigma_3}, \quad z \to 0,
$$

and

$$
\Phi^{(1)}(z) = G_1 (I + O(z - 1)) (z - 1)^{\theta_1 \sigma_3}, \quad z \to 1,
$$

respectively. The matrices $G_0$ and $G_1$ diagonalize the matrix coefficients $A_0$ and $A_1$,

$$
G_0^{-1} A_0 G_0 = \theta_0 \sigma_3, \quad G_1^{-1} A_1 G_1 = \theta_1 \sigma_3,
$$

and are chosen in the form,

$$
G_0 = \frac{1}{\sqrt{-4 k \theta_0}} \begin{pmatrix}
k (2 p q + 2 \theta_\infty + 2 \theta_1 - 2 \theta_0) & k \\
2 p q + 2 \theta_\infty + 2 \theta_1 + 2 \theta_0 & 1
\end{pmatrix} a^{-\frac{q_0}{2}},
$$

and

$$
G_1 = \frac{1}{\sqrt{2 k \theta_1}} \begin{pmatrix}
k (p q + 2 \theta_1) & k q \\
p & 1
\end{pmatrix} b^{-\frac{q_0}{2}}.
$$

The full space

$$
\mathcal{A} = \{ p, q, k, a, b, t, \theta_0, \theta_1, \theta_\infty, \}. \quad (80)
$$

is nine dimensional with $t$ being the isomonodromic time and $\theta_\infty, \theta_0,$ and $\theta_1$ serving as the formal monodromy exponents at the respective singular points. The isomonodromicity with respect to $t$ yields the second differential equation for $\Phi(z)$,

$$
\frac{d \Phi}{d t} = B(z) \Phi, \quad B(z) = B_1 z + B_0,
$$

where

$$
B_1 = \frac{A_2}{t}, \quad B_0 = \begin{pmatrix}
0 & k \frac{2 p q - p q + 2 \theta_1 q + \theta_\infty + \theta_1 - \theta_0}{t} \\
-\frac{1}{t} (p q + \theta_\infty - p + \theta_1 + \theta_0) & 0
\end{pmatrix},
$$

and

$$
A_1 = \begin{pmatrix}
p q + \theta_1 & -k q (p q + 2 \theta_1) \\
\frac{1}{t} & -p q - \theta_1
\end{pmatrix}.
$$
and the compatibility of (67) and (79) implies,

\[
\begin{align*}
\frac{dq}{dt} &= 2p(q - 1)^2 - \frac{a^2}{t} \left(\theta_0 + 3\theta_1 + \theta_\infty\right) + \frac{q^2}{t} \left(t - 2\theta_\infty - 4\theta_1\right) + \frac{1}{t} \left(\theta_\infty + \theta_1 - \theta_0\right), \\
\frac{dp}{dt} &= -\frac{p^2}{t} \left(3q^2 - 4q + 1\right) - p \left(\frac{2q}{t} \left(\theta_0 + 3\theta_1 + \theta_\infty\right) + \frac{1}{t} \left(t - 2\theta_\infty - 4\theta_1\right)\right) + \frac{2\theta_1}{t} \left(\theta_\infty + \theta_1 + \theta_0\right), \\
\frac{dk}{dt} &= -\frac{k}{t} \left(p^2 - 2pq + p + 2\theta_1 q - 2\theta_\infty - 2\theta_1\right), \\
\frac{da}{dt} &= a \left(p - p^2 - 2\theta_1 q - 2\theta_0\right), \\
\frac{db}{dt} &= -\frac{b}{t} \left(3q^2 - p - 4pq + 2\theta_\infty q + 4\theta_1 q + 2\theta_0 q - 2\theta_\infty - 2\theta_1 + t\right), \\
\frac{d\theta_\infty}{dt} &= 0, \quad \frac{d\theta_0}{dt} = 0, \quad \frac{d\theta_1}{dt} = 0.
\end{align*}
\]

(88)

As before, the equations for \(a\) and \(b\) follow from the substitution of (83) into (87) and (84) into (87), respectively, and they simply express the functions \(a(t)\) and \(b(t)\) in terms of \(p\) and \(q\). The third equation in (88) is also trivial – just an expression of \(k\) in terms of \(p\) and \(q\), and the last three equations are the manifestation of the time-independence of the formal monodromy exponents. The nontrivial first two equations are equivalent to the fifth Painlevé equation,

\[
q_{1t} = \left(\frac{1}{2q} + \frac{1}{q - 1}\right)(q_1)^2 - \frac{q_t}{t} + \left(\frac{q - 1}{t}\right)^2 \left(a q + \frac{\beta q}{q} + \gamma q + \delta q(q + 1)(q - 1)\right),
\]

