FINITE TYPE MODULES AND BETHE ANSATZ EQUATIONS

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Abstract. We introduce and study a category $\mathcal{O}^{fin}_{\mathfrak{b}}$ of modules of the Borel subalgebra $U_q\mathfrak{b}$ of a quantum affine algebra $U_q\mathfrak{g}$, where the commutative algebra of Drinfeld generators $h_{i,r}$, corresponding to Cartan currents, has finitely many characteristic values. This category is a natural extension of the category of finite-dimensional $U_q\mathfrak{g}$ modules. In particular, we classify the irreducible objects, discuss their properties, and describe the combinatorics of the q-characters. We study transfer matrices corresponding to modules in $\mathcal{O}^{fin}_{\mathfrak{b}}$. Among them we find the Baxter Q_i operators and T_i operators satisfying relations of the form $T_iQ_i = \prod_j Q_j + \prod_k Q_k$. We show that these operators are polynomials of the spectral parameter after a suitable normalization. This allows us to prove the Bethe ansatz equations for the zeroes of the eigenvalues of the Q_i operators acting in an arbitrary finite-dimensional representation of $U_q\mathfrak{g}$.

1. Introduction

The XXZ models are the celebrated integrable models whose Hamiltonians originate in quantum affine algebras.

Let $U_q\mathfrak{g}$ be a quantum affine algebra. The universal R matrix is a special element $\mathcal{R}$ of the completion $U_q\mathfrak{g}\widehat{\otimes}U_q\mathfrak{g}$ which intertwines the standard comultiplication in $U_q\mathfrak{g}$ with the opposite one. Given a $U_q\mathfrak{g}$ module V, the transfer matrix is the trace $T_V = \mathrm{Tr}_V(\pi_V \otimes \mathrm{id})(\mathcal{R})$. Due to the properties of the R matrix (the Yang-Baxter equation), transfer matrices for various representations commute: $T_VT_W = T_WT_V$, and therefore give rise to a family of commuting operators in a completion of $U_q\mathfrak{g}$. These operators act on any finite-dimensional $U_q\mathfrak{g}$ module. They are called the XXZ Hamiltonians.

The XXZ models received a lot of attention after the pioneering work [Ba] where the $U_q\mathfrak{sl}_2$ transfer matrix was studied in relation to the 6-vertex lattice model. The spectrum of the model is usually obtained via various forms of the Bethe ansatz. In the standard approach of algebraic or coordinate Bethe ansatz, one writes a certain vector, called wave function, depending on parameters. When the parameters satisfy a system of algebraic equations, called Bethe ansatz equations, the wave function becomes an eigenvector of the Hamiltonians.

In this paper we describe the spectrum of the XXZ Hamiltonians using the so-called analytic Bethe ansatz. In particular, using this method, we deduce the Bethe ansatz equations from the absence of poles for a certain operator, as anticipated in [R], [FR], [FH].

The crucial observation is that the R matrix is not only an element of the completed tensor square of $U_q\mathfrak{g}$ but an element of $U_q\mathfrak{b}\widehat{\otimes}U_q\overline{\mathfrak{b}}$, where $U_q\mathfrak{b}$ and $U_q\overline{\mathfrak{b}}$ are the Borel subalgebras of $U_q\mathfrak{g}$. Therefore, in addition to transfer matrices T_V where V is a $U_q\mathfrak{g}$ module, one can consider transfer matrices T_V where V is a $U_q\mathfrak{b}$ module. The algebra $U_q\mathfrak{b}$ has a significantly richer

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representation theory than that of $U_q\mathfrak{g}$ and this additional freedom allows us to solve the problem.

First, we develop the representation theory of $U_q\mathfrak{b}$.

In the category $\mathcal{O}_\mathfrak{b}$ of highest ℓ-weight $U_q\mathfrak{b}$ modules introduced in [HJ], we define a subcategory $\mathcal{O}_\mathfrak{b}^{fin}$ of modules of finite type. By definition, a module is of finite type if the joint spectrum of the Cartan generators $h_{i,r}$ (in the Drinfeld realization of $U_q\mathfrak{g}$) is finite. Here $i \in I$, I is the set of labels of simple roots for the finite-dimensional algebra corresponding to $\mathfrak{g}$, and $r > 0$.

The positive fundamental modules $M^+_{i,a}$, $i \in I$, $a \in \mathbb{C}^\times$, are irreducible modules with simplest possible highest ℓ-weight given by $\Psi_j(z) = (a^{-1/2} - a^{1/2}z)\delta_{i,j} + (1 - \delta_{i,j})$, $j \in I$. They first appeared in [BLZ] in the case $U_q\widehat{\mathfrak{sl}}_2$, and then were studied in [BHK], [BJMST], [HJ], [BFLMS], [FH]. By [FH], the positive fundamental modules are of finite type. We prove that the category $\mathcal{O}_\mathfrak{b}^{fin}$ is topologically generated by finite-dimensional modules (which are essentially restrictions of $U_q\mathfrak{g}$ modules) and the positive fundamental modules.

The category $\mathcal{O}_\mathfrak{b}^{fin}$ naturally extends the category of finite-dimensional $U_q\mathfrak{g}$ modules and has similar properties. In particular, the theory of q-characters provides a powerful tool for the study. We show that the Drinfeld coproduct can be used to construct non-trivial examples of modules of finite type and describe various properties of such modules.

The model example for the analytic Bethe ansatz is the work [Ba], where the $U_q\widehat{\mathfrak{sl}}_2$ transfer matrix was studied in relation to the 6-vertex lattice model. It is based on the existence of the operators $T(a)$ and $Q(a)$, acting in any $U_q\widehat{\mathfrak{sl}}_2$ module W, which satisfy the famous Baxter's TQ relation:

$$T(a)Q(a) = A(a)Q(aq^{-2}) + D(a)Q(aq^2)\,. \tag{1.1}$$

Here $A(a)$ and $D(a)$ are explicit scalar functions depending on W. Since $Q(a)$ is known to be polynomial and $T(a)$ regular, one obtains the equations for the zeroes of $Q(a)$. These are the Bethe ansatz equations.

The T operator was long known to be the suitably normalized transfer matrix corresponding to the 2-dimensional irreducible evaluation $U_q\widehat{\mathfrak{sl}}_2$ module $V(a)$, $T(a) = \bar{T}_{V(a)}$. Recently, the Baxter's operator $Q(a)$ was identified with a suitably normalized transfer matrix as well. This transfer matrix turns out to be related to the positive fundamental module (also called the q oscillator representation [BLZ]). Then the relation (1.1) is interpreted as an equation in the Grothendieck ring of $U_q\mathfrak{b}$ modules (see e.g. [JMS]):

$$[V_{aq^{-1}}][M^+_a] = [M^+_{aq^{-2}}] + [M^+_{aq^2}]\,. \tag{1.2}$$

The functions $A(a)$ and $D(a)$ appear from normalization (depending on W) of the transfer matrices T_{V_a} and $T_{M^+_a}$.

For the general quantum algebra, one has the operators $Q_i(a)$, $i \in I$, which are normalized transfer matrices related to representations $M^+_{i,a}$. They are shown to be polynomial in [FH]. But there are no 2-dimensional modules to complete the argument. In this article we find an appropriate substitute in the category $\mathcal{O}_\mathfrak{b}^{fin}$.

More precisely, we construct modules $N^+_{i,a}$ ($i \in I$, $a \in \mathbb{C}^\times$) which are infinite-dimensional but 2-finite. It means that, similarly to a 2-dimensional $U_q\widehat{\mathfrak{sl}}_2$ module, the algebra generated $h_{j,r}$,

$r > 0$, $j \in I$, has only two different eigenvalues in $N_{i,a}$. Then we show that the modules $N^+_{i,a}$ and $M^+_{i,a}$ satisfy a two term relation similar to (1.2):

$$[N^+_{i,a}][M^+_{i,a}] = \prod_{j:C_{j,i}\neq 0} [M^+_{j,aq_j^{-C_{j,i}}}] + \prod_{j:C_{j,i}\neq 0} [M^+_{j,aq_j^{C_{j,i}}}]\,, \tag{1.3}$$

where $(C_{i,j})_{i,j\in I}$ is the Cartan matrix.

For example, in the case of $U_q\widehat{\mathfrak{sl}}_3$, consider two irreducible $U_q\mathfrak{b}$ modules with highest ℓ-weights

$$\Big(q\frac{1-aq^{-2}z}{1-az}\;,\;1\Big), \quad \text{and} \quad \Big(q\frac{1-aq^{-2}z}{1-az}\;,\;(aq)^{-1/2}(1-aqz)\Big),$$

respectively. The first one is the three dimensional evaluation module. It is a restriction of a $U_q\widehat{\mathfrak{sl}}_3$ module. The second module is not a restriction of any $U_q\widehat{\mathfrak{sl}}_3$ module. It is an infinite dimensional but 2-finite module. We denote it $N^+_{1,a}$. Then equation (1.3) becomes:

$$[N^+_{1,a}][M^+_{1,a}] = [M^+_{1,aq^{-2}}][M^+_{2,aq}] + [M^+_{1,aq^2}][M^+_{2,aq^{-1}}]\,.$$

The two term relation (1.3) immediately leads to the relation for transfer matrices (7.2) similar to (1.1). We show that transfer matrices related to $U_q\mathfrak{b}$ modules $M^+_{i,a}$ and $N^+_{i,a}$ are polynomial on each vector after suitable explicit normalization (see Proposition 7.3 and Proposition 7.4).

Ultimately, it gives a proof of the Bethe ansatz equations (7.6).

In particular, it implies that the spectrum of a transfer matrix related to $U_q\mathfrak{b}$ module of category $\mathcal{O}_\mathfrak{b}$ can be explicitly described in terms of solutions of the Bethe ansatz equation, see Theorem 7.5. For transfer matrices related to the finite-dimensional modules it was conjectured in [FR].

The same method works also for quantum toroidal algebras. The simplest case of type $\mathfrak{gl}_1$ has been studied in our paper [FJMM2].

The paper is constructed as follows. We set up the notation, and remind the standard facts about the algebras in Section 2, and their representations in Section 3. In Section 4 we discuss the q-characters in the context of Borel subalgebras. Then in Section 5 we prove a number of facts about $U_q\mathfrak{b}$ modules which we use later. The main results of Section 5 are Lemma 5.7 and Proposition 5.11. In Section 6 we introduce and study the category of finite type modules. In Section 7 we define the XXZ Hamiltonians, show polynomiality of transfer matrices and deduce the Bethe ansatz equations.

While preparing this paper, there appeared a work [FH2] where closely related functional relations for Q operators are derived. However the polynomial property which is essential for deriving Bethe ansatz equations is not discussed there.

2. Preliminaries

In this section we collect basic definitions concerning quantum loop algebras and their Borel subalgebras. We follow closely the notation of [FH].

2.1. **Notation.** Let $C = (C_{i,j})_{0\le i,j\le n}$ be an indecomposable Cartan matrix of non-twisted affine type. We denote by $\mathfrak{g}$ the Kac-Moody Lie algebra associated with C. Set $I = \{1, \dots, n\}$, and denote by $\mathring{\mathfrak{g}}$ the finite-dimensional simple Lie algebra associated with the Cartan matrix $(C_{i,j})_{i,j\in I}$. Let $\{\alpha_i\}_{i\in I}$, $\{\alpha_i^\vee\}_{i\in I}$, $\{\omega_i\}_{i\in I}$, $\{\omega_i^\vee\}_{i\in I}$ be the simple roots, the simple coroots, the fundamental weights and the fundamental coweights of $\mathring{\mathfrak{g}}$, respectively. We set $Q = \oplus_{i\in I}\mathbb{Z}\,\alpha_i$, $Q^+ = \oplus_{i\in I}\mathbb{Z}_{\ge 0}\,\alpha_i$, $P = \oplus_{i\in I}\mathbb{Z}\,\omega_i$. We denote by Δ the set of all roots and by Δ^+ the set of all positive roots of $\mathring{\mathfrak{g}}$. Let $D = \mathrm{diag}(d_0, \dots, d_n)$ be the unique diagonal matrix such that $B = DC$ is symmetric and that d_i's are relatively prime positive integers. We denote by $(\ ,\ ) : P\times P \to \mathbb{Q}$ the invariant symmetric bilinear form such that $(\alpha_i, \alpha_i) = 2d_i$. Let $a_0, \dots, a_n$ stand for the Kac label ([Kac], pp.55-56). We have $a_0 = 1$.

Throughout this paper, we fix a non-zero complex number q which is not a root of unity. We fix $\hbar \in \mathbb{C}$ such that $q = e^\hbar$, and write $q^\lambda = e^{\lambda\hbar}$ for $\lambda \in \mathbb{C}$. We set $q_i = q^{d_i}$, $q_{i,j} = q^{(\alpha_i,\alpha_j)} = q_{j,i}$. We use the standard symbols for q-integers

$$[m]_v = \frac{v^m - v^{-m}}{v - v^{-1}}\,, \quad [m]_v! = \prod_{k=1}^{m}[k]_v\,, \quad \begin{bmatrix} m \\ k \end{bmatrix}_v = \frac{[m]_v!}{[k]_v![m-k]_v!}\,.$$

2.2. **Quantum loop algebra.** The quantum loop algebra $U_q\mathfrak{g}$ is the $\mathbb{C}$-algebra defined by generators $e_i,\ f_i,\ k_i^{\pm1}$ $(0 \le i \le n)$ and the following relations for $0 \le i, j \le n$.

$$\begin{aligned}
&k_i k_j = k_j k_i, \quad k_0^{a_0} k_1^{a_1} \cdots k_n^{a_n} = 1,\\
&k_i e_j k_i^{-1} = q_{i,j} e_j, \quad k_i f_j k_i^{-1} = q_{i,j}^{-1} f_j,\\
&[e_i, f_j] = \delta_{i,j}\frac{k_i - k_i^{-1}}{q_i - q_i^{-1}},\\
&\sum_{k=0}^{\ell_{i,j}} (-1)^k x_i^{(\ell_{i,j}-k)} x_j x_i^{(k)} = 0 \quad (i \ne j,\ x = e, f).
\end{aligned}$$

In the last relation, we set $\ell_{i,j} = 1 - C_{i,j}$ and $x_i^{(k)} = x_i^k/[k]_{q_i}!$ for $x_i = e_i, f_i$.

The algebra $U_q\mathfrak{g}$ can also be presented in terms of the Drinfeld generators [Dr, Be]

$$x^{\pm}_{i,m}\ (i \in I, m \in \mathbb{Z}), \quad h_{i,r}\ (i \in I, r \in \mathbb{Z}\backslash\{0\}), \quad k_i^{\pm1}\ (i \in I)\,.$$

We shall use the generating series

$$\begin{aligned}
x_i^{\pm}(z) &= \sum_{m\in\mathbb{Z}} x^{\pm}_{i,m} z^m\,,\\
\phi_i^{\pm}(z) &= \sum_{m\in\mathbb{Z}} \phi^{\pm}_{i,m} z^m = k_i^{\pm1} \exp\Bigl(\pm(q_i - q_i^{-1}) \sum_{\pm r>0} h_{i,r} z^r\Bigr)\,.
\end{aligned}$$

Then for all $i,j\in I$ and $r,s\in\mathbb{Z}\backslash\{0\}$ we have

$$\phi_i^\epsilon(z)\phi_j^{\epsilon'}(w)=\phi_j^{\epsilon'}(w)\phi_i^\epsilon(z)\quad(\epsilon,\epsilon'\in\{+,-\})\,, \tag{2.1}$$

$$(q_{i,j}^{\pm1}z-w)\phi_i^\epsilon(z)x_j^\pm(w)=(z-q_{i,j}^{\pm1}w)x_j^\pm(w)\phi_i^\epsilon(z)\quad(\epsilon\in\{+,-\}), \tag{2.2}$$

$$[x_i^+(z),x_j^-(w)]=\delta_{i,j}\delta(z/w)\frac{\phi_i^+(z)-\phi_i^-(z)}{q_i-q_i^{-1}}\,, \tag{2.3}$$

$$\big(q_{i,j}^{\pm1}z-w\big)x_i^\pm(z)x_j^\pm(w)=\big(z-q_{i,j}^{\pm1}w\big)x_j^\pm(w)x_i^\pm(z)\,, \tag{2.4}$$

$$\operatorname*{Sym}_{z_1,\dots,z_{\ell_{i,j}}}\sum_{k=0}^{\ell_{i,j}}(-1)^k\begin{bmatrix}\ell_{i,j}\\ k\end{bmatrix}_{q_i}x_i^\pm(z_1)\cdots x_i^\pm(z_k)x_j^\pm(w)x_i^\pm(z_{k+1})\cdots x_i^\pm(z_{\ell_{i,j}})=0\quad(i\neq j). \tag{2.5}$$

In the last line we set

$$\operatorname*{Sym}_{z_1,\dots,z_\ell}f(z_1,\dots,z_\ell)=\frac{1}{\ell!}\sum_{\sigma\in S_\ell}f\big(z_{\sigma(1)},\dots,z_{\sigma(\ell)}\big)\,.$$

We have in particular

$$k_ix_j^\pm(z)k_i^{-1}=q_{i,j}^{\pm1}x_j^\pm(z)\,,$$

$$[h_{i,r},x_j^\pm(z)]=\pm\frac{[rC_{i,j}]_{q_i}}{r}z^{-r}x_j^\pm(z)\,. \tag{2.6}$$

Let $U_q^\pm\mathfrak{g}$ be the subalgebra of $U_q\mathfrak{g}$ generated by the $x_{i,m}^\pm$ ($i\in I,m\in\mathbb{Z}$), and let $U_q^0\mathfrak{g}$ be the one generated by the $k_i^{\pm1}$, $h_{i,r}$ ($i\in I,r\in\mathbb{Z}\backslash\{0\}$). We have a triangular decomposition

$$U_q\mathfrak{g}\simeq U_q^-\mathfrak{g}\otimes U_q^0\mathfrak{g}\otimes U_q^+\mathfrak{g}\,. \tag{2.7}$$

We denote by $\mathfrak{t}$ the subalgebra of $U_q\mathfrak{g}$ generated by the $k_i^{\pm1}$ ($i\in I$).

