THREE FACETS OF EQUITY IN STEFFE'S RESEARCH PROGRAMS

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The NCTM research committee made a recent, urgent call for mathematics education researchers to "examine and deeply reflect on our research practices through an equity lens." With this in mind, we use this paper to reflect on the ways in which Steffe's work has contributed to three facets of equity. We also suggest opportunities for researchers working within this framework to deepen their commitments to issues of equity.

Keywords: Equity, cognition, radical constructivism

The percentages that Steffe (2017) gives in his plenary paper are alarming because they indicate that current standards, and curricular materials based on these standards, are insufficient for a large percentage of students in grades K-8. For example, a majority of students entering 6th grade are not structuring number and quantity in ways that are required for the significant multiplicative reasoning that is the target of most middle school mathematics standards and curricular materials (e.g., developing proportional reasoning, an understanding of rates, etc.) For us, this phenomenon is fundamentally an issue of equity: As it stands, current standards and curricular materials are inequitable if they do not meet the learning needs of a significant number of elementary and middle school students.[1]

So, we take this opportunity to discuss Steffe's research in relation to the NCTM Research Committee's recent, urgent call for mathematics education researchers to "examine and deeply reflect on our research practices with an equity lens" (Aguirre et al., 2017, p. 125). We start with the important caveat that Steffe has not explicitly analyzed the ways in which race, culture, ethnicity, gender, and socio-economic status impact learning opportunities for students in school mathematics either at a broad level or in his specific interactions with students. This caveat may lead some mathematics education researchers to simply dismiss Steffe's work; after all, isn't this omission simply another way of saying that Steffe has studied mathematics teaching and learning using a colorblind framing that does not account for contextual or cultural factors in the teaching and learning process? We think that this conclusion is far too dismissive given Steffe's: a) profound commitment to unpacking what he terms students' mathematics; b) his drive to work with cognitively diverse students over long periods of time to learn this mathematics; and c) his repeated pushes to interrupt dominant discourses about what mathematicians, mathematics education researchers, and curriculum writers think should constitute school mathematics (e.g., Steffe, 1992, 1994; Steffe & Olive, 2010). We see this paper, then, as an opportunity to reflect on the ways that Steffe's research addresses facets of equity, as well as a space to call for

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1 Steffe (2017) might say that this statement is true for all students because standards and curricular materials under-challenge students who have interiorized three levels of units.

researchers using similar frameworks to deepen their commitment to issues of equity in the context of critiques of de-contextualized and/or colorblind framings of mathematics teaching and learning (e.g., Martin, Gholsson, & Leonard, 2010; Martin, 2009).

In our view, Steffe's research addresses at least three critical facets of equity: positionality and power relations, what counts as mathematics, and access and achievement. We start with an overview of how Steffe's research addresses these facets of equity. Then we provide data excerpts to illustrate each facet of equity. The excerpts are of middle school students who have interiorized one level of unit because their mathematical ways of operating are rarely reflected in current curricula and standards.

Three facets of equity

Positionality and power relations

In sociology, *positionality* refers to the "occupation or adoption of a particular position in relation to others, usually with reference to issues of culture, [race], ethnicity, or gender" (Oxford Dictionary on-line). Those who articulate their positionality are articulating their stance or viewpoint on themselves, others, and interaction between people, often with respect to societal identifiers. In his research, Steffe articulates his stance on himself as a teacher/researcher and on those with whom he interacts (students). This stance starts with self-reflexivity, which is a version of Gutierrez's mirror test (2016): "The principle of self-reflexivity compels teacher/researchers to consider their own knowledge of children's mathematics, including accommodations in it, as constantly being constructed as they interact with children as the children construct mathematical knowledge," where "Self-reflexivity involves applying one's epistemological tenets first and foremost to oneself" (2017).

Although this positionality does not address culture, race, ethnicity, or gender, we argue that it does address relations of power between a teacher and students. As Cobb (2007) points out, Steffe's research paradigm is an actor-oriented perspective, concerned with "small scale" human interactions that are useful (although not sufficient) for instructional design at the classroom level, not an observer-oriented "large scale" view of societal structures that focuses on how people participate (or are barred from participating) in cultural practices. So, the power that Steffe addresses has to do with power relations in student-teacher relationships. Although some may see that as a limited view of power from the perspective of social science more broadly (e.g., Foucault), it nevertheless bears directly on the idea that power is intertwined with knowledge and that those in power (including mathematics education researchers) are those who determine what we count as knowledge.