(89)

where

\[
a = \frac{(\theta_0 - \theta_1 + \theta_\infty)^2}{2}, \quad \beta = \frac{(\theta_0 - \theta_1 - \theta_\infty)^2}{2}, \quad \gamma = (1 - 2\theta_0 - 2\theta_1), \quad \delta = -\frac{1}{2}.
\]

Assuming that \(\theta_\infty, \theta_0,\) and \(\theta_1\) are numerical constants, the function (82) becomes the Hamiltonian of (89) with \(p, q\) being the canonical variables. Also, if we denote

\[
p_1 = p, \quad q_1 = q, \quad p_2 = \ln k, \quad q_2 = \theta_\infty, \quad p_3 = \ln a, \quad q_3 = \theta_0, \quad p_4 = \ln b, \quad q_4 = \theta_1,
\]

(90)

then the whole system (88) becomes Hamiltonian with the same Hamiltonian (82), i.e. with

\[
H = \frac{p_1^2(q_1 - 1)^2 q_1}{t} + p_1 \left(\frac{q_1}{t} \left(q_3 + 3q_4 + q_2\right) + \frac{q_t}{t} \left(t - 2q_2 - 4q_4\right) + \frac{1}{t} \left(q_2 + q_4 - q_3\right)\right) + \frac{2q_1 q_4}{t} \left(q_2 + q_4 + q_3\right) + \frac{q_2^2 - q_3^2 - q_2^2 + 4q_4^2 q_2}{t},
\]

and with respect to the symplectic form,

\[
\Omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2 + dp_3 \wedge dq_3 + dp_4 \wedge dq_4.
\]

The general formulae (3) and (12) transform, in the case of system (79), into the equations,

\[
\omega_{\text{JMU}} = -\text{res}_{z=\infty} \text{Tr} \left(\frac{G^{(\infty)}(z)}{dz} \frac{dG^{(\infty)}(z)}{dt} d\theta_\infty(z)\right) dt
\]

(91)

and

\[
\omega = \text{res}_{z=\infty} \text{Tr} \left(A(z) dG^{(\infty)}(z) G^{(\infty)}(z)^{-1}\right) + \text{res}_{z=0} \text{Tr} \left(A(z) dg^{(0)}(z) G^{(0)}(z)^{-1}\right)
\]

\[
+ \text{res}_{z=1} \text{Tr} \left(A(z) dG^{(1)}(z) G^{(1)}(z)^{-1}\right).
\]

(92)

respectively. Substituting the series \(G^{(\infty)}(z)\) and the exponent \(\Theta_\infty(z)\) from (80) into (91) and using (81), we obtain that, similar to the Painlevé II case,

\[
\omega_{\text{JMU}} = \frac{d\ln t}{dt} = H dt.
\]

(93)
Substituting $G^{(c,0,1)}(z)$ from (89), (93), and (94) into (92) and using after that (91), (95), (96) leads us to the formulae,
\[ \omega = \text{Tr}(A_0 dG_0 G_0^{-1} + A_1 dG_1 G_1^{-1} - A_2 dG_2) \]
\[ = pdq + tdH_1 - \theta_{\infty} \frac{d\theta_1}{\theta} - \theta_0 \frac{a}{a} - \theta_1 \frac{db}{b} + \theta_0 + d\theta_1. \]

Regrouping the last equation, we arrive at the final answer for the form $\omega$,
\[ \omega = pdq - Hdt + d\left( Ht - \theta_{\infty} \ln k - \theta_0 \ln a - \theta_1 \ln b + \theta_0 + \theta_1 \right) + \ln a d\theta_0 + \ln b d\theta_1 + \ln k d\theta_{\infty}, \]

or, using the definition (90) of the canonical coordinates,
\[ \omega = p_1 dq_1 + p_2 dq_2 + p_3 dq_3 + p_4 dq_4 - Hdt + d\left( Ht - q_2 p_2 - q_3 p_3 - q_4 p_4 + q_3 + q_4 \right). \]

Equation (94) proves Conjectures 1 and 2, with $\gamma = 1$, in the case of the $2 \times 2$ system (79) and gives the explicit formula for $G(p_j, q_j, t)$.