The algebra $U_q\mathfrak{g}$ has a $Q\times\mathbb{Z}$-grading given by

$$\deg e_i=(\alpha_i,0),\quad \deg f_i=(-\alpha_i,0),\quad \deg k_i=(0,0)\quad (i\in I),$$
$$\deg e_0=(\mathring{\alpha}_0,1),\quad \deg f_0=(-\mathring{\alpha}_0,-1),\quad \deg k_0=(0,0)\,,$$

where $\mathring{\alpha}_0=-\sum_{i\in I}a_i\alpha_i$. We have $\deg x_{i,m}^\pm=(\pm\alpha_i,m)$ and $\deg h_{i,r}=(0,r)$ for $i\in I$, $m\in\mathbb{Z}$, $r\in\mathbb{Z}\backslash\{0\}$. If $\deg x=(\beta,m)$ then we say that x has *weight* β and *homogeneous degree* m, and write $\beta=\operatorname{wt}x$, $m=\operatorname{hdeg}x$. We denote by $(U_q\mathfrak{g})_\beta$ the graded component of $U_q\mathfrak{g}$ of weight β.

2.3. **Hopf algebra structure.** The algebra $U_q\mathfrak{g}$ has a Hopf algebra structure

$$\Delta(e_i)=e_i\otimes1+k_i\otimes e_i,\quad \Delta(f_i)=f_i\otimes k_i^{-1}+1\otimes f_i,\quad \Delta(k_i)=k_i\otimes k_i\,,$$
$$\varepsilon(e_i)=0,\quad \varepsilon(f_i)=0,\quad \varepsilon(k_i)=1\,,$$
$$S(e_i)=-k_i^{-1}e_i,\quad S(f_i)=-f_ik_i,\quad S(k_i)=k_i^{-1}\,,$$

where $i=0,\dots,n$.

The following Proposition gives partial information about the coproduct of the Drinfeld generators.

Proposition 2.1. [Da] *For $i\in I$ and $r>0$, we have*

$$\Delta\left(\phi_{i,\pm r}^\pm\right)\in\sum_{0\le l\le r}\phi_{i,\pm l}^\pm\otimes\phi_{i,\pm(r-l)}^\pm+\sum_{\beta\in Q^+\backslash\{0\}}\big(U_q\mathfrak{g}\big)_{-\beta}\otimes\big(U_q\mathfrak{g}\big)_\beta\,. \tag{2.8}$$

2.4. **Borel algebras.** We define two subalgebras of $U_q\mathfrak{g}$ as follows.

$$U_q\mathfrak{b} = \langle e_i \ (0 \le i \le n), \quad k_i^{\pm 1}(i \in I)\rangle,$$
$$U_q\overline{\mathfrak{b}} = \langle f_i \ (0 \le i \le n), \quad k_i^{\pm 1}(i \in I)\rangle \,.$$

We call $U_q\mathfrak{b}$ the *positive Borel algebra* (or simply the Borel algebra), and $U_q\overline{\mathfrak{b}}$ the *negative Borel algebra*. Algebras $U_q\mathfrak{b}$, $U_q\overline{\mathfrak{b}}$ are both Hopf subalgebras of $U_q\mathfrak{g}$.

Set $U_q^{\pm}\mathfrak{b} = U_q^{\pm}\mathfrak{g} \cap U_q\mathfrak{b}$ and $U_q^0\mathfrak{b} = U_q^0\mathfrak{g} \cap U_q\mathfrak{b}$. From the result of [Be] it follows that we have the triangular decomposition

$$U_q\mathfrak{b} \simeq U_q^-\mathfrak{b} \otimes U_q^0\mathfrak{b} \otimes U_q^+\mathfrak{b}. \tag{2.9}$$

In terms of the Drinfeld generators we have

$$U_q^+\mathfrak{b} = \langle x^+_{i,m} \ (i \in I, m \ge 0)\rangle \,, \quad U_q^0\mathfrak{b} = \langle h_{i,r}, k_i^{\pm 1} \ (i \in I, r > 0)\rangle \,.$$

We have also $U_q^-\mathfrak{b} \supset \langle x^-_{i,r} \ (i \in I, r > 0)\rangle$ but the inclusion is proper except in the case $\mathring{\mathfrak{g}} = \mathfrak{sl}_2$.

In [Be, Da], certain root vectors $e_\beta \in U_q\mathfrak{b}$, $f_\beta \in U_q\overline{\mathfrak{b}}$ are introduced with each positive real root β of $\mathfrak{g}$. In Appendix we give a review of their definition and basic facts. The subalgebras $U_q^{\pm}\mathfrak{b}$ are generated by root vectors,

$$U_q^+\mathfrak{b} = \langle e_{m\delta+\alpha} \mid m \ge 0, \ \alpha \in \Delta^+\rangle \,,$$
$$U_q^-\mathfrak{b} = \langle k_\alpha e_{m\delta-\alpha} \mid m > 0, \ \alpha \in \Delta^+\rangle \,,$$

where we set $k_\alpha = \prod_{i\in I} k_i^{b_i}$ for $\alpha = \sum_{i \in I} b_i\alpha_i$.

The root vectors have a convexity property (A.10) with respect to a total ordering

$$\beta_0 \prec \beta_{-1} \prec \beta_{-2} \prec \cdots \prec 2\delta \prec \delta \prec \cdots \prec \beta_3 \prec \beta_2 \prec \beta_1 \,, \tag{2.10}$$

where $\{\beta_r \mid r \le 0\} = \{k\delta + \alpha \mid k \ge 0, \ \alpha \in \Delta^+\}$, $\{\beta_r \mid r \ge 1\} = \{k\delta - \alpha \mid k > 0, \ \alpha \in \Delta^+\}$; see (A.1), (A.2), (A.9).

2.5. **Universal R matrix.** It is well known that $U_q\mathfrak{g}$ is equipped with the universal R matrix $\mathcal{R} \in U_q\mathfrak{b}\widehat{\otimes} U_q\overline{\mathfrak{b}}$, which satisfies

$$\mathcal{R}\,\Delta(x) = \Delta^{op}(x)\,\mathcal{R} \quad (x \in U_q\mathfrak{g}),$$
$$\big(\Delta \otimes \mathrm{id}\big)\mathcal{R} = \mathcal{R}_{1,3}\mathcal{R}_{2,3}\,,$$
$$\big(\mathrm{id} \otimes \Delta\big)\mathcal{R} = \mathcal{R}_{1,3}\mathcal{R}_{1,2}\,.$$

Here $\Delta^{op} = \sigma \circ \Delta$, $\sigma(a \otimes b) = b \otimes a$, and $\mathcal{R}_{1,2} = \mathcal{R} \otimes 1$, etc.. It has the product form

$$\mathcal{R} = \mathcal{R}_+\mathcal{R}_0\mathcal{R}_- q^{-t_\infty}\,, \tag{2.11}$$

where each factor is given in terms of root vectors as follows.

$$\mathcal{R}_+ = \prod_{r\le 0} \exp_{q_{\beta_r}}\left(-(q_{\beta_r}-q_{\beta_r}^{-1}) e_{\beta_r}\otimes f_{\beta_r}\right)\,, \tag{2.12}$$

$$\mathcal{R}_0 = \exp\Bigl(-\sum_{\substack{r>0\\ i,j\in I}} \frac{r\widetilde{B}_{i,j}(q^r)}{q^r-q^{-r}}(q_i-q_i^{-1})(q_j-q_j^{-1})\, h_{i,r}\otimes h_{j,-r}\Bigr)\,, \tag{2.13}$$

$$\mathcal{R}_- = \prod_{r\ge 1} \exp_{q_{\beta_r}}\left(-(q_{\beta_r}-q_{\beta_r}^{-1}) e_{\beta_r}\otimes f_{\beta_r}\right)\,. \tag{2.14}$$

In these formulas, we set $q_\beta = q^{(\beta,\beta)/2}$, $\exp_q(x) = \sum_{n=0}^\infty x^n \prod_{j=1}^n (q^2-1)/(q^{2j}-1)$. The matrix $\bigl(\widetilde{B}_{i,j}(q)\bigr)_{i,j\in I}$ is the inverse matrix of $\bigl([d_iC_{i,j}]_q\bigr)_{i,j\in I}$. In (2.12) and (2.14), the product is ordered from left to right in the decreasing order of r as in (2.10). The element t_∞ is formally given by $\sum_{i,j\in I} d_i(C^{-1})_{i,j} h_{i,0}\otimes h_{j,0}$, where we set $k_i = q_i^{h_{i,0}}$. The expression q^{-t_∞} has a well defined meaning on a tensor product of weight modules.

2.6. **Drinfeld coproduct.** Along with the 'standard' coproduct Δ introduced above, we also make use of the so-called Drinfeld coproduct Δ_D given by

$$\begin{aligned} \Delta_D\bigl(x_i^+(z)\bigr) &= x_i^+(z)\otimes 1 + \phi_i^-(z)\otimes x_i^+(z)\,,\\ \Delta_D\bigl(x_i^-(z)\bigr) &= x_i^-(z)\otimes \phi_i^+(z) + 1\otimes x_i^-(z)\,,\\ \Delta_D\bigl(\phi_i^\pm(z)\bigr) &= \phi_i^\pm(z)\otimes\phi_i^\pm(z)\,. \end{aligned} \tag{2.15}$$

Since the terms in the right hand side involve infinite sums of generators, Δ_D is not a coproduct in the usual sense. Nevertheless, under certain circumstances it can be used to define a module structure on tensor products of representations, see Section 5.2.

The Drinfeld coproduct Δ_D and the standard coproduct Δ are related to each other through the factor (2.12) of the universal R matrix:

Proposition 2.2. [EKP] *For any $x\in U_q\mathfrak{g}$ we have*

$$\Delta_D(x) = \sigma(\mathcal{R}_+)^{-1}\cdot\Delta(x)\cdot\sigma(\mathcal{R}_+)\,.$$

The Borel subalgebra $U_q\mathfrak{b}$ is *not* a Hopf subalgebra of $U_q\mathfrak{g}$ with respect to Δ_D. Proposition 2.2 implies that it is rather a coideal, i.e.,

$$\Delta_D\bigl(U_q\mathfrak{b}\bigr)\;\subset\; U_q\mathfrak{g}\,\widehat{\otimes}\,U_q\mathfrak{b}\,. \tag{2.16}$$

3. Representations of $U_q\mathfrak{g}$ and $U_q\mathfrak{b}$

We review known facts about representations of $U_q\mathfrak{g}$ and $U_q\mathfrak{b}$.

3.1. **Weights and ℓ-weights.** We begin by introducing some terminology.

First let $\mathfrak{t}^* = \bigl(\mathbb{C}^\times\bigr)^I$. For $\lambda\in P$ we define elements $\mathfrak{q}^\lambda = \bigl(q_j^{(\lambda,\alpha_j^\vee)}\bigr)_{j\in I}\in\mathfrak{t}^*$. For a $U_q\mathfrak{g}$ module V and $\mu = \bigl(\mu_i\bigr)_{i\in I}\in\mathfrak{t}^*$, we set

$$V_\mu = \{v\in V \mid k_i\, v = \mu_i v \quad (i\in I)\}\,.$$

We have then $\phi^\pm_{i,r}(V_\mu)\subset V_\mu$ and $x^\pm_{i,r}(V_\mu)\subset V_{\mu\,\mathsf{q}^{\pm\alpha_i}}$ for all $i\in I$ and $r\in\mathbb{Z}$. We say that $\mu\in\mathfrak{t}^*$ is a weight of V if $V_\mu\neq 0$. The set of weights of V is denoted by $\mathrm{wt}\,V$. We say that V is $\mathfrak{t}$-diagonalizable if $V=\bigoplus_{\mu\in\mathrm{wt}\,V}V_\mu$.

Next let $\mathfrak{t}^*_{\ell,\mathfrak{g}}$ denote the set of all pairs $\boldsymbol{\Psi}=(\boldsymbol{\Psi}^+,\boldsymbol{\Psi}^-)$ consisting of I-tuples of formal series

$$\boldsymbol{\Psi}^\pm=\bigl(\Psi^\pm_i(z)\bigr)_{i\in I},\quad \Psi^\pm_i(z)=\sum_{\pm m\geq 0}\Psi^\pm_{i,m}z^m\in\mathbb{C}[[z^{\pm1}]]\,,$$

such that $\Psi^+_{i,0}\Psi^-_{i,0}=1$ for all $i\in I$. The sets $\mathfrak{t}^*$ and $\mathfrak{t}^*_{\ell,\mathfrak{g}}$ are both abelian groups by pointwise multiplication, and we have a group homomorphism $\varpi:\mathfrak{t}^*_{\ell,\mathfrak{g}}\to\mathfrak{t}^*$ given by $\varpi(\boldsymbol{\Psi})=\bigl(\Psi^+_{i,0}\bigr)_{i\in I}$.

For a $\mathfrak{t}$-diagonalizable $U_q\mathfrak{g}$ module $V=\bigoplus_{\mu\in\mathrm{wt}\,V}V_\mu$ and $\boldsymbol{\Psi}\in\mathfrak{t}^*_{\ell,\mathfrak{g}}$, we set

$$\begin{aligned}V_{\boldsymbol{\Psi}}=\{v\in V_{\varpi(\boldsymbol{\Psi})}\mid &\text{ there exists a } p\geq 0 \text{ such that}\\ &(\phi^\epsilon_{i,m}-\Psi^\epsilon_{i,m})^p v=0\ (i\in I,\epsilon m\geq 0,\epsilon\in\{+,-\})\}\,.\end{aligned}$$

If $V_{\boldsymbol{\Psi}}\neq 0$, then we call it ℓ-weight space of V of ℓ-weight $\boldsymbol{\Psi}$.

We say that a $U_q\mathfrak{g}$ module V is a highest ℓ-weight module of highest ℓ-weight $\boldsymbol{\Psi}\in\mathfrak{t}^*_{\ell,\mathfrak{g}}$ if it is generated by a non-zero vector $v\in V$ such that

$$U_q^+\mathfrak{g}\cdot v=\mathbb{C}v\,,\quad \phi^\epsilon_i(z)v=\Psi^\epsilon_i(z)v\quad (i\in I,\ \epsilon\in\{+,-\}).$$

If it is the case, we say v is a highest ℓ-weight vector of V. For each $\boldsymbol{\Psi}\in\mathfrak{t}^*_{\ell,\mathfrak{g}}$, there exists a unique simple highest ℓ-weight module of highest ℓ-weight $\boldsymbol{\Psi}$. We denote it by $L(\boldsymbol{\Psi})$. Owing to the triangular decomposition (2.7), $\mathrm{wt}\,L(\boldsymbol{\Psi})$ is contained in the set

$$D(\mu)=\{\mu\,\mathsf{q}^{-\beta}\mid\beta\in Q^+\}$$

where $\mu=\varpi(\boldsymbol{\Psi})$.

For $U_q\mathfrak{b}$ modules, weight and weight space are defined in the same way as above. The notion of ℓ-weight is defined similarly, using a single I-tuple of formal series

$$\boldsymbol{\Psi}=(\Psi_i(z))_{i\in I},\quad \Psi_i(z)=\sum_{m\geq 0}\Psi_{i,m}z^m\in\mathbb{C}[[z]]$$

satisfying $\Psi_{i,0}\neq 0$. We denote by $\mathfrak{t}^*_{\ell,\mathfrak{b}}$ the set of all such $\boldsymbol{\Psi}$'s. A highest ℓ-weight module V of $U_q\mathfrak{b}$ is defined by the conditions that $V=U_q\mathfrak{b}\cdot v$, $v\neq 0$, and

$$U_q^+\mathfrak{b}\cdot v=\mathbb{C}v\,,\quad \phi^+_i(z)v=\Psi_i(z)v\quad (i\in I).$$

The unique simple highest ℓ-weight module of highest ℓ-weight $\boldsymbol{\Psi}\in\mathfrak{t}^*_{\ell,\mathfrak{b}}$ is denote by $L(\boldsymbol{\Psi})$.

3.2. **Category $\mathcal{O}_\mathfrak{g}$.** We consider a full subcategory $\mathcal{O}_\mathfrak{g}$ of the category of all $U_q\mathfrak{g}$ modules. By definition, a $U_q\mathfrak{g}$ module V is an object of category $\mathcal{O}_\mathfrak{g}$ if the following conditions (i)–(iii) hold:

(i) V is $\mathfrak{t}$-diagonalizable,
(ii) $\dim V_\mu<\infty$ for all μ,
(iii) There exist $\mu_1,\cdots,\mu_N\in\mathfrak{t}^*$ such that $\mathrm{wt}\,V\subset D(\mu_1)\cup\cdots\cup D(\mu_N)$.

Category $\mathcal{O}_\mathfrak{g}$ is a monoidal category.

Simple objects of $\mathcal{O}_\mathfrak{g}$ are classified by highest ℓ-weights as explained below. We say that $\boldsymbol{\Psi}\in\mathfrak{t}^*_{\ell,\mathfrak{g}}$ is rational if there exists an I-tuple of rational functions $\{f_i(z)\}_{i\in I}$ such that $f_i(z)$ is regular at $z=0,\infty$, satisfies $f_i(0)f_i(\infty)=1$, and that $\Psi^\pm_i(z)$ are expansions of $f_i(z)$ at $z^{\pm1}=0$

for all $i \in I$. If it is the case, we do not make distinction between the formal series $\Psi_i^{\pm}(z)$ and the rational function $f_i(z)$. We denote by $\mathfrak{r}_{\mathfrak{g}}$ the set of rational elements of $\mathfrak{t}^*_{\ell,\mathfrak{g}}$.