Steffe's orientation is that his own mathematics (his own first-order knowledge) is insufficient to understand children's mathematics (their first-order knowledge). For example, Steffe states: "I usually find it inappropriate to attribute even my most fundamental mathematical concepts and operations to children" (2010b, p. 17). Instead of doing that, he positions students as rational mathematical thinkers who have mathematical knowledge to which he does not have direct access. Steffe positions himself, as a teacher/researcher, as someone who must learn from children "how and in what ways they operate mathematically" (2017) and who must *"create* operations that if a child had those operations, the child would operate as observed" (2017). This statement is a statement about making a second-order model of a student's thinking (the mathematics of students), which he views as mathematical knowledge—as legitimate mathematics. Indeed, for Steffe, second-order knowledge is social knowledge co-constructed by

him and children (2010b). Thus, the students with whom he interacts have power to determine what we count as knowledge—in fact, students are primary in his student-teacher relationships because he could not learn students' mathematics (i.e., create the mathematics of students) without interacting with them. This stance positions students as the generators of knowledge.

What counts as mathematics

There are numerous examples of the creation of second-order knowledge in Steffe's research, starting with the five counting sequences that model how children undergo significant reorganizations in creating and structuring units and quantity in their construction of what we call whole numbers (e.g., Steffe & Cobb, 1988; Steffe, von Glasersfeld, Richards, & Cobb, 1983). In this paper we give an example of fractional knowledge that Steffe learned from students: the partitive fraction scheme (Steffe, 2002, 2010a). Often standards documents and curricular materials define fractions as parts out of wholes (e.g., 4/5 is four parts out of five parts) or as multiples of unit fractions (e.g., 4/5 is 4 x 1/5) (CCSSM, 2010). These definitions do not reflect students' ways and means of operating as they construct fractional knowledge because they omit a lot and, as Steffe (2017) points out, they may ask students to conceive of fractions in ways that are not within their current possibilities in the near term (cf. Norton & Boyce, 2013).

Students who have interiorized only two levels of units have the potential to construct partitive fraction schemes (Steffe, 2010a). Students who construct this scheme create fractions from iterating (repeating) a unit fraction some number of times. So, for example, if asked to draw 4/5 of a granola bar they partition the bar into five equal parts and then take one of those parts four times. This activity looks like these students see 4/5 as 4 x 1/5—they are repeating 1/5 four times, after all. However, the 4-part bar that is the result of their activity is, *for them*, four parts out of five parts—it has a part-whole meaning. So, when these students are asked to draw 7/5 of a granola bar, they will often object that doing so does not make sense because you can't take seven parts out of five (Olive & Steffe, 2001). Constructing partitive fractions is an advance over fractions conceived of only as parts in relation to wholes. However, students who have constructed only partitive fraction schemes do not yet see fractions as consisting of sequences of fractional numbers (e.g., 1/5, 2/5, 3/5, 4/5, 5/5, 6/5, 7/5, etc.), as Steffe (2017) points out. In addition, students who have constructed partitive fraction schemes have just begun to think of fractions as measurable extents—they have not completed this process (Steffe & Olive, 2010).

Yet rather than position students who have constructed partitive fraction schemes as deficient or behind, Steffe argues that these students' mathematics is a legitimate mathematics that should be the basis for developing curricula and instruction in schools. That is, he states: "rather than assume a God-like stance regarding 'school mathematics,' I assume that I must intensively interact with my students to learn what their mathematics might be before I can begin to think about what 'school mathematics' might be" (1992, p. 261). He critiques school mathematics texts—even reform texts—as being based on the writers' first-order knowledge of school mathematics. This phenomenon "places the mathematics of schooling outside of the minds of the students who are to learn it and is manifest in the univocal expression of concepts like multiplication and division. One searches the school mathematics books in vain for a mathematics of children, and school mathematics is taken to be the way it is rather than the way students make it to be" (1992, p. 260). So, Steffe views school mathematics—mathematical knowledge—as something that should be squarely based on students' mathematical ideas.

