Large-x analysis of an operator valued Riemann–Hilbert problem

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Abstract

The purpose of this paper is to push forward the theory of operator-valued Riemann Hilbert problems and demonstrate their effectiveness in respect to the implementation of a non-linear steepest descent method *á la* Deift-Zhou. In the present paper, we demonstrate that the operator-valued Riemann–Hilbert problem arising in the characterisation of so-called c-shifted integrable integral operators allows one to extract the large-x asymptotics of the Fredholm determinant associated with such operators.

1 Introduction

The term integrable integral operator refers to a specific class of integral operators $I + V$ whose integral kernel takes the form

$$V(\lambda,\mu) \;=\; \frac{\sum_{a=1}^{N} e_a(\lambda) f_a(\mu)}{\lambda-\mu} \qquad \text{with} \qquad \sum_{a=1}^{N} e_a(\lambda) f_a(\lambda) \;=\; 0 \tag{1.1}$$

where e_a , f_a, $a = 1, \dots, N$ are functions whose regularity depends on the functional space on which the operator acts. The quite specific structure of their kernels endows integrable integral operators with numerous properties allowing one, in particular, for the construction of the resolvent kernel or computation of the Fredholm determinant of $I + V$ in terms of a solution to a specific $N \times N$ matrix valued Riemann–Hilbert problem [7]. We remind that the jump matrix for this Riemann–Hilbert problem is built out of the functions e_a and f_a, $a = 1, \dots, N$.

Despite the specific form (1.1) imposed on the kernel of integrable integral operators, such operators still arise in many concrete problems of mathematical physics. The Fredholm determinants of specific instances of such operators describe numerous observables, be it in random matrix theory -gap probabilities in the bulk or edge of the spectrum [2, 3, 4] - or quantum integrable models -correlation functions of products of local operators [15, 17]- to name a few.

One can, in fact, consider more general integrable integral operators than those described by (1.1). To generalise the formula, it is enough to replace the discreet variable $a \in \{1, \dots, N\}$ labelling the functions e_a and f_a by a variable s living in some measure space (X, ν). One then replaces the discreet and finite sum in (1.1) by an integral *versus* dν:

$$V(\lambda,\mu) \;=\; \frac{\int_X e(\lambda;s) f(\mu;s)\cdot \mathrm{d}\nu(s)}{\lambda-\mu} \qquad \text{with} \qquad \int_X e(\lambda;s) f(\lambda;s)\cdot \mathrm{d}\nu(s) \;=\; 0\;. \tag{1.2}$$

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Particular, examples of such more general integrable integral operators arose in the context of studying quantum integrable systems at generic value of their interaction strength [6, 7, 12], *viz.* away from their free fermion point. Independently from their existing applications, such more general integrable integral operators are of interest in their own right precisely because of the much larger freedom in the form taken by their kernels and yet the possibility to study them by means of Riemann–Hilbert problems. The price to pay, however, is the complication of the Riemann–Hilbert problem in that one no longer deals with a matrix valued one but rather an operator valued one. Still, in the early days of exploring the correlation functions in quantum integrable systems out of their free fermion point, certain properties of Fredholm determinants of such more general operators were investigated on the basis of operator valued Riemann–Hilbert problems which are associated with these kernels. The Riemann–Hilbert machinery allowed to construct systems of partial differential equations satisfied by specific instances of such operators [7, 11, 13, 14]. It is also important to mention the work [9] where a formal non-linear steepest descent analysis of an oscillatory operator valued Riemann–Hilbert problem was carried out. This allowed the authors to extract the leading asymptotic behaviour in the large parameter out of the logarithm of the Fredholm determinant at stake. However, numerous technical difficulties (the operator nature of the scalar Riemann–Hilbert problem which arises in the very the first step of the analysis, construction of parametrices in terms of special functions with operator index,...) which could not have been overcome stopped, for almost 15 years, any activity related to an asymptotic analysis of operator valued Riemann–Hilbert problems.

Recently in [8] we have proposed a scheme allowing one to extract the large-x asymptotic behaviour of the Fredholm determinant of so-called c-shifted integrable integral operators which belong to the class (1.2), with $X = \mathbb{R}^+ \times [\![1 ; N]\!]$ and e_a, f_a depending on x in an oscillatory way. The method of analysis we developed was completely disconnected from any use of the operator valued Riemann–Hilbert problem that is underlying to such c-shifted operators. Notwithstanding, the very fact that the large-x behaviour of these determinants could have been extracted constituted a strong indication that there must exist a way for overcoming the technical difficulties that constituted a obstruction to the asymptotic analysis of operator valued Riemann–Hilbert problems.

As a matter of fact, the recent progress in the field of Riemann–Hilbert problems brings new ideas and tools which allow one for an effective asymptotic analysis of operator valued Riemann–Hilbert problems. The present paper is precisely devoted to demonstrating this fact. More precisely, we reformulate the original statement of an operator valued Riemann–Hilbert problem [14] what permits us to develop a framework allowing one to discuss the solvability and uniqueness of solutions to operator valued Riemann–Hilbert problems. We demonstrate the effectiveness of our scheme by carrying out the large-parameter non-linear steepest descent analysis of an oscillatory operator-valued Riemann–Hilbert problem which can be though of as the operator-valued generalisation of the Riemann–Hilbert problem associated with the so-called generalised sine kernel [10]. Our analysis allows us to reproduce the results of [8] directly within the operator valued Riemann–Hilbert problem setting. We do stress that the main achievements of this paper is to overcome two technical difficulties which arose previously in the analysis of operator-valued Riemann–Hilbert problems:

- primo, we reduce the problem of constructing solutions to operator valued scalar Riemann–Hilbert problem with jump on I to the one of inverting an integral operator acting on $L^2(\Gamma(I), \mathrm{d}z)$, in which Γ is a small counterclockwise loop around I.