The corresponding equation (18) is
\[ \frac{d\ln r}{dt} = p \frac{dq}{dt} - H + \frac{d}{dt}\left( Ht - \theta_{\infty} \ln k - \theta_0 \ln a - \theta_1 \ln b \right). \]

Again, this equation together with (93) make an identity, which, this time, would not be so easy to check directly.

### 3.6 Painlevé VI

Consider the $2 \times 2$ Fuchsian system with 4 regular singularities at 0, 1, $t$ and $\infty$,
\[ \frac{d\Phi}{dz} = A(z) \Phi, \quad A(z) = \frac{A_0}{z} + \frac{A_1}{z-t} + \frac{A_2}{z-1}, \]

where $A_0$, $A_1$, $A_t \in sl_2(C)$, $A_0 + A_1 + A_t = -\theta_{\infty} \sigma_3$.

Following to (11), we introduce the parametrization,
\[ A_0 = \left( \begin{array}{cc} x_0 + \theta_0 & -u x_0 \\ x_0 + 2\theta_0 & -x_0 - \theta_0 \end{array} \right), \quad A_1 = \left( \begin{array}{cc} x_1 + \theta_1 & -v x_1 \\ x_1 + 2\theta_1 & -x_1 - \theta_1 \end{array} \right), \quad A_t = \left( \begin{array}{cc} x_t + \theta_t & -w x_t \\ x_t + 2\theta_t & -x_t - \theta_t \end{array} \right). \]

Observe that, $\pm \theta_0, \pm \theta_1, \pm \theta_t$ are the eigenvalues of $A_0, A_1, A_t$, and that the following constraints are satisfied,
\[ x_0 + \theta_0 + x_1 + \theta_1 + x_t + \theta_t = -\theta_{\infty}, \]
\[ u x_0 + v x_1 + w x_t = 0, \]
\[ x_0 + 2\theta_0 + x_1 + 2\theta_1 + x_t + 2\theta_t = 0. \]

We also introduce the parameters $k$ and $q$ by writing the entry $A_{12}(z)$ of the matrix $A(z)$ as,
\[ A_{12}(z) = \frac{k(z-q)}{z(z-1)(z-t)}. \]

Notice that,
\[ u x_0 (1+t) + v x_1 + w x_t = k, \quad u x_0 t = k q. \]

Finally we put
\[ p = A_{11}(q) = \frac{x_0 + \theta_0 + x_1 + \theta_1 + x_t + \theta_t}{q-1}, \]
\[ \frac{x_0 + 2\theta_0 + x_1 + 2\theta_1 + x_t + 2\theta_t}{q-t}. \]
Solving equations (97) and (99) with respect to \( u, v, w \), we get

\[
u = \frac{kq}{x_0 t}, \quad v = \frac{k(q-1)}{x_1(1-t)}, \quad w = \frac{k(t-q)}{x_t(1-t)}.
\] (101)

Next, we express \( x, x_1, x_t \) from (98) and (100), and then we express \( x_0 \) from (99). The result is

\[
x_0 = \frac{p^2 q^3 (q-1)(q-t)}{t^2 \theta \theta_\infty} + \frac{p q q(q-1)(q-t)}{t} + \frac{\theta \theta_\infty q(q-t-1)}{2t} + \frac{\theta \theta_\infty^2 t}{2t \theta \theta_\infty(q-t)} - \frac{\theta \theta_\infty^2 t}{2 \theta \theta_\infty(q-t)}
\]

\[
x_1 = \frac{p^2 q^2 q(q-1)(q-t)^2}{2t \theta \theta_\infty(t-1)} + \frac{p q q(q-1)(q-t)}{t-1} + \frac{\theta \theta_\infty q(q-t-1)}{2t-1} + \frac{\theta \theta_\infty^2 t}{2t \theta \theta_\infty(q-t)} - \frac{\theta \theta_\infty^2 t}{2 \theta \theta_\infty(q-t)}
\]

\[
x_t = \frac{p^2 q(q-1)(t-q)^2}{t \theta \theta_\infty(1-t)} + \frac{p q q(q-1)(q-t)}{t-1} + \frac{\theta \theta_\infty q(q-t-1)}{2t-1} + \frac{\theta \theta_\infty^2 t}{2t \theta \theta_\infty(t-1)} - \frac{\theta \theta_\infty^2 t}{2 \theta \theta_\infty(t-1)}
\]

Equations (101) - (102) provide parametrisation of the matrices \( A_0, A_1, A_t \) by the parameters \( q, p, k, \theta_0, \theta_1, \theta_t, \theta_\infty \), which will prove to be the Darboux coordinates, and by the parameter \( t \) which is the isomonodromic time.