Access and achievement

In our view, developing curricular tasks and instructional materials for students who have constructed partitive fraction schemes is about the issue of access. Gutiérrez (2009) characterizes *access* to be about "the resources that students have available to them to participate in mathematics, including such things as: quality mathematics teachers, adequate technology and supplies in the classroom, a rigorous curriculum, a classroom environment that invites participation, and infrastructure for learning outside of class hours" (p. 5). She characterizes *achievement* as about student outcomes, including "participation in a given class, course taking patterns, standardized test scores, and participation in the math pipeline (e.g., majoring in mathematics in college, having a math-based career)" (p. 5). She positions access and achievement at the ends of the "dominant" axis, and power and identity at the ends of the "critical" axis in her framework on equity.

Steffe's research does not directly address some items in Gutiérrez's (2009) list of resources for access, such as adequate technology and supplies in the classroom, or infrastructure for learning outside of class hours. However, Steffe's research is about increasing access to mathematical ideas and participation because it redefines what is being accessed. Rather than position mathematics as something outside of the minds of students to be accessed, he positions mathematics as being created by students, and so it is something that they have access to already, in a sense. Thus, his job as a teacher/researcher to facilitate this access is to create second-order models of students' ways of operating that allow him (and others who work in a similar vein) to interact with students so that their mathematics can surface and so that they can build on their ways of thinking from wherever they are. And, further, his job is to create learning trajectories, which he refers to as third-order models (2017), as curricula that would constitute school mathematics. Steffe's call to base school mathematics on the mathematics of students means students' achievement is defined as making progress from where they are—as learning.

Three Examples

In this section we illustrate each aspect of equity with data of student-teacher interactions. We aim to paint a picture of what student-teacher interactions based on models of students looks like, because we argue that, done well, these interactions open significant opportunities for participation and learning. Although this statement is true for all students, it is striking for those who have interiorized only one level of unit in middle school, because these students' ways of thinking are typically not reflected in or addressed by school mathematics (e.g., Hackenberg, 2013). So, all three examples in this section are of students who have interiorized one level of unit.

Here we present a few aspects of our second-order models of these students to help readers interpret the data: Students who have interiorized one level of unit view numbers as composite units (units of units)—e.g., 5 is five 1s and also one 5. However, for these students there is not a multiplicative relationship between the units of 1 and composite units. In addition, these students have yet to construct disembedding operations, whereby they can lift part of a number out of the number and not destroy the number, e.g., take 10 out of 14 while keeping 14 intact. Since these students cannot yet disembed, they don't reason strategically when combining numbers additively. For example, to determine 14 + 18, they typically count on by 1s from one of the numbers. In contrast, students who have constructed disembedding operations can separate 14 into 10 and 2 and 2, combine one 2 with the 18 to make 20, and then add on the 10 and 2 to get 32. This strategic additive reasoning is not in the province of students who have interiorized one

level of unit. To make assessments of students' levels of units, we often use problems that involve embedded units, as we demonstrate next.

Power and Positionality: Hal coordinating one level of unit

We examine Hal's response to the *Candy Factory Problem* to show what it looks like when a 7th grade student has interiorized only one level of unit. We then analyze how the teacher positioned himself in relation to Hal and the impact of this positionality on power dynamics in a student-teacher relationship.

Candy Factory Problem: A candy factory puts 6 candies in each package, puts 8 packages in each box, and puts 4 boxes in each crate. Make a picture to show the number of candies in one crate.

Data Excerpt 1: Hal solves the *Candy Factory Problem.*[2]
[The teacher reads the problem to Hal. Hal draws Figure 1a.]
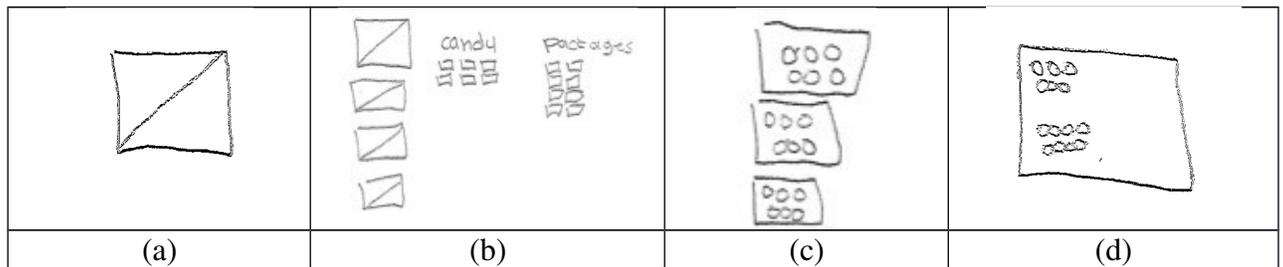