- Secundo, we strongly simplify the construction of local parametrices. More precisely, the setting we propose allows us to construct parameterices in terms of special function (confluent hypergeometric functions in our case) whose auxiliary parameters are scalar-valued holomorphic functions and not holomorphic functions taking values in some infinite dimensional Banach spaces, as it was the case in [9].

In the present paper, we shall develop the formalism on the example of the below integrable integral operator on $L^2([a\,;b])$ of c-shifted type whose integral kernel reads

$$V(\lambda,\mu) \;=\; \frac{\mathrm{i}cF(\lambda)}{2\mathrm{i}\pi(\lambda-\mu)}\cdot\Big\{\frac{\mathrm{e}^{\frac{\mathrm{i}x}{2}[p(\lambda)-p(\mu)]}}{(\lambda-\mu)+\mathrm{i}c}\;+\;\frac{\mathrm{e}^{\frac{\mathrm{i}x}{2}[p(\mu)-p(\lambda)]}}{(\lambda-\mu)-\mathrm{i}c}\Big\}\,. \tag{1.3}$$

Throughout the paper, we shall assume that

- $p([a\,;b])\subset\mathbb{R}$ and that p is a biholomorphism from an open neighbourhood U of $[a\,;b]$ in $\mathbb{C}$ onto some open neighbourhood of $[p(a)\,;p(b)]$ in $\mathbb{C}$ which furthermore satisfies $p'_{|[a\,;b]}>0$;
- F is holomorphic on U and satisfies $\big|\arg(1+F(\lambda))\big|<\pi$ for any $\lambda\in U$.

Our analysis allows us to prove the

Theorem 1.1 *Let p and F be as described above and V_0 denote the integral operator on $L^2([a\,;b])$ whose integral kernl reads*

$$V_0(\lambda,\mu)\;=\;\frac{F(\lambda)}{\pi(\lambda-\mu)}\cdot\sin\Big(\frac{x}{2}[p(\lambda)-p(\mu)]\Big)\,. \tag{1.4}$$

Then the below ratio of Fredholm determinants admits the large-x asymptotic behaviour

$$\frac{\det\,[\mathrm{id}\;+\;V]}{\det\,[\mathrm{id}\;+\;V_0]}\;=\;\det_{\Gamma([a\,;b])}\,[I+\mathcal{U}_+]\cdot\det_{\Gamma([a\,;b])}\,[I+\mathcal{U}_-]\cdot\Big(1\;+\;\mathrm{o}(1)\Big) \tag{1.5}$$

where $\mathcal{U}_\pm$ are integral operators on $L^2(\Gamma([a\,;b]))$, with Γ being a small counterclockwise loop around the interval $[a\,;b]$. The integral kernels $U_\pm$ of $\mathcal{U}_\pm$ read

$$U_\pm(\lambda,\mu)\;=\;\frac{\alpha(\lambda)\cdot\alpha^{-1}(\mu\mp\mathrm{i}c)}{2\mathrm{i}\pi(\lambda-\mu\pm\mathrm{i}c)}\quad\text{with}\quad\alpha(\lambda)\;=\;\exp\Big\{\int_a^b\frac{\ln\,[1+F(\mu)]}{\lambda-\mu}\cdot\frac{\mathrm{d}\mu}{2\mathrm{i}\pi}\Big\}\,. \tag{1.6}$$

We do remind that the large-x asymptotic behaviour of $\det\,[\mathrm{id}\;+\;V_0]$ has been obtained in [10].

The paper is organised as follows. In Section 2 we write down the setting of the operator valued Riemann-Hilbert problem associated with a one-parameter t deformation of the kernel V given in (1.3) and prove its unique solvability under the assumption of non-vanishing of a Fredholm determinant. In Section 3, we discuss an auxiliary scalar operator valued Riemann–Hilbert problem and implement the first step of the non-linear steepest descent method. Then, in Section 4 we construct the parametrices adapted to out problem what allows us to put the original Riemann–Hilbert problem in correspondence with one whose jump matrices are close, in appropriate operator norms, to the identity. We then establish the invertibility, in an appropriate functional space, of the singular integral operator associated with the last operator valued Riemann–Hilbert problem. Finally, in Section 5 we build on the Riemann–Hilbert analysis so as to prove Theorem 1.1. For the reader's convenience, we gather in the Appendix certain of the properties of confuent hypergeometric functions that are of interest to our study.

2 The initial Riemann–Hilbert problem

2.1 A few definitions

We first discuss several notations and conventions that will become handy in the following.

• Throughout the paper, given some oriented curve Σ in $\mathbb{C}$, we agree to denote by $\Gamma(\Sigma)$ a small counterclockwise loop around Σ.