Solutions of (99) have the following behavior at \( z = 0, 1, t, \infty \),

\[ \Phi^{(0)}(z) = (I + O(z^{-1})) z^{-\theta_0 \sigma_3}, \quad z \to \infty. \]
\[ \Phi^{(0)}(0) = G_0 (I + O(z)) z^{\theta_0 \sigma_3}, \quad z \to 0. \]
\[ \Phi^{(1)}(z) = G_1 (I + O(z-1)) (z-1)^{\theta_1 \sigma_3}, \quad z \to 1, \]
\[ \Phi^{(1)}(z) = G_t (I + G_1 (z-t) + O((z-t)^2)) (z-t)^{\theta_t \sigma_3}, \quad z \to t, \]

where

\[
g_1 = \left( \begin{array}{cccc}
G_0 & H & \frac{H}{2} & \frac{H}{2} \\
G_0 & \frac{H}{2} & \frac{H}{2} & \frac{H}{2} \\
G_0 & \frac{H}{2} & \frac{H}{2} & \frac{H}{2} \\
G_0 & \frac{H}{2} & \frac{H}{2} & \frac{H}{2}
\end{array} \right)
\]

and

\[
H = \frac{p}{t} q(q-1)(q-t) \frac{q(q-1)}{t} + \frac{\theta \theta_\infty(q-t)}{t-1} + \frac{\theta \theta_\infty^2 q}{t \theta \theta_\infty(q-t)} + \frac{\theta \theta_\infty^2}{t \theta \theta_\infty(q-t)}
\]

The matrices \( G_0, G_1, G_t \) diagonalize the matrix residues \( A_0, A_1, A_t, \)

\[
G_0^{-1} A_0 G_0 = \theta_0 \sigma_3, \quad G_1^{-1} A_1 G_1 = \theta_1 \sigma_3, \quad G_t^{-1} A_t G_t = \theta_t \sigma_3.
\]

and they are chosen in the form,

\[ G_0 = \sqrt{\frac{kq}{t}} \left( \begin{array}{cc}
\frac{1}{a} & \frac{1}{a + 2 \theta_0} \\
\frac{1}{a + 2 \theta_0} & \frac{1}{a}
\end{array} \right) \]

\[ G_1 = \sqrt{\frac{k(q-1)}{1-t}} \left( \begin{array}{cc}
\frac{1}{b} & \frac{1}{b + 2 \theta_1} \\
\frac{1}{b + 2 \theta_1} & \frac{1}{b}
\end{array} \right) \]

\[ G_t = \sqrt{\frac{k(t-q)}{t}} \left( \begin{array}{cc}
\frac{1}{c} & \frac{1}{c + 2 \theta_t} \\
\frac{1}{c + 2 \theta_t} & \frac{1}{c}
\end{array} \right) \]

(103)
The whole parameter space \( \mathcal{A} \) has dimension 11,

\[
\mathcal{A} = \{ p, q, k, a, b, c, t, \theta_0, \theta_1, \theta_{\infty} \}. \tag{108}
\]

The isomonodromicity with respect to \( t \) yields the second differential equation for \( \Phi(z) \),

\[
\frac{d\Phi}{dt} = -\frac{A_1}{z-i}\Phi. \tag{109}
\]

and the equations

\[
\frac{dG_0}{dt} = \frac{A_1}{t} G_0, \quad \frac{dG_1}{dt} = \frac{A_1}{t-1} G_1, \quad \frac{dG_i}{dt} = \left( \frac{A_0}{t} + \frac{A_1}{t-1} \right) G_i, \tag{110}
\]

for the gauge matrices \( G_0, G_1, G_i \). The compatibility of \( [109] \) and \( [95] \) together with the equations \( [110] \) imply the following dynamical system on \( \{108\} \),

\[
\frac{dq}{dt} = \frac{2pq(q-1)(q-t)}{t(t-1)} + \frac{q(q-1)}{t(t-1)},
\]