| (a) | (b) | (c) | (d) |

Figure 1. Hal's response to the *Candy Factory Problem.*

T: Okay. What do you got there?
H: A crate.
T: And can you, like, the candies…so it says 6 candies in each package, puts 8 packages in each box, and puts 4 boxes in each crate. So can you draw what would be inside the crate? [Hal draws Figure 1b.] So these are your six candies, eight packages, and four crates [points to each part of Figure 1b]? [Hal nods.]
….
[The teacher asks Hal to re-read the question. The teacher then asks Hal to draw a single package containing six candies. The teacher asks Hal to draw a second package, and then a third package (Figure 1c).]
T: How many packages does it say would be in one box?
H: Eight.
T: Yeah. So could you draw everything that would be in one box?
H: Six candies and eight packages [draws Figure 1d.]
….

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2 In the data excerpts, T stands for teacher/researcher, H for Hal, K for Kianna, and W for witness-researcher. Comments enclosed in brackets describe students' nonverbal action or interaction from the teacher/researcher's perspective. Ellipses (…) indicate a sentence or idea that seems to trail off. Four periods (….) denote omitted dialogue.

[The teacher returns to asking Hal to draw a fourth package with six candies in it, adding on to Figure 1c. He then asks Hal how many total candies there would be if he had a fifth package.]

H: Thirty.

T: Thirty? Okay. How did you know it was 30?

H: Because 5 times 6 is 30.

This excerpt illustrates that Hal initially did not consider candies, packages, or boxes to be contained in a crate—he drew a single crate (Figure 1a). He subsequently drew the candies and packages outside of the original crate, re-interpreted the one crate as a box, and drew three more boxes (Figure 1b). These drawings provide indication that he assimilated the situation using a single level of unit (e.g., a crate or a box or a package or a candy). With support from the teacher/researcher, Hal established a drawing where candies were contained within packages (Figure 1c). However, this structure seemed ephemeral for him because when the teacher asked him to use it to show "everything that would be in one box," he drew six candies and eight packages, separately, inside of the box (Figure 1d). Doing so indicates that he did not use a two-levels-of-units structure, a package containing six candies, when he created his box.

Interestingly, the last part of the excerpt demonstrates that Hal could use multiplication facts to solve problems—in fact, later in the interview it was evident that he knew and could use many multiplication facts, which is not atypical for middle grades students, even those that are coordinating solely one level of unit (Norton & Boyce, 2015). However, this phenomenon does not mean that "knowing multiplication facts" for students like Hal results from the structure and imagery that is typically assumed when students use such facts.

We contend that middle grades students like Hal are often silenced or invisible in the classroom; they are often positioned as deficient and behind. In fact, even for the teacher/researcher (who was an experienced middle school teacher), Hal's response to this problem was surprising, and it took significant adjustment on his part in order to be responsive to Hal in the moment. Ultimately the teacher/researcher abandoned the original problem as it was stated in favor of presenting problems that were related but did not involve all of the levels of units as the original problem. The teacher/researcher did so because he interpreted his primary goal of interacting with Hal to be to learn Hal's mathematics. This goal, when taken seriously, can be quite humbling, because even an experienced teacher can quickly realize the insufficiency of his or her own mathematical thinking in bringing forth productive mathematical reasoning on the part of the student. So, positioning oneself as Steffe (2017) does is not at all a simple challenge for mathematics education researchers.

Indeed, we think such an orientation needs to be learned anew in each student-teacher interaction in order for mathematics education researchers and teachers to avoid positioning themselves in a "God-like" role. When a teacher ceases to position themselves as learner (for example, by assuming they know what a student should learn prior to interacting with a student), they reify their prior knowledge as the knowledge to be learned rather than entering interactions openly. Notably, however, this does not mean that teachers or researchers should enter interactions with students unprepared, but rather with a genuine openness to students' contributions to these interactions. In the interaction with Hal, it would have been possible to simply "coach" him through creating a representation for solving the problem where the teacher

would have learned very little about the structure Hal attributed to the situation. We have witnessed many middle school students who have interiorized one level of unit experience this kind of coaching in schools.

Expanding what counts as mathematics: Kianna solves the Coordinate Points Problem

We turn now to a second 7th grade student who was part of the same study, and who had also interiorized one level of unit, to examine how she worked through and solved the *Coordinate Points Problem*.