• The superscript $^{\boldsymbol{T}}$ will denote the transposition of vectors, *viz.*

$$\text{if} \quad \vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix} \qquad \text{then} \quad \vec{v}^{\boldsymbol{T}} = (v_1 \ \dots \ v_N) \ . \tag{2.1}$$

• The space $\mathcal{M}_p(\mathbb{C})$ of $p \times p$ matrices over $\mathbb{C}$ is endowed with the norm $\big\|M\big\| = \max_{a,b} |M_{a,b}|$.

• The space $\mathcal{M}_p\Big(L^2(X, \mathrm{d}\nu)\Big)$ denotes the space of $p \times p$ matrix valued functions on X whose matrix entries belong to $L^2(X, \mathrm{d}\nu)$. This space is endowed with the norm

$$\big\|M\big\|^2_{\mathcal{M}_p\left(L^2\left(X,\mathrm{d}\nu\right)\right)} = \int\limits_X \mathrm{tr}[M^\dagger(x) \cdot M(x)] \cdot \mathrm{d}\nu(x) \quad \text{with} \quad \Big(M^\dagger\Big)_{ab} = \overline{M_{ba}} \tag{2.2}$$

with $\overline{*}$ being the complex conjugation of $*$.

• id refers to the identity operator on $L^2(\mathbb{R}^+, \mathrm{d}s)$, $I_p \otimes \mathrm{id}$ refers to the matrix integral operator on $\oplus_{a=1}^p L^2(\mathbb{R}^+, \mathrm{d}s)$ which has the identity operator on its diagonal.

• Given a vector $\vec{\boldsymbol{E}}$ of functions $\boldsymbol{E}_a \in L^2(\mathbb{R}^+, \mathrm{d}s)$

$$\vec{\boldsymbol{E}} = \begin{pmatrix} \boldsymbol{E}_1 \\ \vdots \\ \boldsymbol{E}_p \end{pmatrix} \quad \text{and a vector of} \ \ 1-\text{forms} \quad \vec{\boldsymbol{\kappa}} = \begin{pmatrix} \boldsymbol{\kappa}_1 \\ \vdots \\ \boldsymbol{\kappa}_p \end{pmatrix} \tag{2.3}$$

on $L^2(\mathbb{R}^+, \mathrm{d}s)$, their scalar product refers to the below sum

$$(\vec{\boldsymbol{\kappa}}, \vec{\boldsymbol{E}}) = \sum_{a=1}^{p} \boldsymbol{\kappa}_a[\boldsymbol{E}_a] \tag{2.4}$$

in which one evaluates the one-form -appearing to the left- on the function -appearing to the right-. Furthermore, the notation $\vec{\boldsymbol{E}} \otimes (\vec{\boldsymbol{\kappa}})^{\boldsymbol{T}}$ refers to the matrix operator on $\oplus_{a=1}^p L^2(\mathbb{R}^+, \mathrm{d}s)$ given as

$$\vec{\boldsymbol{E}} \otimes (\vec{\boldsymbol{\kappa}})^{\boldsymbol{T}} = \Big(\boldsymbol{E}_q \otimes \boldsymbol{\kappa}_r\Big)_{q,r=1,\dots,p} \tag{2.5}$$

where $\boldsymbol{E}_q \otimes \boldsymbol{\kappa}_r$ is the operator on $L^2(\mathbb{R}^+, \mathrm{d}s)$ acting as

$$(\boldsymbol{E}_q \otimes \boldsymbol{\kappa}_r)[g] = \boldsymbol{E}_q \cdot \boldsymbol{\kappa}_r[g] \qquad \text{for any } g \in L^2(\mathbb{R}^+, \mathrm{d}s) \ . \tag{2.6}$$

Definition 2.1 *Let $\widehat{\Phi}(\lambda)$ be an integral operator on $\oplus_{a=1}^p L^2(\mathbb{R}^+, \mathrm{d}s)$ parameterised by an auxiliary variable λ. Let $\widehat{\Phi}(\lambda \mid s, s')$ denote its $p \times p$ matrix integral kernel. Given $\mathcal{D}$ an open subset of $\mathbb{C}$, we say that $\widehat{\Phi}(\lambda)$ is a holomorphic in $\lambda \in \mathcal{D}$ integral operator on $\oplus_{a=1}^p L^2(\mathbb{R}^+, \mathrm{d}s)$ if*

- *point-wise in $(s, s') \in (\mathbb{R}^+)^2$, the $p \times p$ matrix-valued function $\lambda \mapsto \widehat{\Phi}(\lambda \mid s, s')$ is holomorphic in $\mathcal{D}$;*
- *pointwise in $\lambda \in \mathcal{D}$, $(s, s') \mapsto \widehat{\Phi}(\lambda \mid s, s') \in \mathcal{M}_p\Big(L^2(\mathbb{R}^+ \times \mathbb{R}^+, \mathrm{d}s \otimes \mathrm{d}s')\Big)$.*

We also need to define what we mean by $\pm$ boundary values of a holomorphic integral operator. There are two kinds of notions that will be of interest for our analysis. On the one hand L^2 and on the other hand continuous boundary values.