Theorem 3.1. [MY] *Let $\mathbf{\Psi} \in \mathfrak{t}^*_{\ell,\mathfrak{g}}$. Then all weight spaces of $L(\mathbf{\Psi})$ are finite dimensional if and only if $\mathbf{\Psi}$ is rational. The map $\mathbf{\Psi} \mapsto L(\mathbf{\Psi})$ gives a bijection between the set of all rational $\mathbf{\Psi} \in \mathfrak{r}_{\mathfrak{g}}$ and isomorphism classes of simple objects of $\mathcal{O}_{\mathfrak{g}}$.*

We quote also a classical result concerning finite dimensional modules. A $\mathfrak{t}$-diagonalizable $U_q\mathfrak{g}$ module is said to be of type 1 if all weights are of the form $\left(q_i^{(\Lambda,\alpha_i^\vee)}\right)_{i\in I}$ with some $\Lambda \in P$.

Theorem 3.2. [CP] *A $U_q\mathfrak{g}$ module V of type 1 is finite dimensional if and only if $V = L(\mathbf{\Psi})$ with some $\mathbf{\Psi} \in \mathfrak{r}_{\mathfrak{g}}$ which has the form*

$$\Psi_i^{\pm}(z) = q_i^{\deg P_i} \frac{P_i(q_i^{-1}z)}{P_i(q_i z)} \quad (i \in I)\,,$$

where $P_i(z)$ is a monic polynomial satisfying $P_i(0) = 1$.

As in the case of Kac-Moody Lie algebras, [Kac], Section 9.6, the notion of the multiplicity $[V : L(\mathbf{\Psi})]$ of a simple object $L(\mathbf{\Psi})$, $\mathbf{\Psi} \in \mathfrak{r}_{\mathfrak{g}}$, in $V \in \operatorname{Ob}\mathcal{O}_{\mathfrak{g}}$ is well-defined. Following [HL], we define the Grothendieck ring $\operatorname{Rep} U_q\mathfrak{g}$ of category $\mathcal{O}_{\mathfrak{g}}$ as follows. By definition, $\operatorname{Rep} U_q\mathfrak{g}$ is an additive group of maps $c : \mathfrak{r}_{\mathfrak{g}} \to \mathbb{Z}$ such that $\operatorname{supp}(c)$ is contained in a finite union of the $D(\mu)$'s, and that $\operatorname{supp}(c) \cap \varpi^{-1}(\omega)$ is a finite set for any $\omega \in \mathfrak{t}^*$. Here we write $\operatorname{supp}(c) = \{\mathbf{\Psi} \in \mathfrak{r}_{\mathfrak{g}} \mid c(\mathbf{\Psi}) \neq 0\}$. The ring structure is defined by

$$(cc')(\mathbf{\Psi}'') = \sum_{\substack{\mathbf{\Psi},\mathbf{\Psi}' \in \mathfrak{r}_{\mathfrak{g}} \\ \mathbf{\Psi}\mathbf{\Psi}' = \mathbf{\Psi}''}} c(\mathbf{\Psi})c'(\mathbf{\Psi}')[L(\mathbf{\Psi}) \otimes L(\mathbf{\Psi}') : L(\mathbf{\Psi}'')].$$

We write an element c of $\operatorname{Rep} U_q\mathfrak{g}$ also as a formal sum $\sum_{\mathbf{\Psi}\in\mathfrak{r}_{\mathfrak{g}}} c(\mathbf{\Psi})[L(\mathbf{\Psi})]$ where $[L(\mathbf{\Psi})]$ means the map $\mathbf{\Psi}' \mapsto \delta_{\mathbf{\Psi},\mathbf{\Psi}'}$. For $V \in \operatorname{Ob}\mathcal{O}_{\mathfrak{g}}$ we set $[V] = \sum_{\mathbf{\Psi}\in\mathfrak{r}_{\mathfrak{g}}}[V : L(\mathbf{\Psi})][L(\mathbf{\Psi})] \in \operatorname{Rep} U_q\mathfrak{g}$. We have $[V_1 \otimes V_2] = [V_1][V_2]$ and $[V_1 \oplus V_2] = [V_1] + [V_2]$. In addition, for any short exact sequence of $U_q\mathfrak{g}$ modules $0 \to V_1 \to V_2 \to V_3 \to 0$ we have the relation $[V_2] = [V_1] + [V_3]$ in $\operatorname{Rep} U_q\mathfrak{g}$.

3.3. **Category $\mathcal{O}_{\mathfrak{b}}$.** We define category $\mathcal{O}_{\mathfrak{b}}$ of $U_q\mathfrak{b}$ modules by the same conditions (i)–(iii) as in the previous subsection, replacing $U_q\mathfrak{g}$ modules with $U_q\mathfrak{b}$ modules. We denote $\operatorname{Rep} U_q\mathfrak{b}$ the corresponding Grothendieck ring. We say that $\mathbf{\Psi} \in \mathfrak{t}^*_{\ell,\mathfrak{b}}$ is rational if there exists an I-tuple of rational functions $\{f_i(z)\}_{i\in I}$ such that $f_i(z)$ is regular and non-zero at $z = 0$, and that $\Psi_i(z)$ is an expansion of $f_i(z)$ at $z = 0$ for all $i \in I$. We denote by $\mathfrak{r}_{\mathfrak{b}}$ the set of rational elements of $\mathfrak{t}^*_{\ell,\mathfrak{b}}$.

The following result is a counterpart to Theorem 3.1.

Theorem 3.3. [HJ] *Let $\mathbf{\Psi} \in \mathfrak{t}^*_{\ell,\mathfrak{b}}$. Then all weight spaces of $L(\mathbf{\Psi})$ are finite dimensional if and only if $\mathbf{\Psi}$ is rational. The map $\mathbf{\Psi} \mapsto L(\mathbf{\Psi})$ gives a bijection between the set of all rational elements $\mathbf{\Psi} \in \mathfrak{r}_{\mathfrak{b}}$ and isomorphism classes of simple objects of $\mathcal{O}_{\mathfrak{b}}$.*

We have an inclusion of spaces of rational functions $\mathfrak{r}_{\mathfrak{g}} \hookrightarrow \mathfrak{r}_{\mathfrak{b}}$.

Lemma 3.4. *The restriction functor $\operatorname{Res} : \mathcal{O}_{\mathfrak{g}} \to \mathcal{O}_{\mathfrak{b}}$ sends simple objects to simple objects.*

Proof. The proof is similar to those given in Proposition 3.5 of [HJ] and Lemma 3.3 of [FJMM2].

Let V be an object in $\mathcal{O}_{\mathfrak{g}}$. We fix $\mu \in \mathfrak{t}^*$ and $i \in I$.

Consider the set of operators $x^+_{i,m} : V_\mu \to V_{\mu \mathfrak{q}^{\alpha_i}}$ $(m \geq 0)$. Since $\mathrm{Hom}_{\mathbb{C}}(V_\mu, V_{\mu\mathfrak{q}^{\alpha_i}})$ is finite dimensional, we have a linear relation $\sum_{j=a}^{b} c_j x^+_{i,j}|_{V_\mu} = 0$ where $c_j \in \mathbb{C}$ $(0 < a \leq j \leq b)$ and $c_a c_b \neq 0$. Taking commutators with $h_{i,\pm1}$ we obtain $\sum_{j=a}^{b} c_j x^+_{i,j+r}|_{V_\mu} = 0$ for all $r \in \mathbb{Z}$. It follows that operators $x^+_{i,k}|_{V_\mu}$ $(k \in \mathbb{Z})$ belong to the linear span of $\{x^+_{i,m}|_{V_\mu}\}_{m\geq 0}$. By the same argument, operators $x^-_{i,k}|_{V_\mu}$ $(k \in \mathbb{Z})$ belong to the linear span of $\{x^-_{i,m}|_{V_\mu}\}_{m>0}$.

It is clear now that any singular vector of V with respect to $U_q\mathfrak{b}$ is also singular with respect to $U_q\mathfrak{g}$. It is also clear that if V is cyclic with respect to $U_q\mathfrak{g}$ then it is cyclic with respect to $U_q\mathfrak{b}$. The assertion of Lemma follows from these. □

There are objects of $\mathcal{O}_{\mathfrak{b}}$ which cannot be obtained by restricting simple objects of $\mathcal{O}_{\mathfrak{g}}$.

For example, the Borel algebra $U_q\mathfrak{b}$ has a large family of one-dimensional modules labeled by $\mathfrak{t}^*$. These are the modules $L(K)$ whose highest ℓ-weights are constants, $\Psi_i(z) = K_i \in \mathbb{C}^\times$. The only one-dimensional $U_q\mathfrak{b}$ module which is obtained as a restriction corresponds to the choice $K_i^2 = 1$, $i \in I$.

Nontrivial examples are the modules $M^\pm_{i,a}$ $(i \in I,\ a \in \mathbb{C}^\times)$ defined as follows.

$$M^\pm_{i,a} = L(\mathbf{\Psi}) \quad \text{where} \quad \Psi_j(z) = \begin{cases} a^{\mp 1/2}(1-az)^{\pm 1} & \text{for } j = i, \\ 1 & \text{for } j \neq i. \end{cases}$$

We call $M^+_{i,a}$ *positive fundamental module*, and $M^-_{i,a}$ *negative fundamental module*. We note that in [FH] the modules $M^\pm_{i,a} \otimes L\big((a^{\pm\delta_{ij}/2})_{j\in I}\big)$ are called 'prefundamental'.

3.4. **Dual category $\mathcal{O}^\vee_{\mathfrak{b}}$.** We say that a $U_q\mathfrak{b}$ module V is an object of category $\mathcal{O}^\vee_{\mathfrak{b}}$ if the following are satisfied.

(i) V is $\mathfrak{t}$-diagonalizable,
(ii) $\dim V_\mu < \infty$ for all μ,
(iii) There exist $\mu_1, \cdots, \mu_N \in \mathfrak{t}^*$ such that $\mathrm{wt}\, V \subset D(\mu_1)^{-1} \cup \cdots \cup D(\mu_N)^{-1}$.

A $U_q\mathfrak{b}$ module V is said to be of lowest ℓ-weight $\mathbf{\Psi} \in \mathfrak{t}^*_{\ell,\mathfrak{b}}$ if it is generated by a non-zero $v \in V$ satisfying

$$U_q^-\mathfrak{b} \cdot v = \mathbb{C}v\,, \quad \phi^+_i(z)v = \Psi_i(z)v \quad (i \in I)\,.$$

The unique simple $U_q\mathfrak{b}$ module of lowest ℓ-weight $\mathbf{\Psi} \in \mathfrak{t}^*_{\ell,\mathfrak{b}}$ is denoted $L^\vee(\mathbf{\Psi})$.

Let $V = \oplus_{\mu \in \mathrm{wt}\, V} V_\mu$ be a $U_q\mathfrak{b}$ module with finite dimensional weight spaces V_μ. Let $V^* = \oplus_{\mu\in \mathrm{wt}\, V} V^*_\mu$ be the graded dual space. We introduce a structure of left $U_q\mathfrak{b}$ module on V^* by setting

$$(xv^*)(v) = v^*\big(S^{-1}(x)v\big) \quad (v \in V,\ v^* \in V^*,\ x \in U_q\mathfrak{b}).$$

Lemma 3.5. [HJ] *A $U_q\mathfrak{b}$ module V is in category $\mathcal{O}^\vee_{\mathfrak{b}}$ if and only if V^* is in category $\mathcal{O}_{\mathfrak{b}}$. We have*

$$L^\vee(\mathbf{\Psi}) = \big(L(\mathbf{\Psi}^{-1})\big)^*\,.$$

4. The q characters.

The q-characters encode the ℓ-weights of a representation and provide a useful tool [FR]. Following [HJ], we recall their definition in the context of category $\mathcal{O}_{\mathfrak{b}}$ of $U_q\mathfrak{b}$ modules. Then we establish a few useful properties. With obvious changes, all the statements can be similarly proved for the category $\mathcal{O}_{\mathfrak{b}}^{\vee}$ of lowest weight modules as well.

4.1. **The definition of the q-characters.** First we prepare some notation.

In the group ring $\mathbb{Z}[\mathfrak{t}^*]$, we use the letter y_i^b to denote the element $\big(q_i^{b\delta_{i,j}}\big)_{j\in I} \in \mathfrak{t}^*$ where $i \in I$, $b \in \mathbb{C}$. We have $y_i^b y_i^{b'} = y_i^{b+b'}$, and $\mathsf{q}^{\omega_i} = y_i$, $\mathsf{q}^{\alpha_i} = \prod_{j\in I} y_j^{C_{j,i}}$. We identify $\mathbb{Z}[\mathfrak{t}^*]$ with the ring $\mathbb{Z}[y_i^b]_{i\in I, b\in\mathbb{C}}$.

The character of $V \in \mathrm{Ob}\,\mathcal{O}_{\mathfrak{b}}$ is the generating series of weight multiplicities defined by

$$\chi(V) = \sum_{\mu} \dim V_\mu \cdot \mu \,.$$

By the condition (iii) for category $\mathcal{O}_{\mathfrak{b}}$, the right hand side belongs to the ring $\mathcal{X}_0 = \mathbb{Z}[[\mathsf{q}^{-\alpha_i}]][y_i^b]_{i\in I, b\in\mathbb{C}}$ consisting of polynomials in y_i^b's whose coefficients are formal power series in the variables $\mathsf{q}^{-\alpha_i}$.

For each $i \in I$ and $a \in \mathbb{C}^\times$ we introduce further a new independent variable $X_{i,a}$, and set

$$Y_{i,a} = \frac{X_{i,aq_i^{-1}}}{X_{i,aq_i}}\,, \quad A_{i,a} = \prod_{j\in I} \frac{X_{j,aq_{j,i}^{-1}}}{X_{j,aq_{j,i}}}\,.$$

The monomials $Y_{i,a}$ and $A_{i,a}$ are affine analogs of fundamental weights and roots, respectively. We have

$$A_{i,a} = Y_{i,aq_i^{-1}} Y_{i,aq_i} \prod_{j\in I, C_{ji}=-1} Y_{j,a}^{-1} \prod_{j\in I, C_{ji}=-2} Y_{j,aq^{-1}}^{-1} Y_{j,aq}^{-1} \prod_{j\in I, C_{ji}=-3} Y_{j,aq^{-2}}^{-1} Y_{j,a}^{-1} Y_{j,aq^2}^{-1}\,.$$

Note that the $A_{i,a}$'s are algebraically independent.

Define a group isomorphism m between the multiplicative group $\mathfrak{r}_{\mathfrak{b}}$ and the group of monomials in the $X_{i,a}$'s and y_i^b's as follows. If $\mathbf{\Psi} = \big(\Psi_i(z)\big)_{i\in I} \in \mathfrak{r}_{\mathfrak{b}}$ has the form

$$\Psi_i(z) = q_i^{b_i} \prod_{r=1}^{k_i} (a_{i,r}^{-1/2} - a_{i,r}^{1/2} z) \prod_{s=1}^{l_i} (b_{i,s}^{-1/2} - b_{i,s}^{1/2} z)^{-1} \quad (b_i \in \mathbb{C}\,, a_{i,r}\,, b_{i,s} \in \mathbb{C}^\times),$$

then we set

$$m(\mathbf{\Psi}) = \prod_{i\in I} \big(y_i^{b_i} \prod_{r=1}^{k_i} X_{i,a_{i,r}} \prod_{s=1}^{l_i} X_{i,b_{i,s}}^{-1} \big)\,.$$

We use monomials to label ℓ-weights. Namely, if $\mathfrak{m} = m(\mathbf{\Psi})$, then we write the irreducible highest ℓ-weight module $L(\mathbf{\Psi})$ as $L(\mathfrak{m})$, and the ℓ-weight space $V_{\mathbf{\Psi}}$ as $V_{\mathfrak{m}}$. For example, we have $L(X_{i,a}^{\pm 1}) = M_{i,a}^{\pm}$.

The q-character of $V \in \mathrm{Ob}\,\mathcal{O}_{\mathfrak{b}}$ is the generating series of ℓ-weight multiplicities defined by

$$\chi_q(V) = \sum_{\mathfrak{m}} \dim V_{\mathfrak{m}} \cdot \mathfrak{m}\,.$$

We say that a monomial $\mathfrak{m}$ is in $\chi_q(V)$ if $\dim V_{\mathfrak{m}} \neq 0$. It is known that $\chi_q(V)$ belongs to the ring $\mathcal{X} = \mathbb{Z}[[A_{i,a}^{-1}]][X_{i,c}^{\pm 1}, y_i^b]_{i\in I, a,c\in\mathbb{C}^\times, b\in\mathbb{C}}$ consisting of polynomials in $X_{i,c}^{\pm 1}$'s and y_i^b's, whose coefficients are formal power series in the variables $A_{i,a}^{-1}$.

There is a ring homomorphism $\varpi : \mathcal{X} \to \mathcal{X}_0$ given by $\varpi(X_{i,a}) = y_i^{-\log a/(2\log q_i)}$ $(i \in I,\ a \in \mathbb{C}^\times)$, $\varpi(y_i^b) = y_i^b$ $(b \in \mathbb{C})$. We have $\varpi(Y_{i,a}) = y_i$, $\varpi(A_{i,a}) = \mathsf{q}^{\alpha_i}$, and $\varpi(\chi_q(V)) = \chi(V)$ for any $V \in \mathrm{Ob}\, \mathcal{O}_{\mathfrak{b}}$.

The following was stated in [FH].

Proposition 4.1. *The q-character map*

$$\chi_q : \ \mathrm{Rep}\, U_q\mathfrak{b} \to \mathcal{X}, \qquad V \mapsto \chi_q(V),$$

is an injective ring homomorphism.

Proof. The map χ_q is clearly well defined and linear. The property $\chi_q(V_1 \otimes V_2) = \chi_q(V_1)\chi_q(V_2)$ follows from (2.8).