> *Coordinate Points Problem.* You have number cards that have the numbers 1 through 8 on them. You draw a card, replace it, and draw a second card to create a coordinate point (e.g., 1, 2).
> a.   How many coordinate points could you make? Represent these points as an array.
> b.   Suppose you added one additional number card that has the number 9 on it. How many *new* coordinate points could you make?

In typical curricula this problem could be considered to be about "finding the difference of two squares"—the difference between 8^2 and 9^2. Successive iterations of this problem (e.g., starting at 9 and adding the 10 card) could open the possibility for students to consider that the difference between two squares is non-constant, and that the difference of these differences is constant (it is 2). We suggest that seeing the task as univocally about "finding the difference of squares" elides students' mathematics. We use data from Kianna to illustrate what a different characterization affords in terms of seeing what the challenges and successes were for a student who has interiorized one level of unit, and how in turn this characterization serves to expand what counts as mathematics.

We anticipated that, for Kianna, creating pairs and taking them as countable items would be a challenge. Therefore, we asked her to list aloud the new pairs as she was creating them and to keep track of how many pairs she had created. Kianna started the problem by listing aloud the new coordinate points, putting up a finger of her left hand each time she said a new coordinate point and reusing the fingers of her left hand once she had used all five of them. Then the teacher/researcher asked her how many she had created. Kianna seemed uncertain. She listed several calculations that she appeared to think were relevant (e.g., 9 + 9 and 9 x 9). Instead of asking her to compute, the teacher/researcher responded as follows.

> **Data Excerpt 2:** Kianna solves the *Coordinate Points Problem*.
> T: You want to just say them out loud again? You had a pretty cool method before. You want to just keep using that?
> K [smiles]: Yeah. Okay. So, one nine, two nine, three nine, four nine, five nine [puts up a finger of her left hand each time she says a coordinate point until all five fingers on her left hand are raised], six nine [she begins to reuse the fingers of her left hand each time she says a coordinate point], seven nine, eight nine, nine...[is about to say nine-nine, but stops herself]. I mean, one nine [re-states one-nine instead of saying nine-one], two nine [finishes using the fingers on her left hand a second time], three nine, four nine, five nine, six nine, seven nine [finishes using the fingers on her left had a third time], eight nine,

nine nine [puts up the thumb and index finger of her left hand]. That's twelve new ones. Yeah, that's twelve.

T: Twelve? Let's try one more time.

K: Ah. Okay. [Smiles broadly].

T: Do you want to try one more time?

K [emphatically]: Yeah.

T: Okay. You're so close.

K: Okay. One nine, two nine, three nine, four nine, five nine, six nine, seven nine, eight nine, one nine, two nine, three nine, four nine, five nine, six nine [is keeping track on her left hand in a similar manner as the previous attempt]... I said that wrong [realizing she has said one nine, two nine, instead of nine one, nine two, etc.]. Okay. One nine, two nine, three nine, four nine, five nine, six nine, seven nine, eight nine, nine one, nine two, nine three, nine four, nine five, nine six, nine seven, nine eight, nine nine [keeps track in a similar manner on her left hand as previously]. It would be eighteen?

T: Eighteen. You're so close.

W: How'd you get eighteen?

K: I was trying to count them on my fingers. I have a problem when I go past fifteen.

T: You're doing great.

W: Can I ask a quick question?

K: Yeah.

W: How many past fifteen did you get?

K: I think I got two more.

W: Okay. So, what's two more than fifteen?

K: Seventeen.

W: Yeah.

For Kianna, solving the problem appeared to be both challenging and satisfying. Kianna often said she did not like mathematics because she could not "see" herself in her mathematics classes. Her attitude about the interview, however, differed significantly from that: She smiled throughout, acknowledged and accepted the challenges presented to her, and successfully solved problems that were hard for her. We argue that this was possible because the teacher/researcher planned activity for her based on the mathematics of students who have interiorized one level of unit and can make two levels of units in activity, and he used this model as a basis for being responsive to her in the moment. For example, he asked Kianna to verbally list the pairs, which meant he supported her to produce pairs in her activity. He planned this activity for her because creating a pair (coordinate point) in activity is similar to creating a two-levels-of-units structure in activity: Both involve counting two units as a single unit (Tillema, 2013).