Definition 2.2 *Let $\mathcal{D}$ be an open subset of $\mathbb{C}$ and Σ_Φ an oriented smooth curve in $\mathbb{C}$. Let $n(\lambda)$ be the orthogonal to Σ_Φ at the point $\lambda \in \Sigma_\Phi$.*
We say that a holomorphic in $\lambda \in \mathcal{D} \setminus \Sigma_\Phi$ integral operator $\widehat{\Phi}(\lambda)$ on $\oplus_{a=1}^{p} L^2(\mathbb{R}^+, \mathrm{d}s)$ admits L^2 $\pm$-boundary values $\widehat{\Phi}_\pm(\lambda)$ on Σ_Φ if

- *there exists a matrix valued function $(\lambda, s, s') \mapsto \widehat{\Phi}_\pm(\lambda \mid s, s')$ belonging to $L^2(\Sigma_\Phi \times \mathbb{R}^+ \times \mathbb{R}^+)$ and such that*

$$\lim_{\epsilon \to 0^+} \left\| \widehat{\Phi}^{(\pm\epsilon)} - \widehat{\Phi}_\pm \right\|_{\mathcal{M}_p\left(L^2(\Sigma_\Phi \times \mathbb{R}^+ \times \mathbb{R}^+)\right)} = 0 \qquad \text{where} \qquad \widehat{\Phi}^{(\epsilon)}(\lambda \mid s, s') = \widehat{\Phi}(\lambda + \epsilon n(\lambda) \mid s, s') \,.$$

  the operators $\widehat{\Phi}_\pm(\lambda)$ are then defined as the integral operators on $\oplus_{a=1}^{p} L^2(\mathbb{R}^+, \mathrm{d}s)$ characterised by the matrix integral kernel $\widehat{\Phi}_\pm(\lambda \mid s, s')$.

We say that a holomorphic in $\lambda \in \mathcal{D} \setminus \Sigma_\Phi$ integral operator $\widehat{\Phi}(\lambda)$ on $\oplus_{a=1}^{p} L^2(\mathbb{R}^+, \mathrm{d}s)$ admits continuous boundary values $\widehat{\Phi}_\pm(\lambda)$ on $\Sigma'_\Phi \subset \Sigma_\Phi$ if

- *pointwise in $(s, s') \in (\mathbb{R}^+)^2$ the non-tangential limit $\widehat{\Phi}(\lambda \mid s, s') \underset{\lambda \to t}{\longrightarrow} \widehat{\Phi}_\pm(t \mid s, s')$ when λ approaches $t \in \Sigma'_\Phi$ from the $\pm$ side exists and that the map $t \mapsto \widehat{\Phi}_\pm(t \mid s, s')$ is continuous on Σ'_Φ. The operators $\widehat{\Phi}_\pm(\lambda)$ are then defined as the integral operators on $\oplus_{a=1}^{p} L^2(\mathbb{R}^+, \mathrm{d}s)$ characterised by the matrix integral kernel $\widehat{\Phi}_\pm(\lambda \mid s, s')$.*

2.2 The operator-valued Riemann–Hilbert problem

Let $\lambda \mapsto \boldsymbol{m}_k(\lambda)$ be the below one parameter t family of functions taking values in the space of functions on $\mathbb{R}^+$:

$$\boldsymbol{m}_1(\lambda)(s) \equiv \boldsymbol{m}_1(\lambda; s) = \sqrt{c}\, \mathrm{e}^{-\frac{cs}{2}} \mathrm{e}^{ist\lambda} \qquad \text{and} \qquad \boldsymbol{m}_2(\lambda)(s) \equiv \boldsymbol{m}_2(\lambda; s) = \sqrt{c}\, \mathrm{e}^{-\frac{cs}{2}} \mathrm{e}^{-ist\lambda} \,. \tag{2.7}$$

Let $\lambda \mapsto \boldsymbol{\kappa}_k(\lambda)$ be the below one-parameter t family of functions taking values in the space of one-forms on functions on $\mathbb{R}^+$:

$$\boldsymbol{\kappa}_1(\lambda)[f] = \sqrt{c} \int_0^{+\infty} \mathrm{e}^{-\frac{cs}{2}} \mathrm{e}^{-\mathrm{i}st\lambda} f(s) \cdot \mathrm{d}s \qquad \text{and} \qquad \boldsymbol{\kappa}_2(\lambda)[f] = \sqrt{c} \int_0^{+\infty} \mathrm{e}^{-\frac{cs}{2}} \mathrm{e}^{\mathrm{i}st\lambda} f(s) \cdot \mathrm{d}s \,. \tag{2.8}$$

Note that, uniformly in $\lambda \in [a\,; b]$, $s \mapsto \boldsymbol{m}_k(\lambda; s)$ belong to $(L^1 \cap L^\infty)(\mathbb{R}^+, \mathrm{d}s)$ whereas $\boldsymbol{\kappa}_k(\lambda)$ are one-forms on $L^2(\mathbb{R}^+, \mathrm{d}s)$. The one-forms and functions introduced above satisfy to

$$\boldsymbol{\kappa}_k(\lambda)[\boldsymbol{m}_k(\mu)] = \frac{\mathrm{i}c\epsilon_k}{t(\lambda - \mu) + \mathrm{i}\epsilon_k c} \qquad \text{where} \quad k = 1, 2 \quad \text{and} \quad \left\{ \begin{array}{l} \epsilon_1 = -1 \\ \epsilon_2 = 1 \end{array} \right. . \tag{2.9}$$