Suppose $\chi_q(V_1) = \chi_q(V_2)$. Let $\mathfrak{m}$ be a monomial in $\chi_q(V_1)$ such that for any other monomial $\mathfrak{m}'$ in $\chi_q(V_1)$, the monomial $\varpi(\mathfrak{m}'/\mathfrak{m})$ does not belong to $\mathbb{Z}[\mathsf{q}^{\alpha_i}]_{i\in I}$. Choose vectors in V_1 and V_2 corresponding to $\mathfrak{m}$ which are eigenvectors of $\phi_i^+(z)$. Then these vectors are clearly singular. Therefore both V_1 and V_2 contain a subquotient module which is isomorphic to $L(\mathfrak{m})$. We quotient V_1, V_2 in Grothendieck ring by $L(\mathfrak{m})$ and repeat the argument. It follows that V_1 and V_2 give the same class in $\mathrm{Rep}\, U_q\mathfrak{b}$, and thus, χ_q is injective. ☐

For a subset $J \subset I$, let $U_q\mathfrak{b}_J = \langle e_j,\ k_j^{\pm} \mid j \in J\rangle$ denote the corresponding subalgebra of $U_q\mathfrak{b}$. We have the restriction functor $res_J : \ \mathcal{O}_{\mathfrak{b}} \to \mathcal{O}_{\mathfrak{b}_J}$. We define the corresponding ring homomorphism $res_J : \mathcal{X} \to \mathcal{X}_J = \mathbb{Z}[[A_{j,a}^{-1}]][X_{j,c}^{\pm 1}, y_j^b]_{j\in J, a,c\in\mathbb{C}^\times, b\in\mathbb{C}}$ sending $X_{i,a} \mapsto X_{i,a}$, $y_i^b \mapsto y_i^b$ if $i \in J$ and $X_{i,a} \mapsto 1$, $y_i^b \mapsto 1$ if $i \notin J$. Then we clearly have

$$\chi_q(res_J(V)) = res_J(\chi_q(V)), \qquad V \in \mathrm{Ob}\, \mathcal{O}_{\mathfrak{b}}.$$

4.2. **Examples.** The simplest examples are the one dimensional modules. For $\boldsymbol{b} = (b_1, \ldots, b_n) \in \mathbb{C}^I$, we set $y^{\boldsymbol{b}} = \prod_{i\in I} y_i^{b_i}$. Then we have $\chi_q(L(y^{\boldsymbol{b}})) = \chi(L(y^{\boldsymbol{b}})) = y^{\boldsymbol{b}}$.

Important examples of $U_q\mathfrak{b}$ modules and their q-characters are provided by restrictions of $U_q\mathfrak{g}$ modules, see Lemma 3.4 above. For $V \in \mathcal{O}_{\mathfrak{g}}$, we define $\chi_q(V)$ to be the q-character of V considered as a $U_q\mathfrak{b}$ module.

By [FM1], Theorem 4.1, the q-characters of finite-dimensional modules of $U_q\mathfrak{g}$ are polynomials in $Y_{i,a}^{\pm 1}$ of the form $\mathfrak{m}^+(1 + \sum_j \mathfrak{m}_j)$ where $\mathfrak{m}^+$ is a monomial in variables $Y_{i,a}$, $i \in I, a \in \mathbb{C}^\times$, cf. Theorem 3.2, and all $\mathfrak{m}_j$ are monomials in $A_{i,a}^{-1}$, $i \in I, a \in \mathbb{C}^\times$. The finite-dimensional modules $L(Y_{i,a}) \in \mathcal{O}_{\mathfrak{g}}$ are called $U_q\mathfrak{g}$ fundamental modules.

In general, the q-characters of finite-dimensional modules of $U_q\mathfrak{g}$ are difficult to describe, but in some cases they are known.

Example. The $U_q\widehat{\mathfrak{sl}}_{n+1}$ evaluation modules. Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0)$ be a partition with at most n parts and let λ' be the dual partition. A box $\square$ is a pair of positive integers $(i(\square), j(\square))$. We say $\square \in \lambda$ if $\lambda_{i(\square)} \geq j(\square)$. Define the content of a box by $c(\square) = j(\square) - i(\square)$. Let $\mathcal{T}(\lambda)$ be the set of semistandard Young tableaux of shape λ. For $\square \in \lambda$ and $T \in \mathcal{T}(\lambda)$, we have $T(\square) \in \{1, \ldots, n+1\}$.

Consider the $U_q\mathfrak{sl}_{n+1}$ irreducible module with highest ℓ-weight corresponding to a partition λ. Then the corresponding $U_q\widehat{\mathfrak{sl}}_{n+1}$ evaluation module (with an appropriate choice of the evaluation homomorphism) has the highest monomial $\mathfrak{m}^+_{\lambda,a} = \prod_{j=1}^{\lambda_1} Y_{\lambda'_j, aq^{2j-\lambda'_j-1}}$. Then, the q-character is given by (cf. [FM2], Lemma 4.7)

$$\chi_q\big(L(\mathfrak{m}^+_{\lambda,a})\big) = \mathfrak{m}^+_{\lambda,a}\Big(\sum_{T\in\mathfrak{T}(\lambda)}\prod_{\square\in\lambda}\prod_{s=i(\square)}^{T(\square)-1} A^{-1}_{s,aq^{2c(\square)+s}}\Big). \tag{4.1}$$

Example. The q-characters of fundamental modules $\chi_q\big(L(Y_{i,a})\big)$ are known (the answer is very large for, say, E_8 type). See, for example, [FR] for the classical series. Here we write a few top terms which we will need later.

$$\chi_q(L(Y_{i,a})) = Y_{i,a}(1 + A^{-1}_{i,aq_i} + A^{-1}_{i,aq_i}\sum_{j\in I,\ C_{j,i}<0} A^{-1}_{j,aq_iq^{-1}_{j,i}} + \dots), \tag{4.2}$$

where the dots denote terms which contain products of at least three $A^{-1}_{j,b}$'s.

More generally, by [MY], Corollary 3.10, the q-character of an irreducible module $V \in \operatorname{Ob}\mathcal{O}_\mathfrak{g}$ has the form

$$\chi_q(V) = \mathfrak{m}^+(1+\sum_j \mathfrak{m}_j), \qquad \mathfrak{m}^+ = \prod_{i\in I}\prod_{j=1}^{l_i}\frac{X_{i,b_{ij}}}{X_{i,a_{ij}}}, \tag{4.3}$$

where all $\mathfrak{m}_j$ are monomials in the $A^{-1}_{i,a}$ with $i\in I, a\in\mathbb{C}^\times$, and $a_{ij}, b_{ij}\in\mathbb{C}^\times$. In particular, all generalized eigenvalues of $\phi^+_i(z)$ are rational functions in $\mathfrak{r}_\mathfrak{g}$.

Consider the irreducible $U_q\mathfrak{g}$ module with highest monomial

$$\mathfrak{m}^+_{i,a,K} = X_{i,aq_i^K}X^{-1}_{i,a},$$

where $i\in I$, $a\in\mathbb{C}^\times$, $K\in\mathbb{C}$, $K\notin 2\mathbb{Z}_{\le 0}$.

Example. The $U_q\widehat{\mathfrak{sl}}_{n+1}$ parabolic Verma evaluation modules. In the $U_q\widehat{\mathfrak{sl}}_{n+1}$ case the q-character of $L(\mathfrak{m}^+_{i,a,K})$ can be computed using (4.1) and Corollary 5.6 in [MY]. We describe the result.

Consider a strip $S_i = \{(l,j) \mid l\in\{1,\dots,i\}, j\in\mathbb{Z}_{\ge1}\}\subset\mathbb{Z}^2$. A plane partition of height at most h over S_i is a map $T: S_i\to\{0,1,\dots,h\}$ which is zero for all but finitely many points in S_i and which has the property $T(l,j)\ge T(l+1,j)$ and $T(l,j)\ge T(l,j+1)$. Let $\mathfrak{T}_{i,n+1}$ be the set of all plane partitions over S_i of height at most $n+1-i$. We have

$$\chi_q(L(\mathfrak{m}^+_{i,a,K})) = \mathfrak{m}^+_{i,a,K}\Big(\sum_{T\in\mathfrak{T}_{i,n+1}}\prod_{(l,j)\in S_i}\prod_{s=0}^{T(l,j)-1} A^{-1}_{i-l+1+s,aq^{-2j+l+s+1}}\Big). \tag{4.4}$$

In particular, note that the dependence on K is only through the monomial $\mathfrak{m}^+_{i,a,K}$.

For general $\mathfrak{g}$ the q-character of $L(\mathfrak{m}^+_{i,a,K})$ is not known in a closed form, though one can explicitly write an arbitrary number of top terms using the algorithm of [FM1]. We denote $\bar\chi_i$

the corresponding normalized character:

$$\bar{\chi}_i = y_i^{K/2}\chi(L(m^+_{i,a,K})) = 1 + \mathsf{q}^{-\alpha_i} + \sum_{j, C_{j,i}\neq 0} \mathsf{q}^{-\alpha_i-\alpha_j} + \dots\ , \tag{4.5}$$

where the dots denote terms which are product of at least three $\mathsf{q}^{-\alpha_j}$'s. The explicit formula for $\bar{\chi}_i$ was conjectured in [MY], Conjecture 6.3:

$$\bar{\chi}_i = \prod_{\alpha\in\Delta^+} \frac{1}{(1-\mathsf{q}^{-\alpha})^{\langle\omega_i^\vee,\alpha\rangle}}.$$

For type A, B, C, D this is a consequence of known identities in [HKOTY]. It was shown in type G_2 in [LN].

Finally, we discuss the q-characters of positive and negative fundamental $U_q\mathfrak{b}$ modules $M^\pm_{i,a}$. These modules are not restrictions of $U_q\mathfrak{g}$ modules.

Example. The fundamental modules $M^\pm_{i,a}$. The negative fundamental module $M^-_{i,a}$ is obtained as a limit of appropriate $U_q\mathfrak{g}$ modules, see [HJ]. Its q-character is given by

$$\chi_q(M^-_{i,a}) = X^{-1}_{i,aq^K}\ \chi_q(L(\mathfrak{m}^+_{i,a,K}))\,. \tag{4.6}$$

Note that the right hand side is independent of K. In particular, in the case of type A, it can be written explicitly using formula (4.4).

The positive fundamental module $M^+_{i,a}$ is constructed as the dual of the lowest weight version of $M^-_{i,a}$ (see Lemma 3.5). Its q-character was obtained in [HJ] in a special case and in [FH] in general:

$$\chi_q(M^+_{i,a}) = X_{i,a}\bar{\chi}_i\,. \tag{4.7}$$

It is instructive to write (4.6), (4.7) as

$$\chi_q(M^-_{i,a}) = \lim_{q^K\to 0} X^{-1}_{i,aq_i^K}\ \chi_q\bigl(L(X_{i,aq_i^K}X^{-1}_{i,a})\bigr)\,,$$
$$\chi_q(M^+_{i,a}) = \lim_{q^{-K}\to 0} X_{i,aq_i^{-K}}\ \ \chi_q\bigl(L(X_{i,a}X^{-1}_{i,aq_i^{-K}})\bigr)\,,$$

where we impose formally the rule $\lim_{a\to 0} A_{j,a} = \mathsf{q}^{\alpha_j}$.

Quite generally, we call a module $V \in \mathrm{Ob}\,\mathcal{O}_\mathfrak{b}$ *s-finite* if the set of currents $\{k_i^{-1}\phi_i^+(z)\}_{i\in I}$ has s different joint eigenvalues. We call V *finite type* module if it is s-finite for some $s\in\mathbb{Z}_{>0}$.

All finite-dimensional modules V are at most d-finite, where $\dim V = d$. A restriction of a $U_q\mathfrak{g}$ module $V\in \mathrm{Ob}\,\mathcal{O}_\mathfrak{g}$ is finite-type if and only if $\dim V<\infty$. Formula (4.7) says that the positive fundamental module $M^+_{i,a}$ is infinite-dimensional and 1-finite. In contrast, the negative fundamental module $M^-_{i,a}$ is not finite type.

A subquotient of an s-finite module is at most s finite. If V_i is s_i-finite ($i=1,2$), then $V_1\otimes V_2$ is at most s_1s_2-finite, and $V_1\oplus V_2$ is at most (s_1+s_2)-finite. A tensor product (resp. direct sum) is finite type if and only if all factors (resp. summands) are finite type. In particular, an arbitrary tensor product of $M^+_{i,a}$ is 1-finite.

5. Modules of the form $V \otimes M$

In this section we study $U_q\mathfrak{b}$ modules which are tensor products of a restriction module V and a module with polynomial highest ℓ-weight M.

5.1. $U_q\mathfrak{b}$ modules with polynomial highest ℓ-weight.

Let M be an irreducible $U_q\mathfrak{b}$ module. We say that M has polynomial highest ℓ-weight if

$$M = L(\mathbf{\Psi}^M)\,, \quad \mathbf{\Psi}^M \in \mathfrak{r}_{\mathfrak{b}} \cap \mathbb{C}[z]^I. \tag{5.1}$$

In this subsection we investigate special properties of such modules. We write (5.1) also in the monomial notation as

$$M = L(\mathfrak{m}_p)\,, \quad \mathfrak{m}_p = m(\mathbf{\Psi}^M) \in \mathbb{Z}[X_{i,a}, y_i^b]_{i\in I, a\in\mathbb{C}^\times, b\in\mathbb{C}}\,. \tag{5.2}$$

We denote by $|\emptyset\rangle_M$ the highest ℓ-weight vector of M.

First we show that M is 1-finite. For that purpose we need

Lemma 5.1. [FH] *Any tensor product of positive fundamental $U_q\mathfrak{b}$ modules is irreducible. Similarly, any tensor product of negative fundamental $U_q\mathfrak{b}$ modules is irreducible.*

Proof. We give a proof different from the one given in [FH].

Set $M_0 = M^+_{i_1,a_1} \otimes \cdots \otimes M^+_{i_r,a_r}$, $\mathfrak{m} = \prod_{j=1}^r X_{i_j,a_j}$. This module is 1-finite. The dual module M_0^* is isomorphic to $M^{-,\vee}_{i_r,a_r} \otimes \cdots \otimes M^{-,\vee}_{i_1,a_1}$ where $M^{-,\vee}_{i,a} = (M^+_{i,a})^* = L^\vee(X^{-1}_{i,a}) \in \mathcal{O}^\vee_{\mathfrak{b}}$. By the dual versions of (4.6) and (4.3), the multiplicity of the monomial $\mathfrak{m}^{-1}$ in $\chi_q(M_0^*)$ is one.

Suppose that M_0^* contains a non-zero proper submodule N^*. Then either N^* or M_0^*/N^* has a singular vector whose ℓ-weight $\mathfrak{n}^{-1}$ differs from $\mathfrak{m}^{-1}$ by a non-trivial monomial of the $A^{-1}_{j,a}$'s. Let $N \subset M_0$ be the orthogonal complement of N^*. From Lemma 3.5, we conclude that either M_0/N or N has the ℓ-weight $\mathfrak{n}$. This contradicts to the fact that M_0 is 1-finite. Therefore M_0^* is irreducible. Hence M_0 is also irreducible. □

Corollary 5.2. *Any module with polynomial highest ℓ-weight is isomorphic to a tensor product of several positive fundamental modules and a one-dimensional module. In particular it is 1-finite.*

Proof. This follows from Lemma 5.1 and (4.7). □

One of the basic properties of modules with polynomial highest ℓ-weight is the following polynomiality of currents. Introduce the notation for half currents

$$x^+_{i,\geqslant}(z) = \sum_{m\geq 0} x^+_{i,m} z^m\,, \quad x^-_{i,>}(z) = \sum_{r>0} x^-_{i,r} z^r\,.$$

Lemma 5.3. *Let M be as in* (5.1). *Then for all $v \in M$ and $i \in I$ we have*

$$x^+_{i,\geqslant}(z)v\,,\ x^-_{i,>}(z)v \in M \otimes \mathbb{C}[z], \quad \phi^+_i(z)v \in \Psi_i(z)\cdot M \otimes \mathbb{C}[z]. \tag{5.3}$$

Proof. We prove equivalent statements for the (restricted) right dual module M^*. Technically it is simpler, because $M^* = v_0^* \cdot U_q^+\mathfrak{b}$, where v_0^* is the lowest ℓ-weight vector, and $U_q^+\mathfrak{b}$ is generated by $\{x^+_{i,m}\}_{i\in I, m\geq 0}$.

First consider the vector v_0^*. We have $v_0^* x^-_{i,m} = 0$ for $i \in I, m > 0$ and $v_0^*\phi^+_i(z) = v_0^*\Psi_i(z)$. If $m > \deg \Psi_i(z)$, then we have also $v_0^* x^+_{i,m} x^-_{j,r} \in \delta_{i,j}\mathbb{C} v_0^* \phi^+_{i,m+r} = 0$ for $r > 0$. The other generators of $U_q^-\mathfrak{b}$ also kill $v_0^* x^+_{i,m}$ for the weight reason. Since M^* is simple, we must have $v_0^* x^+_{i,m} = 0$ $(m > \deg \Psi_i(z))$.

By induction on the weight, suppose that $w^* \in M^*$ satisfies $w^* x^\pm_{i,m} = 0$ for m large enough and $w^* \phi^+_i(z) \in M^* \otimes \mathbb{C}[z]\Psi_i(z)$ for all $i \in I$. We show that $v^* = w^* x^+_{j,n}$ $(j \in I, n \geq 0)$ has the same property.

For m large, we have $v^* x^-_{i,m} \in \delta_{i,j}\mathbb{C} w^* \phi^+_{i,m+n} = 0$. As for $\phi^+_i(z)$, we can use the following relation which follows from the quadratic relation (2.2)

$$x^+_{j,n}\phi^+_i(z) \in \sum_{r=0}^{p} \mathbb{C} z^r \phi^+_i(z) x^+_{j,n+r} + \mathbb{C} x^+_{j,n+p} z^p \phi^+_i(z)\,,$$

where $p \geq 1$ is arbitrary. Applying this to w^* and choosing p large enough, we find $v^* \phi^+_i(z) \in M^* \otimes \mathbb{C}[z]\Psi_i(z)$. Similarly we can show $v^* x^+_{i,m} = 0$ for m large using the quadratic relation (2.4). □

Corollary 5.4. *For any $v \in M$ and $\alpha \in \Delta^+$ we have $e_{p\delta\pm\alpha} v = 0$ for sufficiently large p.*

Proof. This follows from Lemma 5.3 and Lemma A.1. □

5.2. **Submodules of modules of type $V \otimes M$.** Let now V be an irreducible $U_q\mathfrak{g}$ module with highest ℓ-weight $\boldsymbol{\Psi}^V \in \mathfrak{r}_{\mathfrak{g}}$,

$$V = L(\mathfrak{m}_0)\,, \quad \mathfrak{m}_0 = m(\boldsymbol{\Psi}^V)\,.$$

The following lemma describes the structure of the action of $x^\pm_i(z)$ in V.