After watching Kianna make two attempts at enumerating the pairs, and arriving at 12 and then 18 pairs, the witness-researcher suspected that she was having difficulty coordinating the number of times she had counted by five on her left hand (three) with the number of left over fingers she had (two). Her response of 12 likely stemmed from a lack of differentiation of the number of times she had used all the fingers on her left hand and the two remaining fingers she used—when she reviewed her activity, she equated the two remaining fingers with the number of times she had used all of the fingers on her hand (two): Two hands and two leftover fingers

would give 12. Her response of 18 likely stemmed from a similar lack of differentiation, except that in this response she seemed to substitute the three times she used her hand for the number of leftover fingers; three hands and three leftover fingers would give 18. The witness-researcher interacted responsively with her, assuming that this might be the conflation that she was making and so supported her to review the number of leftover fingers she had beyond 15 to determine she had counted 17 coordinate points.

Kianna's response of 17 coordinate points is numerically equivalent to the difference between 9^2 and 8^2. However, we think that claiming that she found the difference of two squares does not match well with Kianna's mathematics. In fact, we think it conflates the mathematics of the observer with the mathematics of a student by failing to differentiate between the two.

Instead, we think that Kianna's problem entailed creating and counting pairs that contained nine in either the first or second position, where her counting activity involved coordinating the number of times she used the five fingers on her left hand (three) with the number of leftover fingers she had on her left hand at the end of her count (two). There was not evidence that she established in a single structure the total number of pairs that could be created with nine number cards (9^2), the number of pairs that could be created with eight number cards (8^2), and the 17 newly created pairs that had the number nine in either the first or second position. Coordinating these three quantities in a single structure could be initial evidence for considering a students' mathematics to be compatible with something that might be called "finding the difference of two squares." Even though Kianna did not do this, her way of operating was fundamentally interesting to us and to her, involved challenging mathematics for her, and imbued her with a sense of mathematical power—she was a participant in producing the solution to what she considered a challenging problem.

Access and achievement: Alyssa's work on symbolizing her reasoning

We now turn to an example from a different study within a 5-year project to investigate how to differentiate instruction for middle school mathematics students, as well as relationships between students' rational number knowledge and algebraic reasoning. The current phase of the project involves the second author in co-teaching 25-30 day classroom units with a classroom teacher in which the teacher and project team design to differentiate instruction. In the first of these classroom design experiments, the 20-student 8th grade pre-algebra class consisted of five students who had interiorized one level of unit, 13 students who had interiorized two levels of units, and two students who had interiorized three levels of units. The focus of the instruction was equivalence in algebraic contexts, following the *Say It With Symbols* unit from the 3rd edition of the Connected Mathematics Project (Lappan et al., 2014).

One of the students who had interiorized only one level of unit, Alyssa, struggled with most of the ideas in the unit. For example, in class on Day 7 students were learning to factor expressions based on "reversing" the Distributive Property. To factor 6 + 2x Alyssa wrote 3(2 + x). Even after two conversations between the second author and Alyssa's group about what products they were aiming for, Alyssa still wrote 3(2 + x) while her groupmates had expressions like 2(3 + x). Later, on Day 14 students were solving an equation to find the break-even point in a situation that involved a school group selling boxes of greeting cards. Because Alyssa and another group mate were struggling to solve the equation, the second author worked with them on understanding the situation—finding the profit when different amounts of boxes were sold: 1, 5, 10, and 20. After seeing that all of these amounts resulted in losing money, the group mate

was ready to increase the number of boxes to find the break-even point, but Alyssa suggested that they try 3 boxes or 15 boxes. In general, keeping track of the multiple quantities involved in determining profit (number of boxes, revenue, expenses) was challenging for Alyssa.

During a mid-unit interview, the second author posed to Alyssa a question similar to one worked on in class about developing an expression for the amount of money a swimmer raised in a swim-a-thon, where each sponsor gave the swimmer $10 to start and $2 per lap. There were 15 sponsors. Alyssa wrote "$10 + 2x + 15$," where x was the number of laps. Her rationale was that the swimmer was getting more, so "then you're adding, is what I thought." When asked how much money one sponsor gave the swimmer, Alyssa suggested "$10*15 + 2x$", She explained as follows: "the 15 is how many sponsors and then they start with $10 so I did 10 times 15 to give the amount of money that she's getting." She added the $2x$ because "for every lap they're giving her more money." But then she was concerned about the $150 because it seemed like too much money from one sponsor. So, although she had just identified the 150 as coming from 15 sponsors, she then thought it was from just one.