We are now in position to introduce the vector-valued function $\vec{\boldsymbol{E}}_R(\mu)$ and the vector valued one-forms $\vec{\boldsymbol{E}}_L(\mu)$:

$$\vec{\boldsymbol{E}}_L(\mu) = F(\mu) \begin{pmatrix} \mathrm{e}^{-\frac{\mathrm{i}x}{2} p(\mu)} \cdot \boldsymbol{\kappa}_1(\mu) \\ -\mathrm{e}^{\frac{\mathrm{i}x}{2} p(\mu)} \cdot \boldsymbol{\kappa}_2(\mu) \end{pmatrix} \qquad \text{and} \qquad \vec{\boldsymbol{E}}_R(\mu) = \frac{-1}{2\mathrm{i}\pi} \begin{pmatrix} \mathrm{e}^{\frac{\mathrm{i}x}{2} p(\mu)} \cdot \boldsymbol{m}_1(\mu) \\ \mathrm{e}^{-\frac{\mathrm{i}x}{2} p(\mu)} \cdot \boldsymbol{m}_2(\mu) \end{pmatrix} . \tag{2.10}$$

These allow one to construct the integrable integral kernel $V_t(\lambda,\mu)$ of the integral operator V_t on $L^2([a\,;b])$ as

$$V_t(\lambda,\mu) \;=\; \frac{\big(\vec{\boldsymbol{E}}_L(\lambda),\vec{\boldsymbol{E}}_R(\mu)\big)}{\lambda-\mu} \;=\; \frac{\mathrm{i}cF(\lambda)}{2\mathrm{i}\pi(\lambda-\mu)}\cdot\Big\{\frac{\mathrm{e}^{\frac{\mathrm{i}x}{2}[p(\lambda)-p(\mu)]}}{t(\lambda-\mu)+\mathrm{i}c}+\frac{\mathrm{e}^{\frac{\mathrm{i}x}{2}[p(\mu)-p(\lambda)]}}{t(\lambda-\mu)-\mathrm{i}c}\Big\}\,. \tag{2.11}$$

Note that the one-parameter t family of integral kernels $V_t(\lambda,\mu)$ contains the kernel $V(\lambda,\mu)$ introduced in (1.3) as a special case; indeed one has $V(\lambda,\mu)=V_1(\lambda,\mu)$.
The kernel $V_t(\lambda,\mu)$ gives rise to the Riemann–Hilbert problem for a 2×2 operator-valued matrix $\chi(\lambda)=I_2\otimes\mathrm{id}+\widehat{\chi}(\lambda)$

- $\widehat{\chi}(\lambda)$ is a holomorphic in $\lambda\in\mathbb{C}\setminus[a\,;b]$ integral operator on $L^2(\mathbb{R}^+,\mathrm{d}s)\oplus L^2(\mathbb{R}^+,\mathrm{d}s)$;
- $\widehat{\chi}(\lambda)$ admits continuous $\pm$-boundary values $\widehat{\chi}_{\pm}(\lambda)$ on $]a\,;b[$;
- uniformly in $(s,s')\in\mathbb{R}^+\times\mathbb{R}^+$ and for any compact K such that $\overset{\circ}{K}\supset\{a,b\}$, there exist a constant $C>0$ such that

$$\big\|\widehat{\chi}(\lambda\mid s,s')\big\| \;\le\; \frac{C}{1+|\lambda|}\cdot\mathrm{e}^{-\frac{c}{4}(s+s')}\quad \text{on}\quad \mathbb{C}\setminus K\qquad \text{for some }\; C>0\,. \tag{2.12}$$

- there exists λ-independent vectors $\vec{\boldsymbol{N}}_{\varsigma}$, $\varsigma\in\{a,b\}$ whose entries are functions in $(L^1\cap L^\infty)(\mathbb{R}^+,\mathrm{d}s)$ and an integral operator $\widehat{\chi}^{(\varsigma)}_{\mathrm{reg}}(\lambda)$ on $L^2(\mathbb{R}^+,\mathrm{d}s)\oplus L^2(\mathbb{R}^+,\mathrm{d}s)$ such that

$$\chi(\lambda) \;=\; I_2\otimes\mathrm{id} \;+\; \ln[w(\lambda)]\cdot\vec{\boldsymbol{N}}_{\varsigma}\otimes\big(\vec{\boldsymbol{E}}_L(\varsigma)\big)^{\boldsymbol{T}} \;+\; \widehat{\chi}^{(\varsigma)}_{\mathrm{reg}}(\lambda)\qquad \text{where}\quad w(\lambda)\;=\;\frac{\lambda-b}{\lambda-a}\,. \tag{2.13}$$

  The integral kernel $\widehat{\chi}^{(\varsigma)}_{\mathrm{reg}}(\lambda\mid s,s')$ satisfies to the bound

$$\big\|\widehat{\chi}^{(\varsigma)}_{\mathrm{reg}}(\lambda\mid s,s')\big\| \;\le\; C\mathrm{e}^{-\frac{c}{4}(s+s')}(s+1)(s'+1)\qquad \text{uniformly in }\; \lambda\in U_{\varsigma}\;\text{ and }\;(s,s')\in\mathbb{R}^+\times\mathbb{R}^+ \tag{2.14}$$

  for some open neighbourhood U_{ς} of $\varsigma\in\{a,b\}$.