Lemma 5.5. *([Y],Proposition 3.1) Let $V \in \mathrm{Ob}\,\mathcal{O}_{\mathfrak{g}}$, $\mathfrak{m}$ a monomial in $\chi_q(V)$ and $v \in V_{\mathfrak{m}}$. Then the formal series $x^\pm_i(z)v$ has the form*

$$x^\pm_i(z)v = \sum_a \sum_{k=0}^{r_a} v_{k,a} (\partial_a)^k \delta(za), \quad v_{k,a} \in V_{\mathfrak{m}A^{\pm 1}_{i,a}}\,,$$

where a runs over a finite subset of $\mathbb{C}^\times$ and $r_a = \dim V_{\mathfrak{m}} + \dim V_{\mathfrak{m}A^{\pm 1}_{i,a}} - 2$. □

We study the properties of the module $V \otimes M$. To this aim, it is convenient to use the Drinfeld coproduct (2.15).

Lemma 5.6. *On $V \otimes M$, the Drinfeld coproduct Δ_D gives a well defined structure of a $U_q\mathfrak{b}$ module which we denote $V \otimes_D M$. As $U_q\mathfrak{b}$ modules, $V \otimes_D M$ and $V \otimes M$ are isomorphic.*

Proof. We show that $\sigma(\mathcal{R}_+)$ is a well-defined linear operator on $V \otimes M$. Consider its action on a weight vector $v \otimes w \in V \otimes M$. Expanding (2.12), we obtain a linear combination of terms of the form

$$f^{n_1}_{-k_1\delta-\beta_1} \cdots f^{n_N}_{-k_N\delta-\beta_N} v \otimes e^{n_1}_{k_1\delta+\beta_1} \cdots e^{n_N}_{k_N\delta+\beta_N} w\,,$$

where $k_i, n_i \geq 0$ and $\beta_i \in \Delta^+$, $1 \leq i \leq N$. The second component stays in a finite dimensional subspace of M of weight $\geq \mathrm{wt}\, w$. For this term to be non-zero, there are only finitely many choices of (n_i, β_i). From Corollary 5.4, it is non-zero only for finitely many k_i's. Therefore $\sigma(\mathcal{R}_+) v \otimes w$ comprises only finitely many non-zero terms. In view of Proposition 2.2, $\Delta_D(x)$

has a well-defined action on $V \otimes M$ for any $x \in U_q\mathfrak{b}$, and $\sigma(\mathcal{R}_+)$ gives an intertwiner between $V \otimes_D M$ and $V \otimes M$. □

The next lemma shows that submodules of $V \otimes_D M$ have a very special form.

Lemma 5.7. *Let $W \subset V \otimes_D M$ be a non-zero submodule. Then there exists a linear subspace $V^{(0)} \subset V$ such that $W = V^{(0)} \otimes M$. The highest ℓ-weight vector v_0 of V belongs to $V^{(0)}$.*

Proof. Let $w \in W$ be a vector in an ℓ-weight subspace. Since W is a submodule,

$$\Delta_D(x^-_{i,k})w = \Big(\sum_{j\geq 0} x^-_{i,k-j} \otimes \phi^+_{i,j}\Big)w + (1 \otimes x^-_{i,k})w$$

belongs to W for any $i \in I$, $k > 0$. By Lemma 5.5, the first term is a sum of terms which belong to ℓ-weight subspaces different from that of w. On the other hand, by Corollary 5.2, the second term belongs to the same ℓ-weight subspace as that of w. Hence both terms separately belong to W. Since w is arbitrary, we conclude that

$$\Big(\sum_{j\geq 0} x^-_{i,k-j} \otimes \phi^+_{i,j}\Big)W \subset W\,, \quad (1 \otimes x^-_{i,k})W \subset W\,. \tag{5.4}$$

Let now $x \in U_q^-\mathfrak{b}$. Then we claim that $\Delta_D(x) = 1 \otimes x + \cdots$ where $\cdots$ is a sum of terms whose first component contains at least one $x^-_{i,j}$. Indeed, the element $x \in U_q^-\mathfrak{b} \subset U_q^-\mathfrak{g}$ can be written in terms of $x^-_{i,m}$, $i \in I$, $m \in \mathbb{Z}$. We have $\Delta_D(x^-_{i,m}) = 1 \otimes x^-_{i,m} + \cdots$ and $U_q\mathfrak{b}$ is a coideal, see (2.16). Therefore the claim follows.

Hence, we can generalize (5.4) to

$$(1 \otimes x)W \subset W \quad (x \in U_q^-\mathfrak{b}). \tag{5.5}$$

By the same argument leading to (5.4), we have also

$$(x^+_{i,k} \otimes 1)W \subset W\,, \quad \Big(\sum_{j\geq 0} \phi^-_{i,-j} \otimes x^+_{i,k+j}\Big)W \subset W\,. \tag{5.6}$$

Consider the linear subspace $V^{(0)} = \{v \in V \mid v \otimes |\emptyset\rangle_M \in W\}$. Using (5.5), we obtain that $V^{(0)} \otimes M \subset W$. We prove the equality by showing the following statement: If $w = \sum_{r=1}^N v_r \otimes m_r \in W$ and $\{v_r\}_{r=1}^N \subset V$, $\{m_r\}_{r=1}^N \subset M$ are linearly independent weight vectors, then $v_r \in V^{(0)}$ for all r.

Suppose $N = 1$, so that $w = v_1 \otimes m_1$, $v_1 \neq 0$, $m_1 \neq 0$. If $m_1 \in \mathbb{C}|\emptyset\rangle_M$, there is nothing to show. Otherwise there exists an $i \in I$ and $k \geq 0$ such that $x^+_{i,k}m_1 \neq 0$. By Lemma 5.3, there is the largest k with this property. For this k we obtain $\phi^-_{i,0}v_1 \otimes x^+_{i,k}m_1 \in W$ by using (5.6). Repeating this process, we arrive at $v_1 \otimes |\emptyset\rangle_M \in W$.

Next let $N > 1$, and assume that the statement is proved for $N' < N$. Arguing similarly as above, we can find an $i \in I$ and $k \geq 0$ such that $x^+_{i,k}m_r \neq 0$ for some r and that $x^+_{i,l}m_s = 0$ for all $l > k$, $1 \leq s \leq N$. Applying (5.6), we obtain that $\sum_{r=1}^N \phi^-_{i,0}v_r \otimes x^+_{i,k}m_r \in W$. If $\{x^+_{i,k}m_r\}_{r=1}^N$ is linearly independent, then we repeat this procedure. After a finite number of steps, we obtain vectors $m'_r \in M$ where $\sum_{r=1}^N v_r \otimes m'_r \in W$, $m'_r \neq 0$ for some r, and $\{m'_r\}_{r=1}^N$ is not linearly independent. Renumbering indices, we may assume that $\{m'_s\}_{s=1}^{N'}$ $(0 < N' < N)$ is linearly independent and that $m'_r = \sum_{s=1}^{N'} m'_s a_{s,r}$ $(N' < r \le N)$ with some $a_{s,r} \in \mathbb{C}$. Then

$$\sum_{s=1}^{N'} v'_s \otimes m'_s \in W\,, \quad v'_s = v_s + \sum_{r=N'+1}^{N} v_r a_{r,s}\,.$$

It follows that $v'_s \in V^{(0)}$ $(1 \le s \le N')$ by the induction hypothesis. Since $V^{(0)} \otimes M \subset W$, we see that $\sum_{s=1}^{N'} v'_s \otimes m_s$ belongs to W. This in turn implies that

$$\sum_{r=N'+1}^{N} v_r \otimes \bigl(m_r - \sum_{s=1}^{N'} m_s a_{s,r}\bigr) \in W\,.$$

Using again the induction hypothesis, we conclude that $v_r \in V^{(0)}$ for all r.

Finally, since $V^{(0)} \neq 0$, we can use (5.6) to show that $v_0 \in V^{(0)}$. □

Corollary 5.8. *Let $\mathfrak{m}_0 = m(\boldsymbol{\Psi}^V)$, $\boldsymbol{\Psi}^V \in \mathfrak{r}_{\mathfrak{g}}$, and let $\mathfrak{m}_p$ be a monomial in $\mathbb{Z}[X_{i,a}, y_i^b]_{i\in I, a\in\mathbb{C}^\times, b\in\mathbb{C}}$. If $\mathfrak{m}_p \notin \mathbb{Z}[y_i^b]_{i\in I, b\in\mathbb{C}}$, then the module $L(\mathfrak{m}_0\mathfrak{m}_p)$ is infinite-dimensional.* □

Proof. Module $L(\mathfrak{m}_0\mathfrak{m}_p)$ is a subquotient of $L(\mathfrak{m}_0) \otimes_D L(\mathfrak{m}_p)$. Lemma 5.7 says that it is actually a submodule of the form $V^{(0)} \otimes L(\mathfrak{m}_p)$. Hence $L(\mathfrak{m}_0\mathfrak{m}_p)$ is finite dimensional only if $L(\mathfrak{m}_p)$ is one-dimensional. □

Next we give a sufficient condition for $V \otimes M$ to be irreducible. The following lemma shows that a cancellation in q-characters *must* happen in order for $V \otimes M$ to be reducible.

Lemma 5.9. *Let*

$$\chi_q(V) = \mathfrak{m}_0\Bigl(1 + \sum_s \prod_t A^{-1}_{i'_{t,s}, a'_{t,s}}\Bigr)\,, \quad \chi_q(M) = y^{\boldsymbol{b}} \prod_{j=1}^{r} X_{i_j,a_j} \cdot \bar{\chi}_{i_j}\,.$$

If $(i_j, a_j) \neq (i'_{t,s}, a'_{t,s})$ for all j and all t, s, then $V \otimes M$ is irreducible.

Proof. We again work with $V \otimes_D M$. Let W be a non-zero submodule of $V \otimes_D M$. By Lemma 5.7, it has the form $V^{(0)} \otimes M$ where $V^{(0)} = \{v \in V \mid v \otimes |\emptyset\rangle_M \in W\}$. By (5.6), we have $x^+_{i,p} V^{(0)} \subset V^{(0)}$ $(i \in I,\, p \ge 0)$. We show that $x^-_{i,p} V^{(0)} \subset V^{(0)}$ $(i \in I,\, p > 0)$.

To this aim, let $v \in V^{(0)} \cap V_{\mathfrak{m}}$. By Lemma 5.5, we can write $x^-_i(z)v = \sum_{k,a} v_{k,a} \partial_a^k\bigl(\delta(za)\bigr)$, where $v_{k,a} \in V_{\mathfrak{m}A^{-1}_{i,a}}$. Then

$$\Delta_D(x^-_{i,>}(z))(v \otimes |\emptyset\rangle_M) \equiv \sum_{k,a} \Bigl(\partial_a^k\bigl(\delta(za)\bigr) \prod_{j,\ i_j = i} (1 - za_j)\Bigr)_{>} (v_{k,a} \otimes |\emptyset\rangle_M) \quad \bmod V^{(0)} \otimes M\,.$$

Here, for a formal series $r(z) = \sum_{n\in\mathbb{Z}} r_n z^n$, we set $r(z)_> = \sum_{n>0} r_n z^n$. If $v_{k,a} \neq 0$, then by the assumption $a_j \neq a$ for all j in the product. Hence $v_{k,a} \in V^{(0)}$.

From the proof of Lemma 3.4, it follows that $x^\pm_{i,p} V^{(0)} \subset V^{(0)}$ for all $i \in I$ and $p \in \mathbb{Z}$. Since V is irreducible, we have $V^{(0)} = V$ and hence $W = V \otimes M$. This shows that $V \otimes_D M$ is irreducible. □

We finish the section with a technical lemma which is useful in the study of submodules of $V \otimes_D M$. The algebra $U_q^- \mathfrak{b}$ is not generated by $x^-_{i,k}$ with $k > 0$, however, in many cases, it is possible to avoid checking the invariance of the submodule with respect to the other generators.

We denote the highest ℓ-weight vector of V by v_0 and that of M by $|\emptyset\rangle_M$.

Lemma 5.10. *Assume that $V_0 \subset V$ is an ℓ-weighted subspace such that the space $V_0 \otimes_D M$ is invariant under the action of $\{x^+_{i,k}\}_{i\in I, k\geq 0}$ and $\{x^-_{i,k}\}_{i\in I,k>0}$. Assume that $v_0 \in V_0$ and $V_0 \otimes_D |\emptyset\rangle_M \subset U_q\mathfrak{b}(v_0 \otimes_D |\emptyset\rangle_M)$. Finally, assume that V_0 and the ℓ-weighted complement of V_0 in V have no common ℓ-weights. Then $V_0 \otimes_D M$ is a submodule in $V \otimes_D M$.*

Proof. Let W be the submodule generated by $V_0 \otimes_D M$. The module W is the cyclic submodule of $V \otimes_D M$ generated by $v_0 \otimes |\emptyset\rangle_M$. Indeed, by assumption, $V_0 \otimes_D |\emptyset\rangle_M \subset U_q\mathfrak{b}(v_0 \otimes_D |\emptyset\rangle_M)$ and we have (5.5). We wish to show $W = V_0 \otimes_D M$.

By Lemma 5.3, and since $V_0 \otimes_D M$ is invariant by $\{x^-_{i,k}\}_{i\in I,k>0}$, there exists a $K > 0$ such that $x^-_{i,k}(V_0 \otimes |\emptyset\rangle_M) \subset V_0 \otimes |\emptyset\rangle_M$ for all $i \in I$ and $k > K$. Consider the following subalgebras of $U_q\mathfrak{g}$,

$$E_K = \langle x^-_{i,k}, i \in I, k > K\rangle \cdot U^0_q\mathfrak{b} \cdot U^+_q\mathfrak{b}\,, \qquad E = \langle x^-_{i,k}, i \in I, k \in \mathbb{Z}\rangle \cdot U^0_q\mathfrak{b} \cdot U^+_q\mathfrak{b}\,.$$

Note that we have the inclusions of subalgebras $E_K \subset U_q\mathfrak{b} \subset E$.

Then $V_0 \otimes |\emptyset\rangle_M$ is an E_K-module. Consider the E module $\bar{W} = E \otimes_{E_K} (V_0 \otimes |\emptyset\rangle_M)$ induced from the E_K-module $V_0 \otimes |\emptyset\rangle_M$. Clearly W is a subquotient of $\bar{W}$ considered as $U_q\mathfrak{b}$ module. Therefore, to prove the lemma it is sufficient to show that all ℓ-weights of $\bar{W}$ effectively coincide with the ℓ-weights of $V_0 \otimes |\emptyset\rangle_M$. More precisely, we show that for each ℓ-weight vector $\bar{w} \in \bar{W}$ there exists an $S_{\bar{w}} > 0$ and $v_{\bar{w}} \in V_0$ such that the eigenvalues of $h_{i,s}$ with $i \in I$ and $s > S_{\bar{w}}$ on $\bar{w}$ equal to the corresponding eigenvalues of $h_{i,s}$ on $v_{\bar{w}} \otimes |\emptyset\rangle_M$.

Define a filtration $\bar{W}_0 \subset \bar{W}_1 \subset \bar{W}_2 \subset \cdots$ of $\bar{W}$, where $\bar{W}_0 = V_0 \otimes |\emptyset\rangle_M$, and $\bar{W}_r$ $(r \geq 1)$ is spanned by $\bar{W}_{r-1}$ and elements

$$\bar{w} = x(v \otimes |\emptyset\rangle_M)\,, \quad x = x^-_{j_1,m_1} \cdots x^-_{j_r,m_r}\,, \tag{5.7}$$

with $j_t \in I$, $m_t \in \mathbb{Z}$, and $v \in V_0$. Then for any $\bar{w}$ of the form (5.7), there exists an $S_{\bar{w}} > 0$ such that $[h_{i,s}, x](v \otimes |\emptyset\rangle_M)$ belongs to $\bar{W}_{r-1}$ if $s > S_{\bar{w}}$. This follows from (2.6) with the help of the relation (2.4) and the fact that $x^-_{i,k}$ with large k preserves $V_0 \otimes |\emptyset\rangle_M$. Thus the action of $h_{i,s}$ on $\bar{W}$ is represented by a triangular matrix whose diagonal entries are the eigenvalues of $h_{i,s}$ on $V_0 \otimes |\emptyset\rangle_M$. Hence we obtain the desired result.

□

5.3. **Grading.** As an application of Lemma 5.6, we show that the module M has a sort of homogeneous grading. A similar result was proved in [FH] (for the dual module and with $N = 1$) by a different method.

Proposition 5.11. *Let $M = M^+_{i_1,a_1} \otimes \cdots \otimes M^+_{i_N,a_N}$, and let $\mathbf{\Psi}^M = (\Psi^M_i(z))_{i\in I}$ be its highest ℓ-weight. Set $\omega^\vee = \omega^\vee_{i_1} + \cdots + \omega^\vee_{i_N}$, $\mu = (\mu_i)_{i\in I} \in \mathfrak{t}^*$, $\mu_i = \prod_{j:i_j=i} a_j^{-1/2}$. Then there exists a* *grading* $M=\oplus_{m=0}^{\infty}M[m]$ *as vector space with the following properties.*

$$\mathbb{C}|\emptyset\rangle_M = M[0]\,,\quad M_{\mu\mathsf{q}^{-\beta}}=\bigoplus_{m=0}^{\infty}M_{\mu\mathsf{q}^{-\beta}}\cap M[m]\quad (\beta\in Q^+)\,, \tag{5.8}$$

$$e_{k\delta-\alpha}M[m]\subset M[m+k]\quad (k>0,\alpha\in\Delta^+)\,, \tag{5.9}$$

$$e_{k\delta+\alpha}M[m]\subset\sum_{j=0}^{(\omega^\vee,\alpha)}M[m+k-j]\quad (k\ge 0,\alpha\in\Delta^+)\,, \tag{5.10}$$

$$\overline{\phi}^{+}_{i,k}M[m]\subset M[m+k]\quad (k\ge 0, i\in I). \tag{5.11}$$

In the last line we set $\Psi_i^M(z)^{-1}\phi_i^+(z)=\sum_{k\ge 0}\overline{\phi}^{+}_{i,k}z^k$.