\[
\frac{dp}{dt} = \frac{1}{4t(t-1)} \left( 4p^2(2tq-3q^2-t+2q) + 4p(1-2q) + 4\theta_{\infty}(\theta_{\infty}-1) \right) - \frac{\theta_0^2}{q^2(t-1)} + \frac{\theta_1^2}{(q-1)^2} - \frac{\theta_2^2}{(q-t)^2}, \tag{111}
\]

\[
\frac{dk}{dt} = k \frac{[2\theta_{\infty}-1](q-t)}{t(t-1)},
\]

\[
\frac{da}{dt} = -\frac{2\theta_0(q-t)a}{qt(t-1)}, \quad \frac{db}{dt} = -\frac{2\theta_1(q-t)b}{qt(t-1)}, \quad \frac{dc}{dt} = -\frac{2\theta_1(q(2t-1)-t^2)c}{(q-t)t(t-1)}, \quad \frac{d\theta_0}{dt} = -\frac{\theta_1}{t}, \quad \frac{d\theta_1}{dt} = \frac{\theta_0}{t}, \quad \frac{d\theta_{\infty}}{dt} = 0. \tag{112}
\]

As before, the only non-trivial equations are the first two, and they are equivalent to the sixth Painlevé equation for the function \( q(t) \),

\[
\frac{d^2q}{dt^2} = \frac{1}{2} \left( \frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-t} \right) \left( \frac{dq}{dt} \right)^2 - \left( \frac{1}{t} + \frac{1}{t-1} + \frac{1}{q-t} \right) \frac{dq}{dt} + \frac{q(q-1)(q-t)}{t^2(t-1)^2} \left( \alpha + \frac{t}{q^2} + \gamma \frac{t-1}{(q-1)^2} + \delta \frac{t(t-1)}{(q-t)^2} \right), \tag{113}
\]

where

\[
\alpha = \frac{(2\theta_{\infty}-1)^2}{2}, \quad \beta = -2\theta_0^2, \quad \gamma = 2\theta_1^2, \quad \delta = \frac{1-4\theta_{\infty}^2}{2}.
\]

Assuming that \( \theta_j, j = 0, 1, t, \infty \) are numerical constants, the function \( [104] \) becomes the Hamiltonian of \( [113] \) with \( p, q \) being the canonical variables. Also, if we denote

\[
p_1 = p, \quad q_1 = q, \quad p_2 = \ln k, \quad q_2 = \theta_{\infty}, \quad p_3 = \ln a, \quad q_3 = \theta_0, \quad p_4 = \ln b, \quad q_4 = \theta_1, \quad p_5 = \ln c, \quad q_5 = \theta_t, \tag{114}
\]

then the whole system \( [111] - [112] \) becomes Hamiltonian with the same Hamiltonian \( [104] \), that is with,

\[
H = p_1^2 q_1 q_1(t-1)(q_1-t) + p_2 q_1 q_1(t-1) + q_2(1-q_2)q_1(t-1) + q_3^2 q_1(t-1) + \frac{q_4^2 q_1(t-1)}{(q_1-1)(t-1)} + \frac{q_5^2 q_1(2t-1)}{(q_1-1)t(t-1)},
\]

and with respect to the symplectic form,

\[
\Omega = \sum_{j=1}^5 dp_j \wedge dq_j.
\]
The general formulae (3) and (12) transform, in the case of system (95), into the equations,

$$\omega_{\text{MU}} = - \text{res}_{z=1} \text{Tr} \left( \left( G^{(i)}(z) \right)^{-1} \frac{dG^{(i)}(z)}{dz} \frac{d\theta_i(z)}{dt} \right) dt$$

and

$$\omega = \text{res}_{z=\infty} \text{Tr} \left( A(z) dG^{(\infty)}(z) G^{(\infty)}(z)^{-1} \right) + \text{res}_{z=0} \text{Tr} \left( A(z) dG^{(0)}(z) G^{(0)}(z)^{-1} \right)$$

res$_{z=1} \text{Tr} \left( A(z) dG^{(1)}(z) G^{(1)}(z)^{-1} \right) + \text{res}_{z=1} \text{Tr} \left( A(z) dG^{(i)}(z) G^{(i)}(z)^{-1} \right) \right)$.