- the $\pm$ boundary values satisfy $\chi_+(\lambda)\cdot G_{\chi}(\lambda) \;=\; \chi_-(\lambda)$ where the jump matrix reads

$$G_{\chi}(\lambda) \;=\; \begin{pmatrix} \mathrm{id}-F(\lambda)\cdot\boldsymbol{m}_1(\lambda)\otimes\boldsymbol{\kappa}_1(\lambda) & F(\lambda)\,\mathrm{e}^{\mathrm{i}xp(\lambda)}\cdot\boldsymbol{m}_1(\lambda)\otimes\boldsymbol{\kappa}_2(\lambda) \\ -F(\lambda)\,\mathrm{e}^{-\mathrm{i}xp(\lambda)}\cdot\boldsymbol{m}_2(\lambda)\otimes\boldsymbol{\kappa}_1(\lambda) & \mathrm{id}+F(\lambda)\cdot\boldsymbol{m}_2(\lambda)\otimes\boldsymbol{\kappa}_2(\lambda) \end{pmatrix}\,. \tag{2.15}$$

Proposition 2.1 *The Riemann–Hilbert problem for χ admits, at most, a unique solutions. Furthermore, there exists $\delta>0$ and small enough such that for any t such that $|\Im(t)|<\delta$ and $\det[I+V_t]\neq 0$, this unique solution exists and takes the explicit form*

$$\chi(\lambda) \;=\; I_2\otimes\mathrm{id}-\int_a^b \frac{\vec{\boldsymbol{F}}_R(\mu)\otimes\big(\vec{\boldsymbol{E}}_L(\mu)\big)^{\boldsymbol{T}}}{\mu-\lambda}\cdot\mathrm{d}\mu \qquad \textit{and}\qquad \chi^{-1}(\lambda) \;=\; I_2\otimes\mathrm{id}+\int_a^b \frac{\vec{\boldsymbol{E}}_R(\mu)\otimes\big(\vec{\boldsymbol{F}}_L(\mu)\big)^{\boldsymbol{T}}}{\mu-\lambda}\cdot\mathrm{d}\mu \tag{2.16}$$

where $\vec{\boldsymbol{F}}_R(\lambda)$ and $\vec{\boldsymbol{F}}_L(\lambda)$ correspond to the solutions to the below linear integral equations

$$\vec{\boldsymbol{F}}_R(\lambda) \;+\; \int_a^b V_t(\mu,\lambda)\cdot\vec{\boldsymbol{F}}_R(\mu)\cdot\mathrm{d}\mu \;=\; \vec{\boldsymbol{E}}_R(\lambda)\qquad \textit{and}\qquad \vec{\boldsymbol{F}}_L(\lambda) \;+\; \int_a^b V_t(\lambda,\mu)\cdot\vec{\boldsymbol{F}}_L(\mu)\cdot\mathrm{d}\mu \;=\; \vec{\boldsymbol{E}}_L(\lambda)\,. \tag{2.17}$$

The solutions $\vec{\boldsymbol{F}}_{R/L}(\lambda)$ can be constructed in terms of χ as

$$\vec{\boldsymbol{F}}_R(\mu) \;=\; \chi(\mu)\cdot\vec{\boldsymbol{E}}_R(\mu)\qquad \textit{and}\qquad \big(\vec{\boldsymbol{F}}_L(\mu)\big)^{\boldsymbol{T}} \;=\; \big(\vec{\boldsymbol{E}}_L(\mu)\big)^{\boldsymbol{T}}\cdot\chi^{-1}(\mu)\qquad \textit{with}\quad \lambda\in]a\,;b[\,. \tag{2.18}$$

Note that the reconstruction formulae (2.18) are independent of the + or − boundary values of χ as a consequence of the specific form taken by the jump matrix for χ.

Furthermore, we do insist that solutions to the Riemann–Hilbert problem for χ do exist for larger values of $|\Im(t)|$ then those stated in the proposition above. However, for larger values of $|\Im(t)|$, they define integral operators on weighted L^2 spaces $\oplus L^2(\mathbb{R}^+, \mathrm{e}^{\alpha s}\mathrm{d}s)$ for some $\alpha > 0$ whose magnitude depends on $|\Im(t)|$. Since the conclusions of the above proposition are already enough for the purpose developed in the present paper, we chose not to venture deeper in such technicalities.

Proof —

• Uniqueness

For any $\lambda \in \mathbb{C}\setminus[a\,;b]$, the matrix-valued operator $\chi(\lambda)$ decomposes as $\chi(\lambda) = I_2 \otimes \mathrm{id} + \widehat{\chi}(\lambda)$, with an integral kernel $\widehat{\chi}(\lambda \mid s, s')$ that satisfies to (2.12). This guarantees that its Fredholm determinant $\gamma(\lambda) = \det[I_2 \otimes \mathrm{id} + \widehat{\chi}(\lambda)]$ is well defined, *cf.* [5]. Likewise, it is readily seen by applying Fubbini's and Morera's theorems that γ is holomorphic on $\mathbb{C} \setminus [a\,;b]$. By applying the dominated convergence theorem and the estimates (2.12) it is readily seen that γ admits continuous-boundary values on $]a\,;b[$, which furthermore satisfy

$$\gamma_{\pm}(\lambda) \;=\; \det\left[\chi_{\pm}(\lambda)\right] , \tag{2.19}$$

ie. one can exchange the ± boundary values with the operation of computing the determinant.