Proof. Let $r=(\theta,\theta)/2$ where θ is the maximal root of $\mathring{\mathfrak{g}}$, and let $r_i=r/d_i\in\{1,2,3\}$ for $i\in I$ so that $q_i^{r_i}=q^r$. Let $V_{i,a}=L(X_{i,a}^{-1}X_{i,aq_i^{-2r_i}})$. Then $V_{i,a}$ is the restriction of a finite-dimensional $U_q\mathfrak{g}$ module. We set $V=V_{i_1,a_1}\otimes\cdots\otimes V_{i_N,a_N}$, and denote by $v_0\in V$ the tensor product of highest ℓ-weight vectors of V_{i_j,a_j}.

By Lemma 5.7, a non-zero submodule of $V\otimes_D M$ has the form $V_0\otimes M$ with $v_0\in V_0$. Let S be the irreducible submodule containing $v_0\otimes|\emptyset\rangle_M$. Comparing highest ℓ-weights, we must have $S=\mathbb{C}v_0\otimes M$; furthermore

$$S\simeq M^+_{i_1,a_1q^{-2r}}\otimes\cdots\otimes M^+_{i_N,a_Nq^{-2r}}\simeq L(K)\otimes\tau^*M\,,$$

where $L(K)$ is a one-dimensional module corresponding to the weight of v_0, and τ^*M denotes the twist of M by the automorphism $\tau(x)=q^{-2r\,\mathrm{hdeg}\,x}x$ $(x\in U_q\mathfrak{b})$. Let $\tau:\tau^*M\to M$ be the unique linear isomorphism such that $\tau\circ x=\tau(x)\circ\tau$ and $\tau|\emptyset\rangle_{\tau^*M}=|\emptyset\rangle_M$. Let Φ be the composition of the natural maps

$$M\longrightarrow\mathbb{C}v_0\otimes M\xrightarrow{\sim}L(K)\otimes\tau^*M\xrightarrow{\mathrm{id}\otimes\tau}L(K)\otimes M\longrightarrow M$$

sending $|\emptyset\rangle_M$ to $|\emptyset\rangle_M$. By construction Φ commutes with k_i, and we have

$$\Phi\circ x=q^{-2r\,\mathrm{hdeg}\,x}x\circ\Phi\quad (x\in U_q^-\mathfrak{b}), \tag{5.12}$$

$$\Phi\circ\frac{\phi_i^+(z)}{\Psi_i^M(z)}=\frac{\phi_i^+(q^{-2r}z)}{\Psi_i^M(q^{-2r}z)}\circ\Phi\,. \tag{5.13}$$

Eq. (5.12) implies that Φ is diagonalizable with eigenvalues of the form q^{-2rm}, $m\in\mathbb{Z}_{\ge0}$. Let $M[m]$ denote the eigenspace corresponding to the eigenvalue q^{-2rm}. We have $M=\oplus_{m=0}^{\infty}M[m]$ and $M[0]=\mathbb{C}|\emptyset\rangle_M$. Since Φ commutes with k_i, (5.8) is clear. Also (5.9), (5.11) are obvious from the properties (5.12) and (5.13), respectively. Note that (5.11) implies

$$\phi^+_{i,k}M[m]\subset\sum_{j=0}^{(\omega^\vee,\alpha_i)}M[m+k-j]\quad (k\ge 0,i\in I). \tag{5.14}$$

It remains to prove (5.10). Let $\Psi_i^{M,-}(z)$ be the expansion of $\Psi_i^M(z)$ at $z=0$. Consider the series

$$\overline{x}^{+}_{i,\geqslant}(z)=\sum_{k\ge0}\overline{x}^{+}_{i,k}z^{-k}=\left(\frac{x_i^+(z)}{\Psi_i^{M,-}(z)}\right)_{\geqslant}\,.$$

The coefficients have the form $\overline{x}^+_{i,k} = \sum_{l\geq 0} c_l x^+_{i,k+l}$, and thanks to Lemma 5.3 they are well defined operators on M. Similarly to (5.13), we have the relation

$$\Phi \circ \overline{x}^+_{i,\geqslant}(z) = \overline{x}^+_{i,\geqslant}(q^{-2r}z) \circ \Phi\,.$$

Arguing as in (5.14), we deduce (5.10) for $e_{k\delta+\alpha_i} = x^+_{i,k}$. Since these elements generate $U_q^+\mathfrak{b}$, the proof is now complete.

□

6. Finite type modules

6.1. **Definition of finite type modules.** We denote $\mathcal{O}^{fin}_{\mathfrak{b}} \subset \mathcal{O}_{\mathfrak{b}}$ the category of all highest ℓ-weight modules of finite type. We denote $\mathrm{Rep}_F\, U_q\mathfrak{b} \subset \mathrm{Rep}\, U_q\mathfrak{b}$ the corresponding Grothendieck ring.

If $J \subset I$, then the restriction map preserves the property of being finite type, and we have $res_J : \mathcal{O}^{fin}_{\mathfrak{b}} \to \mathcal{O}^{fin}_{\mathfrak{b}_J}$ and $res_J : \mathrm{Rep}_F\; U_q\mathfrak{b} \to \mathrm{Rep}_F\; U_q\mathfrak{b}_J$.

6.2. **Classification of finite type modules.** For $i \in I$, we call a monomial $\mathfrak{m} \in \mathcal{X}$ *i-dominant* if $\mathfrak{m} \in \mathbb{Z}[Y_{i,a}, X_{i,a}, y^b_i, X^{\pm 1}_{j,a}, y^b_j]_{j\in I\setminus\{i\}, a\in\mathbb{C}^\times, b\in\mathbb{C}}$. We call a monomial $\mathfrak{m} \in \mathcal{X}$ *dominant* if it is i-dominant for all $i \in I$.

Theorem 6.1. *The irreducible module $L(\mathfrak{m})$ is finite type if and only if $\mathfrak{m}$ is dominant.*

Proof. If $\mathfrak{m}$ is dominant, then write $\mathfrak{m} = \mathfrak{m}_1\mathfrak{m}_0\mathfrak{m}_p$, where $\mathfrak{m}_1$ is a monomial in y^b_i, $\mathfrak{m}_0$ is a monomial in $Y_{i,a}$, and $\mathfrak{m}_p$ is a monomial in $X_{i,a}$. Thus, $L(\mathfrak{m})$ is a subquotient of the tensor product of three finite type modules: one-dimensional $L(\mathfrak{m}_1)$, finite-dimensional $L(\mathfrak{m}_0)$, and 1-finite $L(\mathfrak{m}_p)$. Therefore it is finite.

To prove the only if statement, it is sufficient to prove it in the case $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$. We write y^b, X_a, Y_a for $y^b_1, X_{1,a}, Y_{1,a}$, respectively.

Let

$$\mathfrak{m} = y^b \prod_{i=1}^{i_0} X_{a_i} \prod_{j=1}^{j_0} X^{-1}_{b_j}\,. \tag{6.1}$$

Suppose $i_0 < j_0$. Then we claim that $L(\mathfrak{m})$ is not finite type. Indeed, considering the dual module $L(\mathfrak{m})^* = L^\vee(\mathfrak{m}^{-1})$ (see Lemma 3.5) and using Corollary 5.8, we see that $L(\mathfrak{m})$ is infinite dimensional. Moreover, $L(\mathfrak{m})$ is a subquotient of a tensor product of several negative fundamental modules $L(X^{-1}_b)$, restriction of a $U_q\mathfrak{g}$ module, and a one-dimensional module. All ℓ-weight spaces in such modules are finite-dimensional, so $L(\mathfrak{m})$ is not finite type.

Suppose $i_0 = j_0$. Then $L(\mathfrak{m})$ is a tensor product of restriction of a $U_q\mathfrak{g}$ module and a one-dimensional module and theorem follows.

Finally suppose $i_0 > j_0$. We show that there is a way to write $\mathfrak{m} = \mathfrak{m}_1\mathfrak{m}_0\mathfrak{m}_p$, where $L(\mathfrak{m}_1)$ is one-dimensional, $L(\mathfrak{m}_0)$ is the restriction of a $U_q\mathfrak{g}$ module, $L(\mathfrak{m}_p)$ is 1-finite, and $L(\mathfrak{m}) = L(\mathfrak{m}_1) \otimes L(\mathfrak{m}_0) \otimes L(\mathfrak{m}_p)$. The procedure is similar to writing a highest ℓ-weight of a finite-dimensional $U_q\widehat{\mathfrak{sl}}_2$ module as a product of highest ℓ-weights of evaluation modules.

We call a pair (a, b), $a, b \in \mathbb{C}^\times$, a string with head a and tail b. The string is of finite size $s \in \mathbb{Z}_{\geq 1}$ if and only if $a = bq^{-2s+2}$. In other words, the restriction module $L(X_aX^{-1}_b)$ is of finite dimension s if and only if (a,b) is of finite size s. Moreover, it is an evaluation module and we have

$$\chi_q(L(X_aX_b^{-1})) = X_aX_b^{-1}(1+\sum_{r=1}^{s-1}\prod_{t=0}^{r-1} A^{-1}_{bq^{-2t}})\,,$$

where s is the size of the string. If the size is not finite, the same equation holds with $s=\infty$.

We say that a point c is in a generic position relative to the string (a,b) if $c\notin bq^{-2\mathbb{Z}}$, or if (a,b) is of finite size s and $c\neq bq^{-2t}$, $0\le t\le s-2$. In other words, if c is in a generic position then the module $L(X_aX_b^{-1})\otimes L(X_c)$ is irreducible by Lemma 5.9.

Starting with the monomial $\mathfrak{m}$ (6.1) we form strings with tails (a_{i_j},b_j) $(j=1,\dots,j_0)$ such that all i_j are distinct and such that all (i_0-j_0) remaining a_i are in a generic position relative to all j_0 strings. It is clear that this can be done (sometimes in several ways).

Then we set

$$\mathfrak{m}_1 = y^b\,,\qquad \mathfrak{m}_0=\prod_{j=1}^{j_0} X_{a_{i_j}}X_{b_j}^{-1}\,,\qquad \mathfrak{m}_p=\prod_{i,\ a_i\notin\{a_{i_1},\dots,a_{i_{j_0}}\}} X_{a_i}\,.$$

Then by Lemma 5.9 we have $L(\mathfrak{m})=L(\mathfrak{m}_1)\otimes L(\mathfrak{m}_0)\otimes L(\mathfrak{m}_p)$. It follows that $L(\mathfrak{m})$ is finite type if and only if $L(\mathfrak{m}_0)$ is finite type, and, therefore, if and only if $L(\mathfrak{m}_0)$ is finite dimensional. □

Corollary 6.2. *The ring* $\mathrm{Rep}_F\, U_q\mathfrak{b}$ *is topologically generated by one-dimensional modules* $L(y^{\boldsymbol{b}})$*, restrictions of* $U_q\mathfrak{g}$ *fundamental modules* $L(Y_{i,a})$*, and positive fundamental modules* $M^+_{i,a}$*.*

A $U_q\mathfrak{b}$ module V is called *prime* if $V=V_1\otimes V_2$ implies $\dim V_1=1$ or $\dim V_2=1$.

The following follows from the proof of Theorem 6.1.

Corollary 6.3. *Let* $\mathfrak{g}=\widehat{\mathfrak{sl}}_2$*. If* $L(\mathfrak{m})\in\mathcal{O}^{fin}_{\mathfrak{b}}$ *is an irreducible module of finite type, then* $L(\mathfrak{m})=L(\mathfrak{m}_0)\otimes L(\mathfrak{m}_p)$ *is a tensor product of restriction of a finite-dimensional module* $L(\mathfrak{m}_0)$ *and a 1-finite module* $L(\mathfrak{m}_p)$*, where* $\mathfrak{m}_0$ *and* $\mathfrak{m}_p$ *are uniquely determined by* $\mathfrak{m}$*.*

In particular, $L(\mathfrak{m})\in\mathcal{O}^{fin}_{\mathfrak{b}}$ *is prime if and only if* $\mathfrak{m}=X_a$ *or* $\mathfrak{m}=\prod_{i=0}^{s-1}Y_{aq^{-2s}}$*.* □

For a monomial $\mathfrak{m}\in\mathcal{X}$, define $d_i(\mathfrak{m})$ by setting $d_i(y_j^{\boldsymbol{b}})=0$, $d_i(X^{\pm1}_{j,a})=\pm\delta_{ij}$ and requiring $d_i(\mathfrak{m}'\mathfrak{m}'')=d_i(\mathfrak{m}')+d_i(\mathfrak{m}'')$.

Corollary 6.4. *Let* $V=L(\mathfrak{m})\in\mathcal{O}^{fin}_{\mathfrak{b}}$ *be an irreducible highest* ℓ*-weight* $U_q\mathfrak{b}$ *module of finite type. Then* $d_i(\mathfrak{m})\ge 0$ *for all* $i\in I$*, and* $\chi_q(V)$ *has the form*

$$\chi_q(V)=\mathfrak{m}(1+\sum_{r=1}^{N}\mathfrak{m}_r)\prod_{i\in I}\bar{\chi}_i^{d_i(\mathfrak{m})},$$

where $\mathfrak{m}_r\in\mathbb{Z}[A^{-1}_{i,a}]_{i\in I,a\in\mathbb{C}^\times}$ *for all* r*.*

In particular, $L(\mathfrak{m})\in\mathcal{O}^{fin}_{\mathfrak{b}}$ *is finite-dimensional if and only if* $d_i=0$ *for all* $i\in I$*.*

If $\mathfrak{m}$ *does not depend on* y_i*, then* $\chi_q(V)$ *can be written as a Laurent polynomial in* $\chi_q(M^+_{i,a})$*.*

As a byproduct we also recover a result of [B].

Corollary 6.5. *Let* W *be a finite-dimensional* $U_q\mathfrak{b}$ *module. Then* $W=L(y^{\boldsymbol{b}})\otimes V$ *where* V *is the restriction of a finite-dimensional* $U_q\mathfrak{g}$ *module.*

Proof. If W is finite-dimensional, then W is finite type and Theorem 6.1 applies. Moreover, by Corollary 6.4, we have $d_i = 0$ for all $i \in I$. □

6.3. **2-finite modules.** In this section we discuss examples of non-trivial modules of finite type.

The q-characters of all $U_q\widehat{\mathfrak{sl}}_2$ finite type modules can be written immediately from Corollary 6.3. Moreover, we have a notion of dominant monomials. Therefore the q-characters of many modules could be built recursively by the algorithm described in [FM1]. However, we also have an alternative approach due to results of Section 5, which we now use.

If W is an irreducible module of finite type, then by Theorem 6.1 W is (up to tensoring by a one dimensional module) a subquotient of $V \otimes M$, where V is the restriction of a $U_q\mathfrak{g}$ module and M is a tensor product of positive fundamental modules.

We start with the classification of 1-finite modules.

Proposition 6.6. *Let $M \in \mathrm{Ob}\,\mathcal{O}_\mathfrak{b}$ be an irreducible module. Then M is 1-finite if and only if it has polynomial highest ℓ-weight.*

Proof. It is enough to show the 'only if' part.

By Theorem 6.1, there exists a dominant monomial $\mathfrak{m} \in \mathcal{X}$ such that $M = L(\mathfrak{m})$. Tensoring M with an appropriate one dimensional module, we may assume that $\mathfrak{m} = \mathfrak{m}_0\mathfrak{m}_p$, where $\mathfrak{m}_0$ is a monomial in the $Y_{i,a}$'s and $\mathfrak{m}_p$ is a monomial in the $X_{i,a}$'s. Replacing $X_{i,aq_i}Y_{i,a}$ with $X_{i,aq_i^{-1}}$ as necessary, we may assume further that if $\mathfrak{m}_0$ is divisible by $Y_{i,a}$ then $\mathfrak{m}_p$ is not divisible by X_{i,aq_i}. Under this condition we claim that $\mathfrak{m}_0 = 1$.

Indeed, let $V = L(\mathfrak{m}_0)$, $M' = L(\mathfrak{m}_p)$. By Lemma 5.7, any non-zero submodule of $V \otimes_D M'$ contains $v_0 \otimes |\emptyset\rangle_{M'}$ where v_0 is the highest ℓ-weight vector of V. Therefore M is a submodule of $V \otimes_D M'$ containing $v_0 \otimes |\emptyset\rangle_{M'}$. If $\dim V > 1$, then there exist $i \in I$ and $a \in \mathbb{C}^\times$ such that $Y_{i,a}$ divides $\mathfrak{m}_0$. We have

$$\Delta_D\big(x^-_{i,>}(z)\big)(v_0 \otimes |\emptyset\rangle_{M'}) = \Big(x^-_i(z)v_0 \otimes \phi^+_i(z)|\emptyset\rangle_{M'}\Big)_> + v_0 \otimes x^-_{i,>}(z)|\emptyset\rangle_{M'}\,.$$

Since X_{i,aq_i} does not divide $\mathfrak{m}_p$, Lemma 5.5 implies that the first term in the right hand side is a non-zero vector. Since it is a sum of terms of ℓ-weight different from that of $v_0 \otimes |\emptyset\rangle_{M'}$, this contradicts to the assumption that M is 1-finite. Hence we have $\dim V = 1$. □

Next we describe 2-finite modules. For $i \in I$ and $a \in \mathbb{C}^\times$, let

$$N^+_{i,a} = L(\mathfrak{m}^{(2)}_{i,a}), \qquad \mathfrak{m}^{(2)}_{i,a} = X^{-1}_{i,a} \prod_{j \in I, C_{j,i} \neq 0} X_{j,aq^{-1}_{j,i}}\,. \tag{6.2}$$

The properties of the module $N^+_{i,a}$ are given in the following proposition.