From (115) it follows that

$$\omega_{\text{MU}} = \theta_i \text{Tr} \left[ g_1 \sigma_3 \right].$$

and taking into account (103), we obtain that,

$$\omega_{\text{MU}} \equiv \frac{d\ln r}{dt} dt = Hdt - p q(q-1) t(t-1) dt - \frac{\theta_\infty(q-t)}{t(t-1)} dt.$$

Similarly, (116) reduces to the equation,

$$\omega = \text{Tr} \left[ G_0^{-1} A_0 dG_0 + G_1^{-1} A_1 dG_1 + G_2^{-1} A_2 dG_2 - A_0 G_0 G_1^{-1} dt \right],$$

which after using (103), (107) and simplifying yields the formula

$$\omega = p dq - Hdt - \theta_\infty \frac{dk}{k} - \theta_0 \frac{da}{a} - \theta_1 \frac{db}{b} - \theta_1 \frac{dc}{c} + d \theta_\infty.$$

This, in turn, can be rewritten as

$$\omega = p dq - Hdt + d \left( \theta_\infty - \theta_0 \ln a - \theta_1 \ln b - \theta_1 \ln c - \theta_\infty \ln k \right) + \ln k d \theta_\infty + \ln a d \theta_0 + \ln b d \theta_1 + \ln c d \theta_1;$$

or, remembering the definitions (114) of the canonical coordinates,

$$\omega = \sum_{j=1}^5 p dq - Hdt + d \left( q_2 - q_3 p_3 - q_4 p_4 - q_5 p_5 - q_2 p_2 \right).$$

Equation (117) proves Conjectures 1 and 2, with $\gamma = 1$, in the case of the 2 × 2 system (95) and gives the explicit formula for $G(p, q, t)$.

$$G(p_1, p_2, p_3, p_4, p_5, q_1, q_2, q_3, q_4, q_5, t) = q_2 - q_3 p_3 - q_4 p_4 - q_5 p_5 - q_2 p_2.$$

The corresponding equation (118) and the truncated action are

$$\frac{d \ln r}{dt} = p \frac{dq}{dt} - H \frac{d}{dt} \left( \theta_0 \ln a + \theta_1 \ln b + \theta_1 \ln c + \theta_\infty \ln k \right),$$

and

$$Hdt = pdq - Hdt + d \left( \frac{1}{2} \ln \left( \frac{k(q-t)}{t(t-1)} \right) - \theta_0 \ln a - \theta_1 \ln b - \theta_1 \ln c - \theta_\infty \ln k \right),$$

respectively.

### 3.7 Schlesinger system

This section reproduces the result of Mal2 (Subsection 5.6, Remark 5.5). Once again, we are grateful to M. Mazzocco for informing us about this part of Malgrange's work.

Consider the Fuchsian system

$$\frac{d\Phi}{dz} = A(z) \Phi(z), \quad A(z) = \sum_{v=1}^n \frac{A_v}{z - a_v}, \quad A_v \in \mathfrak{sl}_N(\mathbb{C}).$$

(118)
We assume that all matrix coefficients $A_\nu$ are diagonalizable

$$A_\nu = G_\nu \Theta_\nu G_\nu^{-1} ; \quad \Theta_\nu = \text{diag} \{ \theta_{\nu,1}, \ldots, \theta_{\nu,N} \},$$

and that their eigenvalues are distinct and non-resonant:

$$\theta_{\nu,\mu} \neq \theta_{\nu,\beta} \mod \mathbb{Z}.$$

We also assume that the residue of $A(z)$ at $z = \infty$ is diagonal, i.e.

$$A_\infty = -\sum_{\nu=1}^n A_\nu = \Theta_\infty.$$

Solutions of (118) have the following behavior at the singular points

$$\Phi^{(\nu)}(z) = (I + O(z^{-1}))z^{-\Theta_\nu}, \quad z \to \infty,$$

$$\Phi^{(\nu)}(z) = G_\nu(I + g_{\nu,1}(z-a_\nu) + O((z-a_\nu)^2))(z-a_\nu)^{\Theta_\nu} C_\nu, \quad z \to a_\nu.$$

The isomonodromic times are now positions of the singular points $a_\nu$. The isomonodromic deformations with respect to these times yields the equation,

$$\frac{d\Phi}{da_\nu} = B_\nu(z)\Phi(z), \quad B_\nu(z) = -A_\nu z - a_\nu.$$

The compatibility conditions give the Schlesinger system

$$\frac{dA_\mu}{da_\nu} = \frac{[A_\mu, A_\nu]}{a_\mu - a_\nu}, \quad \mu \neq \nu, \quad \frac{dA_\nu}{da_\nu} = -\sum_{\mu \neq \nu} \frac{[A_\mu, A_\nu]}{a_\mu - a_\nu}, \quad (119)$$

and also the equations (cf. equations (110)),

$$\frac{dG_\mu}{da_\nu} = \frac{A_\nu}{a_\nu - a_\mu} G_\mu, \quad \mu \neq \nu, \quad \frac{dG_\nu}{da_\nu} = -\sum_{\mu \neq \nu} \frac{A_\mu}{a_\mu - a_\nu} G_\nu. \quad (120)$$