We now focus on the behaviour of γ near the endpoints a, b. Starting from (2.13) one obtains

$$\det[\chi(\lambda)] \;=\; \det\Big[I_2 \otimes \mathrm{id} \,+\, \widehat{\chi}^{(\varsigma)}_{\mathrm{reg}}(\lambda)\Big] \;+\; \ln[w(\lambda)] \cdot \Big(\vec{\boldsymbol{E}}_L(\varsigma), M(\lambda) \cdot \vec{\boldsymbol{N}}_{\varsigma}\Big) \quad \varsigma \in \{a, b\} \tag{2.20}$$

where $w(\lambda)$ is as in (2.13). The operator

$$M(\lambda) \;=\; \lim_{\eta \to 1} \left\{ \det\Big[I_2 \otimes \mathrm{id} \,+\, \eta\widehat{\chi}^{(\varsigma)}_{\mathrm{reg}}(\lambda)\Big] \cdot \Big(I_2 \otimes \mathrm{id} \,+\, \eta\widehat{\chi}^{(\varsigma)}_{\mathrm{reg}}(\lambda)\Big)^{-1} \right\} \;=\; I_2 \otimes \mathrm{id} \,+\, \widehat{M}(\lambda) \tag{2.21}$$

is well defined even if $\det\Big[I_2 \otimes \mathrm{id} \,+\, \widehat{\chi}^{(\varsigma)}_{\mathrm{reg}}(\lambda)\Big] \;=\; 0$. This can be readily seen from its series of multiple integral representation, see *eg*. [5] and the use of the bounds (2.14). The latter ensures that the function

$$\lambda \;\mapsto\; \Big(\vec{\boldsymbol{E}}_L(\varsigma), M(\lambda) \cdot \vec{\boldsymbol{N}}_{\varsigma}\Big) \tag{2.22}$$

is bounded in some open neighbourhood of $\lambda = \varsigma$, hence leading to

$$\big|\gamma(\lambda)\big| \;\le\; C \cdot \Big|\ln|\lambda - a| \cdot \ln|\lambda - b|\Big| \qquad \text{for some } \; C > 0 \text{ and } \; \text{when } \; \lambda \to \varsigma \in \{a, b\}. \tag{2.23}$$

Finally, independently of $\lambda \in [a\,;b]$, the integral operator $\widehat{G}_{\chi}(\lambda) \;=\; G_{\chi}(\lambda) - I_2 \otimes \mathrm{id}$ has a 2×2 matrix integral kernels that is smooth and such that

$$\Big\|\widehat{G}_{\chi}(\lambda \mid s, s')\Big\| \;\le\; C\mathrm{e}^{-\frac{c(s+s')}{2}} \; . \tag{2.24}$$

This ensures that the Fredholm determinant of $G_{\chi}(\lambda)$ is well defined. Then, the multiplicative property of Fredholm determinants along with

$$\det\Big[G_{\chi}(\lambda)\Big] \;=\; 1 \qquad \text{for any} \quad \lambda \in [a\,;b] \tag{2.25}$$

ensure that γ solves the scalar Riemann–Hilbert problem

- γ is holomorphic on $\mathbb{C} \setminus [a\,;b]$;
- γ admits continuous $\pm$-boundary values on $]a\,;b[$ which satisfy $\gamma_+(\lambda) \;=\; \gamma_-(\lambda)$;
- there exists a constant $C > 0$ such that when $\lambda \to \varsigma \in \{a, b\}$, γ satisfies to the bound

$$\big|\gamma(\lambda)\big| \;\leq\; C \cdot \Big|\ln|\lambda - a| \cdot \ln|\lambda - b|\Big| \;; \tag{2.26}$$

- $\gamma(\lambda) \;=\; 1 \;+\; \mathrm{O}(\lambda^{-1})$ when $\lambda \to \infty$.

The Riemann–Hilbert problem for γ is uniquely solvable, its solution being $\gamma \;=\; 1$. As a consequence, the matrix-valued operator $\chi(\lambda)$ is invertible for any $\lambda \in \mathbb{C} \setminus [a\,;b]$. Its $\pm$-boundary values $\chi_\pm(\lambda)$ are likewise invertible for any $\lambda \in]a\,;b[$. Assume that $\chi^{(1)}$ and $\chi^{(2)}$ are two solutions to the Riemann–Hilbert problem in question. The operator $G_\chi(\lambda)$ is invertible due to (2.25). Therefore, $\Phi = \chi^{(1)} \cdot \big(\chi^{(2)}\big)^{-1} = I_2 \otimes \mathrm{id} + \widehat{\Phi}$ solves a Riemann–Hilbert problem analogous to the one for χ with the sole exception that

- $\Phi_+(\lambda) = \Phi_-(\lambda)$ on $]a\,;b[$;
- $\Phi(\lambda)$ admits continuous boundary values on $[a\,;b]$;
- $\widehat{\Phi}(\lambda \mid s, s')$ has, at most, $\mathrm{O}(\ln^2|\lambda - \varsigma|)$ singularities at the endpoints $\varsigma \in \{a, b\}$ in the sense of (2.13) .

This means that, for any $(s, s') \in \mathbb{R}^+ \times \mathbb{R}^+$ and lying outside of a set of measure zero, the holomorphic matrix-valued functions $\lambda \mapsto \widehat{\Phi}(\lambda \mid s, s')$ are continous across $[a\,;b]$. Being bounded by 0 at infinity, they are identically zero by Liouville's theorem, *viz.* $\Phi(\lambda) = I_2 \otimes \mathrm{id}$ implying uniqueness.