Proposition 6.7. *The module $N^+_{i,a}$ is 2-finite. Moreover we have*

$$\chi_q(N^+_{i,a}) = \mathfrak{m}^{(2)}_{i,a}(1 + A^{-1}_{i,a}) \prod_{j: C_{j,i} < 0} \bar{\chi}_j. \tag{6.3}$$

Proof. Noting that $\mathfrak{m}^{(2)}_{i,a} = Y_{i,aq_i^{-1}} \prod_{j\in I, C_{j,i}<0} X_{j,aq_{j,i}^{-1}}$, we set

$$V = L(Y_{i,aq_i^{-1}}), \quad M = \bigotimes_{j:C_{j,i}<0} M^+_{j,aq_{j,i}^{-1}}.$$

We use an argument similar to that of the proof of Proposition 5.11. Let $\mathfrak{m} = Y_{i,aq_i^{-1}}$. By (4.2), the ℓ-weight subspaces $V_{\mathfrak{m}}$, $V_{\mathfrak{m}A^{-1}_{i,a}}$ are both one-dimensional. Choose their generators v_0, v_1 and consider the two dimensional subspace $V^{(0)} = \mathbb{C}v_0 + \mathbb{C}v_1$ of V. We claim that $V^{(0)} \otimes M$ is a submodule of $V \otimes_D M$.

Indeed, we have $\dim L(\mathfrak{m})_{\mathfrak{m}A^{-1}_{i,a}} = 1$, and if $C_{i,j} < 0$, then $\dim L(\mathfrak{m})_{\mathfrak{m}_j} = 1$ where $\mathfrak{m}_j = \mathfrak{m}A^{-1}_{i,a}A^{-1}_{j,aq_{j,i}^{-1}}$. Therefore, by Lemma 5.5 and Lemma 5.3, we see that $\Delta_D\big(x^-_{j,>}(z)\big)(v_1 \otimes M) \subset V^{(0)} \otimes M$. It is also clear that $V^{(0)} \otimes M$ is stable under $\Delta_D\big(x^+_{j,\geqslant}(z)\big)$. Therefore $V^{(0)} \otimes M$ is a submodule by Lemma 5.10.

By Lemma 5.7, the only possibility for a non-zero proper submodule of $V^{(0)} \otimes M$ is $\mathbb{C}v_0 \otimes M$. However, it is easy to see that the latter is not a submodule. Hence $V^{(0)} \otimes M$ is irreducible. Comparing the highest ℓ-weights, we conclude that $N^+_{i,a} = V^{(0)} \otimes M$, from which follows (6.3). □

The module $N^+_{i,a}$ should be thought of as an analog of $U_q\widehat{\mathfrak{sl}}_2$ module in the direction i. We will use modules $N^+_{i,a}$ in Section 7 to establish Bethe ansatz equations for XXZ Hamiltonians in the same way as it is done in the case of $U_q\widehat{\mathfrak{sl}}_2$. For that we will need the following lemma.

Proposition 6.8. *We have the equality in the Grothendieck ring* $\mathrm{Rep}_F\, U_q\mathfrak{b}$*.*

$$[N^+_{i,a}][M^+_{i,a}] = \prod_{j,\ C_{j,i}\neq 0} [M^+_{j,aq_{j,i}^{-1}}] + \prod_{j,\ C_{j,i}\neq 0} [M^+_{j,aq_{j,i}}].$$

Proof. The lemma follows from the comparison of q-characters. □

More generally, we have the following construction. Let $V \in \mathcal{O}^{fin}_{\mathfrak{b}}$ be an irreducible $U_q\mathfrak{b}$ module of finite type. Then by Corollary 6.4, $\chi_q(V) = \mathfrak{m}(1 + \sum_{r=1}^N \mathfrak{m}_r)\prod_{i\in I}\bar{\chi}_i^{-d_i(\mathfrak{m})}$. We call $\chi_q^{ess} = 1 + \sum_{r=1}^N \mathfrak{m}_r \in \mathbb{Z}_{\geq 0}[A^{-1}_{i,a}]_{i\in I, a\in\mathbb{C}^\times}$ the *normalized essential* q*-character* of V.

Let $J \subset I$. We have the obvious inclusion of rings $\iota_J : \mathbb{Z}[A^{-1}_{i,a}]_{i\in J, a\in\mathbb{C}^\times} \to \mathbb{Z}[A^{-1}_{i,a}]_{i\in I, a\in\mathbb{C}^\times}$.

Proposition 6.9. *Let W_J be a finite type $U_q\mathfrak{b}_J$ module. Then there exists a finite type $U_q\mathfrak{b}$ module W such that $\iota_J(\chi_q^{ess}(W_J)) = \chi_q^{ess}(W)$.*

Proof. Let $W_J = L(\mathfrak{m}_J)$. Let $V = L(\mathfrak{m}_J) \in \mathcal{O}^{fin}_{\mathfrak{b}}$ be the highest ℓ-weight $U_q\mathfrak{b}$ module where we consider $\mathfrak{m}_J$ as a monomial in $\mathcal{X}$. We denote v_0 the highest ℓ-weight vector of V. Then $(U_q\mathfrak{b}_J)v_0 \simeq W_J$. Now similarly to the construction of $N^+_{i,a}$, we consider the tensor product $V \otimes_D M$. The module M is a tensor product of sufficiently many factors $M^+_{i,a}$, $i \notin J$, so that its highest ℓ-weight $\mathbf{\Psi}^M(z)$ satisfies $\big(\Psi^M_i(z)x^-_i(z)v\big)_> = 0$ for all $i \notin J$ and $v \in V_{\mathfrak{m}}$ such that $\mathfrak{m}$ appears in $\chi_q\big(U_q\mathfrak{b}_J)v_0\big)$ but $\mathfrak{m}A^{-1}_{i,a}$ does not appear in $\chi_q\big((U_q\mathfrak{b}_J)v_0\big)$.

Let W be the irreducible submodule of $V \otimes_D M$ containing $v_0 \otimes |\emptyset\rangle_M$. Then the equality $\iota_J(\chi_q^{ess}(W_J)) = \chi_q^{ess}(W)$ follows from Lemma 3.6 in [FM1]. □

The module W in Proposition 6.9 is not unique, it is defined up to tensor multiplication by various $M^+_{i,a}$. We denote the W which corresponds to the minimal choice of such factors by W^I_J and call it the *lift of module* W_J. Therefore we have a map $\mathcal{P}^I_J:\ \mathcal{O}^{fin}_{\mathfrak{b}_J}\to\mathcal{O}^{fin}_{\mathfrak{b}}$, $W_J\mapsto W^I_J$. Note that it is not a ring homomorphism.

In particular, if we take $J=\{i\}\subset I$, then $\mathfrak{g}_J\simeq U_q\widehat{\mathfrak{sl}}_2$, the fundamental module $L(Y_a)$ is just a 2-dimensional evaluation module and we have $\mathcal{P}^I_J(L(Y_a))=N^+_{i,aq}$.

Example. Let $\mathfrak{g}=\widehat{\mathfrak{sl}}_{n+1}$, $J=\{i\}\subset I$. Let $W_J=L(\prod_{p=0}^{s-1}Y_{aq^{-2p}})$ be the evaluation $U_q\widehat{\mathfrak{sl}}_2$ module of dimension $s+1$. In this case formula (4.1) becomes

$$\chi_q(W_J)=\prod_{p=0}^{s-1}Y_{aq^{-2p}}\Big(1+\sum_{j=1}^{s}\prod_{r=1}^{j}A^{-1}_{aq^{-2r+3}}\Big).$$

Then we have $\mathcal{P}^I_J(W_J)=L(X_{i-1,aq^2}X_{i+1,aq^2}\prod_{p=0}^{s-1}Y_{i,aq^{-2p}})$.

This module is $(s+1)$-finite and it has a similar q-character:

$$\begin{aligned}&\chi_q(L(X_{i-1,aq^2}X_{i+1,aq^2}\prod_{p=0}^{s-1}Y_{i,aq^{-2p}}))\\&=X_{i-1,aq^2}X_{i+1,aq^2}(\prod_{p=0}^{s-1}Y_{i,aq^{-2p}})\Big(1+\sum_{j=1}^{s}\prod_{r=1}^{j}A^{-1}_{i,aq^{-2r+3}}\Big)\bar{\chi}_{i-1}\bar{\chi}_{i+1}.\end{aligned}$$

Example. Let $\mathfrak{g}=\widehat{\mathfrak{sl}}_3$. Consider $L(Y_{1,aq^{-2}}Y_{1,a})$. It is the 6-dimensional evaluation module related to partition $\lambda=(2\geq 0)$, see (4.1). The module $L(Y_{1,aq^{-2}}Y_{1,a}X_{2,aq^2})$ is the 3-finite module which is a lift of the 3-dimensional $U_q\widehat{\mathfrak{sl}}_2$ evaluation module $L(Y_{aq^{-2}}Y_a)$. The module $L(Y_{1,aq^{-2}}Y_{1,a}X_{2,a})$ is a 5-finite module and it is not a lift of a $U_q\widehat{\mathfrak{sl}}_2$ module.

7. Bethe Ansatz

7.1. **Normalized R matrix.** In this section, we take u to be an indeterminate.

We define $s_u: U_q\mathfrak{g}\to U_q\mathfrak{g}[u^{\pm1}]$ by setting $s_u(x)=u^{\mathrm{hdeg}\,x}x$ for any homogeneous element $x\in U_q\mathfrak{g}$. We set also

$$\mathcal{R}(u)=\big(s_u\otimes\mathrm{id}\big)\big(\mathcal{R}\big)\qquad\in U_q\mathfrak{b}\widehat{\otimes}U_q\overline{\mathfrak{b}}[[u]]\,.$$

In formula (2.12) for $\mathcal{R}_+$, the first tensor component of each term acts as annihilation operator on modules from $\mathcal{O}_\mathfrak{b}$. Likewise, in formula (2.14) for $\mathcal{R}_-$, the second tensor component of each term acts as annihilation operator on modules from $\mathcal{O}_\mathfrak{g}$. Therefore, if $V\in\mathrm{Ob}\,\mathcal{O}_\mathfrak{b}$ and $W\in\mathrm{Ob}\,\mathcal{O}_\mathfrak{g}$, then each coefficient of the formal series $\mathcal{R}(u)$ is a well defined operator on $V\otimes W$.

Suppose further that V is a tensor product of highest ℓ-weight $U_q\mathfrak{b}$ modules, and W is a tensor product of highest ℓ-weight $U_q\mathfrak{g}$ modules. Denote by $v_0\in V$ the tensor product of highest ℓ-weight vectors, and by $w_0\in W$ the tensor product of highest ℓ-weight vectors. We write the eigenvalues of $h_{i,r}$ on these vectors as $\langle h_{i,r}\rangle_V$, $\langle h_{i,r}\rangle_W$, respectively. From the remark above, we see that $\mathcal{R}(u)\big(v_0\otimes w_0\big)=f_{V,W}(u)\big(v_0\otimes w_0\big)$, where

$$f_{V,W}(u)=q^{-(\mathrm{wt}\,v_0,\mathrm{wt}\,w_0)}\exp\Big(-\sum_{\substack{r>0\\ i,j\in I}}\frac{r\widetilde{B}_{i,j}(q^r)}{q^r-q^{-r}}(q_i-q_i^{-1})(q_j-q_j^{-1})\,u^r\langle h_{i,r}\rangle_V\langle h_{j,-r}\rangle_W\Big)\,. \tag{7.1}$$

We have

$$f_{V_1\otimes V_2,W}(u)=f_{V_1,W}(u)f_{V_2,W}(u)\,,\quad f_{V,W_1\otimes W_2}(u)=f_{V,W_1}(u)f_{V,W_2}(u)\,.$$

We define the normalized R matrix $\overline{R}_{V,W}(u)\in\mathrm{End}\big(V\otimes W\big)[[u]]$ by

$$\overline{R}_{V,W}(u)\big(v\otimes w\big)=f_{V,W}(u)^{-1}\mathcal{R}(u)\big(v\otimes w\big)\quad(v\in V,w\in W).$$

Lemma 7.1. *Let $V=M^+_{i,1}$, and let $W\in\mathrm{Ob}\,\mathcal{O}_{\mathfrak{g}}$ be an irreducible $U_q\mathfrak{g}$ module. Then the normalized R matrix $\overline{R}_{M^+_{i,1},W}(u)$ does not have a pole in $u\in\mathbb{C}^\times$.*

Proof. Recall that $\overline{R}_{M^+_{i,1},W}(u)(v_0\otimes w_0)=v_0\otimes w_0$, where $v_0\in M^+_{i,1}$, $w_0\in W$ are the highest ℓ-weight vectors. Suppose that $\overline{R}_{M^+_{i,1},W}(u)$ has a pole at $u=u_0$. Let $P:V\otimes W\to W\otimes V$ be the map $P(v\otimes w)=w\otimes v$. Then $\mathrm{res}_{u=u_0}P\,\overline{R}_{M^+_{i,1},W}(u):V\otimes W(u_0^{-1})\to W(u_0^{-1})\otimes V$ is an intertwiner of $U_q\mathfrak{b}$ modules, where $W(u_0^{-1})$ is the module W twisted by $s^{-1}_{u_0}$. Its image is a non-zero submodule of $W(u_0^{-1})\otimes M^+_{i,1}$, which does not contain $w_0\otimes v_0$ due to the normalization of $\overline{R}_{M^+_{i,1},W}(u)$. However Lemma 5.7 shows that there is no such submodule. This is a contradiction. $\square$

7.2. **Transfer matrices.** Let $\tilde p=(\tilde p_i)_{i\in I}$ be indeterminates, and set $p=(p_i)_{i\in I}$, $p_i=\prod_{j\in I}\tilde p_j^{C_{j,i}}$. For $V\in\mathrm{Ob}\,\mathcal{O}_{\mathfrak{b}}$, denote by $\tilde p^h\in\mathrm{End}\,V$ the operator which acts on each weight vector $v\in V_\mu$ by $\tilde p^h v=(\prod_{i\in I}\tilde p_i^{\log\mu_i/\log q_i})v$.

For an object $V\in\mathrm{Ob}\,\mathcal{O}_{\mathfrak{b}}$, the twisted transfer matrix associated with the 'auxiliary space' V is a formal series defined by

$$\mathfrak{T}_V(u;p)=\mathrm{Tr}_{V,1}\left(\tilde p^{-h}\otimes\mathrm{id}\cdot\mathcal{R}(u)\right)\quad\in U_q\overline{\mathfrak{b}}[\tilde p_i^{\pm1}][[u,p_i]]_{i\in I}\,.$$

Here $\mathrm{Tr}_{V,1}$ means that the trace is taken on the first tensor component. Clearly we have

$$\mathfrak{T}_{V_1\oplus V_2}(u;p)=\mathfrak{T}_{V_1}(u;p)+\mathfrak{T}_{V_2}(u;p)\,,$$
$$\mathfrak{T}_{V_1\otimes V_2}(u;p)=\mathfrak{T}_{V_1}(u;p)\mathfrak{T}_{V_2}(u;p)\,,$$

hence the assignment $V\mapsto\mathfrak{T}_V$ gives a homomorphism of rings from $\mathrm{Rep}\,U_q\mathfrak{b}$ to $U_q\overline{\mathfrak{b}}[\tilde p_i^{\pm1}][[u,p_i]]_{i\in I}$.

Element $\mathfrak{T}_V(u;p)$ gives rise to a formal series of operators which act on any given 'quantum space' $W\in\mathrm{Ob}\,\mathcal{O}_{\mathfrak{g}}$. It is convenient to use the normalized R matrix and define

$$T_{V,W}(u;p)=\mathrm{Tr}_{V,1}\big((\tilde p^{-h}\otimes\mathrm{id})\overline{R}_{V,W}(u)\big)\quad\in\mathrm{End}(W)[\tilde p_i^{\pm1}][[u,p_i]]_{i\in I}\,,$$

so that $\mathfrak{T}_V(u;p)\big|_W=f_{V,W}(u)T_{V,W}(u;p)$. Note that $T_{V,W}(u;p)$ acts on each subspace of W of fixed weight.

7.3. **Bethe Ansatz.** From now on, we choose W to be a tensor product of irreducible finite dimensional $U_q\mathfrak{g}$ modules. Let $w_0 \in W$ be the tensor product of highest ℓ-weight vectors, with weight $\mu = \mathrm{wt}\, w_0$ and highest ℓ-weight $\mathfrak{m} = \prod_{i\in I, b\in\mathbb{C}^\times} Y_{i,b}^{m_{i,b}}$. Then $\phi_i^+(z)w_0 = \prod_{b\in\mathbb{C}^\times}((q_i - bz)/(1-q_i bz))^{m_{i,b}} w_0$. We set

$$a_i(u) = \prod_b \Bigl(q_i u - b\Bigr)^{m_{i,b}}, \quad d_i(u) = \prod_b \Bigl(u - q_i b\Bigr)^{m_{i,b}},$$

and introduce the notation

$$Q_i(u;p) = T_{M^+_{i,1},W}(u;p)\,, \quad \mathcal{T}_i(u;p) = a_i(u) T_{N^+_{i,1},W}(u;p)\,.$$

The following is an analog of Baxter's relation well known for $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$.

Lemma 7.2. *We have the relation*

$$\mathcal{T}_i(u;p)Q_i(u;p) = a_i(u) \prod_{j:C_{j,i}\neq 0} Q_j(q_{j,i}^{-1}u;p) + p_i d_i(u) \prod_{j:C_{j,i}\neq 0} Q_j(q_{j,i}u;p)\,. \tag{7.2}$$

Proof. This follows from Proposition 6.8 and the following calculation:

$$\begin{aligned}\frac{d_i(u)}{a_i(u)} &= \prod_{j:C_{j,i}\neq 0} \frac{f_{M^+_{j,1},W}(q_{j,i}u)}{f_{M^+_{j,1},W}(q_{j,i}^{-1}u)}\\ &= q_i^{(\alpha_i, \mathrm{wt}\, w_0)} \exp\Bigl(\sum_{\substack{l,j\in I\\ r>0}} \frac{q_{i,l}^r - q_{i,l}^{-r}}{q^r - q^{-r}} \tilde{B}_{l,j}(q^r)(q_j - q_j^{-1})\langle h_{j,-r}\rangle_W u^r\Bigr)\\ &= \langle \phi_i^-(u^{-1})^{-1}\rangle_W\,.\end{aligned}$$

□

We show next that on each weight subspace of W the operators $Q_i(u;p)$, $\mathcal{T}_i(u;p)$ are polynomials in u. Since they are defined as traces on infinite dimensional spaces, we need certain estimates on the growth of the matrix elements to ensure the polynomiality.