With questioning support similar to what we have shown in the prior two data excerpts, Alyssa developed correct numerical responses for the swimmer swimming 4 laps with 1 sponsor and then 2 sponsors. However, she appeared to be "in" the activity of reasoning through these specific outcomes and did not stand above them in order to abstract a structure that she could represent algebraically. The second author expected this phenomenon, to some degree, based on her second-order model of students like Alyssa working on algebraic problems (Hackenberg, 2013). So, the second author drew from evolving second-order knowledge of Alyssa's ways of thinking that opened possibilities for Alyssa to be mathematically active—i.e., to access her mathematical ways of thinking in the context of the problem, and thereby to participate mathematically. In contrast, in math class Alyssa often followed along with the responses of group mates and did not seem mathematically active. In other words, she often did not seem to access her mathematical ways of thinking.

Interestingly, like Kianna, Alyssa appeared to find the interview pleasing in that in the school days that followed she asked the second author why her math class couldn't be like the interview because what they were doing in math class did not make sense to her, implying that the interview was sensible and even enjoyable. So, in the interaction during the interview, Alyssa appeared to experience herself as capable of doing mathematics in a way that she did not regularly experience in mathematics classrooms. Her comments and demeanor further support the claim that in the interview she had access to mathematical activity in a way that was pleasing and unusual for her.

If the second author had had more time with Alyssa on the swim-a-thon problem (e.g., if the problem were a classroom task), she would have continued to work with numerical examples for the amount of money earned in swimming 4 laps with different numbers of sponsors to learn whether Alyssa could abstract a pattern from her activity that she could represent algebraically. Exploring these possibilities with Alyssa would have promoted Alyssa's achievement in the sense of learning. If Alyssa's classroom tasks were designed similarly to this task, how would her access to mathematical activity and her mathematical achievement, or learning, change? We can't say for sure, of course. However, students who tend to feel like mathematics makes sense and who feel that their ideas are valued are certainly more likely to participate regularly and actively, in comparison with students who generally feel that they don't understand in mathematics classrooms, which was the case for Alyssa. Being active mathematically is certainly

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necessary for learning and achievement more generally, although it does not guarantee any particular learning or achievement.

Concluding Remarks

We have aimed to show how Steffe's research programs address three aspects of equity: positionality and power relations in student-teacher relationships, what counts as mathematics, and access and achievement. In doing so, we have seen how intertwined these three aspects are —it is hard to draw a boundary between them, because each mutually influences the other. For example, by positioning students as the generators of mathematical knowledge, Steffe expands what counts as mathematical knowledge: Students' mathematics counts as mathematics, or rather, the mathematics of students, since that is what he creates based on his interactions with students. Steffe advocates that this mathematics become the basis for curricular design, which has implications for access in the sense of how mathematically active a student might be in their classroom interactions with a teacher, and achievement in the sense that this helps to re-define what success might look like in mathematics classrooms.

We do note that, throughout his work, Steffe focuses on cognitive diversity. Over the course of his career he has worked with students from diverse backgrounds including different racial, cultural, ethnic, gender, and socio-economic backgrounds. Thus the participants in his studies have been diverse in these ways, but this has not been the focus of his analyses. We see this observation as an opportunity for researchers working within this tradition to continue to expand their analytic lens. We see at least three promising possibilities for such an expansion: a) explicit analyses of student-teacher interactions that account for how race, culture, ethnicity, gender, or socio-economic status impact the mathematics that a teacher-researcher is able to bring forth in interactions with students; b) design or teaching experiments that embed the goal of making second order models of students mathematics in situations that address a substantial social issue; and c) explicit attempts, based on second order models of students' mathematics, to influence policy discussions.

To examine what letter *b* might look like, we highlight one of our current graduate students who has used Steffe's framework as a basis for selecting students into a design experiment in which he created mathematical problems that opened the way for students to consider racial bias in jury selection (Gatza, in press). Gatza is working to unpack how, for example, different students' understanding of randomness or a limiting process (including differences based on level of units coordination) impact how they reason about issues of racial bias, and in turn how their understanding of issues of racial bias impact their understanding of randomness or a limiting process (in ways that may not strictly be accounted for based on differences in units coordination). We see unpacking these complex relationships as one avenue for researchers working in this tradition to deepen their commitments to issues of equity.

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