Following [JMMS] we introduce matrix Darboux coordinates

$$Q_\nu = G_\nu \Theta_\nu, \quad P_\nu = G_\nu^{-1}$$

and Hamiltonians

$$H_\nu = \sum_{\mu \neq \nu} \text{Tr}(Q_\mu P_\mu Q_\nu P_\nu) / (a_\nu - a_\mu).$$

Notice that $A_\nu = Q_\nu P_\nu$. Then, as it is shown in [Mal2], the isomonodromic equations (119–120) are equivalent to the Hamiltonian system,

$$\frac{dP_{\mu,jk}}{da_\nu} = -\frac{\partial H_\nu}{\partial Q_{\mu,kj}}, \quad \frac{dQ_{\mu,jk}}{da_\nu} = \frac{\partial H_\nu}{\partial P_{\mu,kj}}.$$

Moreover, by a rather straightforward calculation, one can show that the general formulae (9) and (12) in the case of the Fuchsian system (118) produce the following expressions of the forms $\omega_{\text{JMUI}}$ and $\omega$,

$$\omega_{\text{JMUI}} = \sum_{\nu=1}^n H_\nu d a_\nu,$$

and

$$\omega = \sum_{\nu=1}^n \text{Tr}(P_\nu dQ_\nu) - H_\nu d a_\nu.$$

In other words, we have validity of Conjectures 1 and 2, with $\gamma = 1$, in the case of the Fuchsian system (118). Moreover, the form $\omega$ just coincides with $\omega_{\text{cla}}.$
4 Appendix.

4.1 The proof of Lemma 1

The proof of Lemma 1 is rather short. Indeed, noticing that

\[
(G^{(v)})^{-1} \frac{dG^{(v)}}{dz} = (G^{(v)})^{-1} AG^{(v)} - \frac{d\Theta_v}{dz}
\]

and plugging this into the right hand side of (3), we have,

\[
\omega_{\mu\nu} = - \sum_{k=1}^L \sum_{v=1,...,n,\infty} \text{res}_{z=a_v} \text{Tr} \left[ (G^{(v)})^{-1} AG^{(v)} \frac{d\Theta_v}{dt_k} \right] dI_k + \sum_{k=1}^L \sum_{v=1,...,n,\infty} \text{res}_{z=a_v} \text{Tr} \left( \frac{d\Theta_v}{dz} \frac{d\Theta_v}{dt_k} \right) dI_k.
\]

(122)

The expression \( \frac{d\Theta_v}{dz} \frac{d\Theta_v}{dt_k} \) has poles of order at least 2, so it does not have residues and hence the second sum in (122) vanishes. We also have,

\[
\frac{d\Theta_v}{dt_k} = (G^{(v)})^{-1} B_k G^{(v)} - (G^{(v)})^{-1} \frac{dG^{(v)}}{dt_k}.
\]

(123)

Substituting (123) into (122) we transform it to the equation,

\[
\omega_{\mu\nu} = - \sum_{k=1}^L \sum_{v=1,...,n,\infty} \text{res}_{z=a_v} \text{Tr} \left( AB_k \right) dI_k + \sum_{k=1}^L \sum_{v=1,...,n,\infty} \text{res}_{z=a_v} \text{Tr} \left( A \frac{dG^{(v)}}{dt_k} (G^{(v)})^{-1} \right) dI_k.
\]

The function \( AB_k \) is rational, therefore sum of its residues is zero. So we get (10).