For vectors $w^* \in W^*$ and $w \in W$, introduce the notation $L_{w^*,w}(u)$ for the matrix coefficients in the second component,

$$v^* L_{w^*,w}(u) v = v^* \otimes w^* \overline{R}_{V,W}(u) v \otimes w \quad (v^* \in V^*, v \in V).$$

We regard V^*, W^* as right $U_q\mathfrak{b}$ modules. The intertwining property of the R matrix and Lemma A.2 entail the following relations.

$$\begin{aligned} L_{w^* x^+_{i,k},w}(u) &= L_{w^*,x^+_{i,k}w}(u) k_i + u^k (L_{w^*,w}(u) x^+_{i,k} - x^+_{i,k} L_{w^* k_i,w}(u))\\ &\quad + \sum_j u^{\mathrm{hdeg}\, a_j} L_{w^*,b_j w}(u) a_j - \sum_j u^{\mathrm{hdeg}\, b_j} b_j L_{w^* a_j,w}(u)\,, \end{aligned} \tag{7.3}$$

$$\begin{aligned} L_{w^*,x^-_{i,k}w}(u) &= k_i^{-1} L_{w^* x^-_{i,k},w}(u) - u^k (L_{w^*,k_i^{-1}w}(u) x^-_{i,k} - x^-_{i,k} L_{w^*,w}(u))\\ &\quad - \sum_j u^{\mathrm{hdeg}\, a'_j} L_{w^*,b'_j w}(u) a'_j + \sum_j u^{\mathrm{hdeg}\, b'_j} b'_j L_{w^* a'_j,w}(u)\,. \end{aligned} \tag{7.4}$$

Here $a_j \in U_q\mathfrak{b}$, $b_j \in U_q^+\mathfrak{b}$ are such that wt $b_j > 0$, wt a_j + wt $b_j = \alpha_i$, and $a'_j \in U_q^-\mathfrak{b}$, $b'_j \in U_q\mathfrak{b}$ are such that wt $a'_j < 0$, wt a'_j + wt $b'_j = -\alpha_i$. Even though $U_q^-\mathfrak{b}$ is not generated by the $x^-_{j,k}$, they are enough to generate the space W from w_0.

Proposition 7.3. *On each weight subspace $W_{\mu\mathfrak{q}^{-\beta}}$ ($\beta \in Q^+$), $Q_i(u;p)$ is a polynomial in u of degree at most $(\omega_i^\vee, \beta)$.*

Proof. Set $V = M^+_{i,a}$, and let $V = \oplus_{m=0}^\infty M[m]$ be the grading as in Proposition 5.11. Denote by u^∂ the operator $u^\partial\big|_{M[m]} = u^m \times \mathrm{id}_{M[m]}$. The assertion is proved if we show that as $u \to \infty$

$$u^{-\partial} L_{w^*,w}(u) u^\partial = O(u^{(\omega_i^\vee,\beta)}) \qquad (w^* \in W^*_{\mu\mathfrak{q}^{-\beta}},\ w \in W). \tag{7.5}$$

We set $\bar{h}_{i,r,V} = h_{i,r} - \langle h_{i,r}\rangle_V$. Proposition 5.11 implies that the operators $u^r u^{-\partial}\bar{h}_{i,r,V}u^\partial$ and $u^{\mathrm{hdeg}\,x}u^{-\partial}xu^\partial$ ($x \in U_q^-\mathfrak{b}$) are independent of u, while

$$u^{\mathrm{hdeg}\,x}u^{-\partial}xu^\partial = O(u^{(\omega_i^\vee,\mathrm{wt}\ x)}) \quad (x \in U_q^+\mathfrak{b}).$$

Consider the case where $w^* = w_0^*$ is the lowest weight vector and $w = w_0$ is the highest weight vector. Then we have

$$L_{w_0^*,w_0}(u) = \exp\Big(-\sum_{\substack{r>0\\ k,j\in I}} \frac{r\widetilde{B}_{k,j}(q^r)}{q^r - q^{-r}}(q_k - q_k^{-1})(q_j - q_j^{-1})\, u^r \bar{h}_{k,r,V}\langle h_{j,-r}\rangle_W\Big)\,.$$

From the remark above we see that $u^{-\partial}L_{w_0^*,w_0}(u)u^\partial$ is independent of u. Hence (7.5) holds in this case.

By induction on the height of β, and using (7.4), we can show that $u^{-\partial}L_{w_0^*,w_2}(u)u^\partial = O(1)$ for all $w \in W$. Finally (7.5) in the general case follows from (7.3). □

Proposition 7.4. *On each weight subspace of W, $\mathcal{T}_i(u;p)$ is a polynomial in u.*

Proof. Set $V = N^+_{i,a}$, and let $V = N_0 \oplus N_1$ be the decomposition into ℓ-weight subspaces. In the notation of the proof of Proposition 6.7, we have $N_s = \mathbb{C}v_s \otimes M \subset V^{(0)} \otimes_D M$, $M = \bigotimes_{j:C_{j,i}<0} M^+_{j,aq_{j,i}^{-1}}$. Set $\omega^\vee = \sum_{j:C_{j,i}<0}\omega_j^\vee$. Using the grading $M = \oplus_{m=0}^\infty M[m]$ of M in Proposition 5.11 we set $N_s[m] = v_s \otimes M[m]$. Let u^∂ be the operator which act as scalar u^m on $N_j[m]$. We have then

$$\begin{aligned}
L_{w_0^*,w_0}(u) &= \exp\Big(-\sum_{\substack{r>0\\ k,j\in I}} \frac{r\widetilde{B}_{k,j}(q^r)}{q^r - q^{-r}}(q_k - q_k^{-1})(q_j - q_j^{-1})\, u^r \bar{h}_{k,r,N_0}\langle h_{j,-r}\rangle_W\Big)\\
&= \frac{d_i(u)}{a_i(u)}\exp\Big(-\sum_{\substack{r>0\\ k,j\in I}} \frac{r\widetilde{B}_{k,j}(q^r)}{q^r - q^{-r}}(q_k - q_k^{-1})(q_j - q_j^{-1})\, u^r \bar{h}_{k,r,N_1}\langle h_{j,-r}\rangle_W\Big)\,.
\end{aligned}$$

Hence $a_i(u)L_{w_0^*,w_0}(u)$ is a polynomial on each vector.

Acting with $\Delta_D(x^{\pm}_{j,k})$ we find

$$u^k u^{-\partial} x^{-}_{j,k} u^{\partial} N_s[m] \subset N_s[m+k] + \delta_{s,0}\delta_{i,j} \sum_{l\geq 0} u^{k-l} N_1[m+l]\,,$$

$$u^k u^{-\partial} x^{+}_{j,k} u^{\partial} N_s[m] \subset \sum_{r=0}^{(\omega^\vee,\alpha_j)} u^r \sum_{l\geq k} u^{k-l} N_s[m+l-r] + \delta_{s,1}\delta_{i,j} u^k N_0[m]\,.$$

Note that we need only a finitely many of the $x^{\pm}_{j,k}$'s in order to generate W (resp. W^*) from w_0 (resp. w_0^*). The rest of the argument is the same as in the proof of Proposition 7.3; we use the intertwining relations (7.3), (7.4) and induction on the weight to prove that $u^{-\partial} L_{w^*,w}(u) u^{\partial} = O(u^K)$ for some $K = K_{w^*,w}$. □

Let w be an eigenvector of $Q_i(u;p)$ with eigenvalue $Q_{i,w}(u;p)$. By Proposition 7.3, we can write $Q_{i,w}(u;p) = Q_{i,w}(0;p)\prod_{\nu=1}^{N_i}(1-u/\zeta_{i,\nu})$. Substituting $u=\zeta_{i,\nu}$ into (7.2), and using the polynomiality of $\mathcal{T}_i(u;p)$, we obtain the Bethe ansatz equations

$$p_i \frac{d_i(\zeta_{i,\nu})}{a_i(\zeta_{i,\nu})} \prod_{j\in I}\prod_{\mu=1}^{N_j} \frac{1-q_{j,i}\zeta_{i,\nu}/\zeta_{j,\mu}}{1-q_{j,i}^{-1}\zeta_{i,\nu}/\zeta_{j,\mu}} = -1 \quad (i\in I, \nu=1,\cdots,N_i). \tag{7.6}$$

The corresponding eigenvalue of the normalized transfer matrix can be obtained from the q-character of the 'auxiliary space' V. The recipe is given as follows [FH].

Theorem 7.5. *Let w be an eigenvector of $T_{V,W}(u;p)$ of weight $(q_i^{\nu_i})$. Then the corresponding eigenvalue of $f_{V,W}(u;p)T_{V,W}(u;p)$ is obtained from $\chi_q(V)$ by the substitution*

$$X_{i,a} \to f_{M^+_{i,1},W}(a;p) Q_{i,w}(a;p)\,, \quad y_i^{b_i} \to q^{-(b_i\omega_i, \sum_j \nu_j\omega_j)}.$$

Appendix A. Root vectors

Following [Be, Be2, Da], we review the definition and known facts about root vectors of $U_q\mathfrak{g}$. In this section we take q to be an indeterminate and work over $\mathbb{C}(q)$.

Let $\Omega, \Xi : U_q\mathfrak{g} \to U_q\mathfrak{g}$ be anti-isomorphisms of $\mathbb{C}$-algebras defined by

$$\Omega e_i = f_i\,, \quad \Omega f_i = e_i\,, \quad \Omega k_i = k_i^{-1} \quad (0\leq i\leq n), \quad \Omega q = q^{-1},$$
$$\Xi e_i = e_i\,, \quad \Xi f_i = f_i\,, \quad \Xi k_i = k_i^{-1} \quad (0\leq i\leq n), \quad \Xi q = q\,.$$

Denote by s_i $(0\leq i\leq n)$ the simple reflections. For $\alpha = \sum_{i\in I} b_i\alpha_i \in Q$ we set $k_\alpha = \prod_{i\in I} k_i^{b_i}$. Lusztig's automorphisms $\{T_i\}_{0\leq i\leq n}$ of $U_q\mathfrak{g}$ are characterized by the following properties:

$$T_i k_\alpha = k_{s_i\alpha} \quad (\alpha\in Q)\,,$$
$$T_i e_j = \begin{cases} -f_i k_i & (i=j), \\ \sum_{r+s=-C_{i,j}} (-1)^s q_i^{-r} e_i^{(s)} e_j e_i^{(r)} & (i\neq j), \end{cases}$$
$$\Omega T_i = T_i \Omega\,, \quad \Xi T_i = T_i^{-1}\Xi\,.$$

For an element w of the affine Weyl group $W = \langle s_i \ (0\leq i\leq n)\rangle$, we define $T_w = T_{i_1}\cdots T_{i_m}$ where $w = s_{i_1}\cdots s_{i_m}$ is a reduced expression. This definition does not depend on the chosen expression of w.

There is an embedding of groups $t : Q^\vee = \sum_{i\in I}\mathbb{Z}\alpha_i^\vee \to W$, $x^\vee \mapsto t_{x^\vee}$, such that $t_{x^\vee}(\alpha) = \alpha - (x^\vee,\alpha)\delta$ ($\alpha \in Q$). Fix an element $x^\vee \in Q^\vee$ which satisfies $(x^\vee,\alpha_i) > 0$ for all $i \in I$. Fix also a reduced decomposition $t_{x^\vee} = s_{i_1}\cdots s_{i_N}$ such that $i_1 = 0$, and introduce a sequence $\{i_r\}_{r\in\mathbb{Z}}$ by setting $i_{r+N} = i_r$ for $r \in \mathbb{Z}$. We define $\{\beta_r\}_{r\in\mathbb{Z}}$ by

$$\beta_r = \begin{cases} s_{i_1}\cdots s_{i_{r-1}}\alpha_{i_r} & (r \ge 1),\\ s_{i_0}\cdots s_{i_{r+1}}\alpha_{i_r} & (r \le 0). \end{cases}$$

By construction we have $\beta_{r+N} = \beta_r - (x^\vee,\beta_r)\delta$. For any $s \ge 1$ the following hold [Be2]:

$$\{\beta_r \mid 1 \le r \le sN\} = \{m\delta - \alpha \mid \alpha \in \Delta_+\,,\ 1 \le m \le s(x^\vee,\alpha)\}\,, \tag{A.1}$$

$$\{\beta_r \mid 0 \ge r \ge -sN+1\} = \{m\delta + \alpha \mid \alpha \in \Delta_+\,,\ 0 \le m \le s(x^\vee,\alpha)-1\}\,. \tag{A.2}$$

In particular, $\{\beta_r\}_{r\in\mathbb{Z}}$ coincides with the set of positive real roots of $\mathfrak{g}$.

The root vectors are defined for positive real roots by

$$e_{\beta_r} = \begin{cases} T_{i_1}\cdots T_{i_{r-1}}e_{i_r} & (r \ge 1)\,,\\ T_{i_0}\cdots T_{i_{r+1}}e_{i_r} & (r \le 0)\,, \end{cases} \tag{A.3}$$

$$f_{\beta_r} = \Omega e_{\beta_r}\,. \tag{A.4}$$

Positive imaginary roots have multiplicities. Labeling them as $(m\delta, i)$ ($m > 0$, $i \in I$) we set

$$e_{(m\delta,i)} = q_i^{-2}e_i e_{m\delta-\alpha_i} - e_{m\delta-\alpha_i}e_i \quad (m > 0)\,, \qquad f_{(m\delta,i)} = \Omega e_{(m\delta,i)}\,. \tag{A.5}$$

The root vectors are related to the Drinfeld generators by

$$x^+_{i,m} = o(i)^m \times \begin{cases} e_{m\delta+\alpha_i} & (m \ge 0),\\ -f_{-m\delta-\alpha_i}k_i^{-1} & (m < 0), \end{cases} \tag{A.6}$$

$$\phi^+_{i,m} = -o(i)^m(q_i - q_i^{-1})k_i e_{(m\delta,i)} \ (m > 0)\,, \tag{A.7}$$

$$x^-_{i,m} = \Omega x^+_{i,-m}\,, \quad \phi^-_{i,r} = \Omega\phi^+_{i,-r}\,. \tag{A.8}$$

Here $o(i) = \pm 1$ is chosen in such a way that $o(i)o(j) = -1$ holds for $C_{i,j} < 0$. The Borel subalgebras are generated by root vectors,

$$U_q^+\mathfrak{b} = \langle e_{m\delta+\alpha} \mid m \ge 0,\ \alpha \in \Delta^+\rangle\,,$$
$$U_q^-\mathfrak{b} = \langle k_\alpha e_{m\delta-\alpha} \mid m > 0,\ \alpha \in \Delta^+\rangle\,.$$

Define a total order $\prec$ on the set of positive roots as follows. For real roots we set $\beta_r \prec \beta_s$ if $r > s > 0$ or $s < r \le 0$. Positive imaginary roots are mutually ordered in an arbitrary way. We set further $\beta_s \prec (m\delta,i) \prec \beta_r$ for any $m > 0$, $r > 0 \ge s$. Altogether we have

$$\beta_0 \prec \beta_{-1} \prec \beta_{-2} \prec \cdots \prec (m\delta,i) \prec \cdots \prec \beta_3 \prec \beta_2 \prec \beta_1\,. \tag{A.9}$$

The root vectors satisfy the convexity property [Be2]

$$e_\beta e_\alpha - q^{(\alpha,\beta)}e_\alpha e_\beta = \sum_{\{\gamma_i\},\{n_i\}} a^{\{n_i\}}_{\{\gamma_i\}} e^{n_1}_{\gamma_1}\cdots e^{n_m}_{\gamma_m}\,, \tag{A.10}$$

where $a^{\{n_i\}}_{\{\gamma_i\}} \in \mathbb{C}(q)$, and the sum is taken over γ_i and $n_i \in \mathbb{Z}_{>0}$ such that $\alpha \prec \gamma_1 \prec \cdots \prec \gamma_m \prec \beta$, $\sum_i n_i\gamma_i = \alpha + \beta$. If $\alpha = \beta_r$, $\beta = \beta_s$ and $r, s > 0$ or $r, s \le 0$, then the coefficients are Laurent polynomials of q [BCP]. If in addition $\alpha+\beta$ is a root, then $e_{\alpha+\beta}$ appears with non-zero coefficient.

Lemma A.1. *For any $\alpha\in\Delta^+$ and $l\ge 1$, there exists a $p_0\ge 1$ such that if $p\ge p_0$ then $e_{p\delta\pm\alpha}$ belongs to the algebra generated by $x^{\pm}_{i,m}$ and $k_i^{\pm 1}$ ($i\in I$, $m\ge l$).*

Proof. This can be verified by induction on the height of α, using (A.1), (A.2), and (A.10). □

The following can be extracted from [Da], Theorem 4:

Lemma A.2. *We have*

$$\Delta(x^+_{i,k}) = x^+_{i,k}\otimes 1 + k_i\otimes x^+_{i,k} + \sum_j a_j\otimes b_j\,,$$

where $a_j\in U_q\mathfrak{b}$, $b_j\in U_q^+\mathfrak{b}$ are such that $\mathrm{wt}\, b_j>0$, $\mathrm{wt}\, a_j+\mathrm{wt}\, b_j=\alpha_i$. *Similarly*

$$\Delta(x^-_{i,k}) = x^-_{i,k}\otimes k_i^{-1} + 1\otimes x^-_{i,k} + \sum_j a'_j\otimes b'_j\,,$$

where $a'_j\in U_q^-\mathfrak{b}$, $b'_j\in U_q\mathfrak{b}$ are such that $\mathrm{wt}\, a'_j<0$, $\mathrm{wt}\, a'_j+\mathrm{wt}\, b'_j=-\alpha_i$.

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