4.2 The proof of Lemma 2

Denote

\[
I = \sum_{k=1}^L \sum_{j=1}^d \sum_{v=1,...,n,\infty} \text{res}_{z=a_v} \text{Tr} \left[ \frac{\partial}{\partial m_j} \left( A \frac{dG^{(v)}}{dt_k} (G^{(v)})^{-1} \right) - \frac{\partial}{\partial t_k} \left( A \frac{dG^{(v)}}{dt_k} (G^{(v)})^{-1} \right) \right].
\]

We have

\[
I = \sum_{k=1}^L \sum_{j=1}^d \sum_{v=1,...,n,\infty} \text{res}_{z=a_v} \text{Tr} \left[ \frac{\partial A}{\partial m_j} \frac{dG^{(v)}}{dt_k} (G^{(v)})^{-1} - A \frac{\partial A}{\partial m_j} \frac{dG^{(v)}}{dt_k} (G^{(v)})^{-1} \frac{dG^{(v)}}{dt_k} (G^{(v)})^{-1} \right] - \frac{d\Theta_v}{dt_k} (G^{(v)})^{-1}
\]

\[
+ \sum_{k=1}^L \sum_{j=1}^d \sum_{v=1,...,n,\infty} \text{res}_{z=a_v} \text{Tr} \left( A \frac{\partial G^{(v)}}{\partial m_j} (G^{(v)})^{-1} - \frac{\partial A}{\partial m_j} \frac{dG^{(v)}}{dt_k} (G^{(v)})^{-1} \right).
\]

We use the formula (124) to get rid of \( \frac{dG^{(v)}}{dt_k} \) and equation (10) to replace \( \frac{\partial A}{\partial m_j} \), Omitting terms with zero residue and after some cancellations we have

\[
I = \sum_{k=1}^L \sum_{j=1}^d \sum_{v=1,...,n,\infty} \text{res}_{z=a_v} \text{Tr} \left[ - \frac{\partial A}{\partial m_j} G^{(v)} \frac{d\Theta_v}{dt_k} (G^{(v)})^{-1} + A G^{(v)} \frac{d\partial \Theta_v}{\partial m_j} (G^{(v)})^{-1} \frac{dG^{(v)}}{dt_k} (G^{(v)})^{-1} - \frac{dA}{dz} \frac{dG^{(v)}}{dt_k} (G^{(v)})^{-1} \right]
\]

\[
+ \sum_{k=1}^L \sum_{j=1}^d \sum_{v=1,...,n,\infty} \text{res}_{z=a_v} \text{Tr} \left( -A \frac{d\Theta_v}{\partial m_j} (G^{(v)})^{-1} \right).
\]

We replace \( B_k \) using again formula (124). After that, we notice that the residue of the derivative with respect to \( z \) of formal series is zero. Therefore we can “integrate by parts”, moving the derivative from one term to another. We do that with the term, where we replaced \( B_k \). Using (121), we have

\[
I = \sum_{k=1}^L \sum_{j=1}^d \sum_{v=1,...,n,\infty} \text{res}_{z=a_v} \text{Tr} \left[ - \frac{\partial A}{\partial m_j} G^{(v)} \frac{d\Theta_v}{dt_k} (G^{(v)})^{-1} + A G^{(v)} \frac{d\partial \Theta_v}{\partial m_j} (G^{(v)})^{-1} \frac{dG^{(v)}}{dt_k} (G^{(v)})^{-1} - \frac{dA}{dz} \frac{dG^{(v)}}{dt_k} (G^{(v)})^{-1} \right]
\]

\[
+ \sum_{k=1}^L \sum_{j=1}^d \sum_{v=1,...,n,\infty} \text{res}_{z=a_v} \text{Tr} \left( -A \frac{d\Theta_v}{\partial m_j} (G^{(v)})^{-1} \right).
\]

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\[
+ \sum_{k=1}^{L} \sum_{j=1}^{d} \sum_{\nu=1,...,n,\infty} \text{res}_{z=a_{\nu}} \text{Tr} \left( -G^{(\nu)} \frac{\partial \Theta_{\nu}}{\partial k} (G^{(\nu)})^{-1} \frac{\partial G^{(\nu)}}{\partial m_{j}} (G^{(\nu)})^{-1} A + \frac{\partial \Theta_{\nu}}{\partial k} (G^{(\nu)})^{-1} \frac{\partial G^{(\nu)}}{\partial m_{j}} \frac{d \Theta_{\nu}}{dz} \right) \\
+ \sum_{k=1}^{L} \sum_{j=1}^{d} \sum_{\nu=1,...,n,\infty} \text{res}_{z=a_{\nu}} \text{Tr} \left( G^{(\nu)} \frac{\partial \Theta_{\nu}}{\partial k} (G^{(\nu)})^{-1} \frac{\partial^{2} G^{(\nu)}}{\partial z \partial m_{j}} (G^{(\nu)})^{-1} \right).
\]

Finally using (121) one more time we get \( I = 0 \).

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References