• Existence

We chose $\delta > 0$ and assume the open neighborhood U on which F and p are analytic to be relatively compact and small enough so that

$$\Big|\mathrm{e}^{\pm \mathrm{i} s t \lambda}\Big| \;\leq\; \mathrm{e}^{\frac{c}{4}s} \quad \text{for any} \quad \lambda \in U \;\;,\; \delta \quad |\Im(t)| \;\leq\; \delta \quad \text{and} \quad s \in \mathbb{R}^+. \tag{2.27}$$

We first show that the integral operator defined by (2.16) is indeed a holomorphic in $\lambda \in \mathbb{C} \setminus [a\,;b]$ integral operator on $L^2(\mathbb{R}^+, \mathrm{d}s) \oplus L^2(\mathbb{R}^+, \mathrm{d}s)$. Let $R_t(\lambda, \mu)$ be the resolvent kernel of the inverse operator $\mathrm{id} - \mathcal{R}_t$ to $\mathrm{id} + \mathcal{V}_t$. This operator exists since $\det[\mathrm{id} + \mathcal{V}_t] \neq 0$. Then, one has the representation:

$$\vec{\boldsymbol{F}}_R(\lambda; s) \;=\; \vec{\boldsymbol{E}}_R(\lambda; s) \;-\; \int_a^b R_t(\lambda, \mu)\, \vec{\boldsymbol{E}}_R(\mu; s) \cdot \mathrm{d}\mu \;. \tag{2.28}$$

It further follows from (2.27) that,

$$\max_a \Big|\big[\vec{\boldsymbol{E}}_R(\lambda; s)\big]_a\Big| \;\leq\; C\mathrm{e}^{-\frac{sc}{4}} \;. \tag{2.29}$$

The bounds on $\vec{\boldsymbol{E}}_R(\lambda; s)$ and the regularity of the resolvent kernel $R_t(\lambda, \mu)$ ensure that

$$\Big|\vec{\boldsymbol{F}}_R(\lambda; s)\Big| \;\leq\; \mathrm{e}^{-\frac{cs}{4}} \cdot C \qquad \text{uniformly in } \lambda \in U \;\text{ and } |\Im(t)| \;\leq\; \delta \;. \tag{2.30}$$

Therefore, $\widehat{\chi}(\lambda)$ as defined through (2.16) does indeed correspond to a holomorphic in $\lambda \notin [a\,;b]$ integral operator on $L^2(\mathbb{R}^+, \mathrm{d}s) \oplus L^2(\mathbb{R}^+, \mathrm{d}s)$.

We now establish the overall bounds (2.12) uniformly away from the endpoints a and b as well as the local ones (2.13)-(2.14) in some neighbourhood thereof. Since R_t is holomorphic on $U \times U$, one obtains from (2.28) that

$$\left|\frac{\vec{\boldsymbol{F}}_R(\lambda; s) \cdot \left(\vec{\boldsymbol{E}}_L(\lambda; s')\right)^{\boldsymbol{T}} - \vec{\boldsymbol{F}}_R(\mu; s) \cdot \left(\vec{\boldsymbol{E}}_L(\mu; s')\right)^{\boldsymbol{T}}}{\lambda - \mu}\right| \leq \mathrm{e}^{-\frac{c(s+s')}{4}}(1 + s)(1 + s')C \tag{2.31}$$

uniformly in $\lambda, \mu \in U$ and $|\Im(t)| \leq \delta$. The latter informations along with the representation

$$\chi(\lambda) = \mathrm{id} \otimes I_2 - \vec{\boldsymbol{F}}_R(\lambda) \otimes \left(\vec{\boldsymbol{E}}_L(\lambda)\right)^{\boldsymbol{T}} \cdot \ln[w(\lambda)] - \int_a^b \frac{\vec{\boldsymbol{F}}_R(\lambda; s) \cdot \left(\vec{\boldsymbol{E}}_L(\lambda; s')\right)^{\boldsymbol{T}} - \vec{\boldsymbol{F}}_R(\mu; s) \cdot \left(\vec{\boldsymbol{E}}_L(\mu; s')\right)^{\boldsymbol{T}}}{\lambda - \mu} \cdot \mathrm{d}\mu \tag{2.32}$$

ensure that $\widehat{\chi}(\lambda)$ does indeed admit continuous $\pm$-boundary values on $]a\,;b[$ and that it furthermore satisfies to the local (2.13)-(2.14) and overall (2.12) bounds.

It now solely remains to prove that χ, as defined through (2.16), does indeed satisfy to the jump condition. In fact, this follows from the manipulations outlined in [14], where the operator valued Riemann–Hilbert problem description of integrable integral operators of c-shifted type has been proposed for the first time. For the readers convenience, we recall these arguments below.

It follows directly from the integral representation (2.16) that

$$\chi_+(\lambda) - \chi_-(\lambda) = -2\mathrm{i}\pi \cdot \vec{\boldsymbol{F}}_R(\lambda) \otimes \left(\vec{\boldsymbol{E}}_L(\lambda)\right)^{\boldsymbol{T}} . \tag{2.33}$$

Furthermore, by using the explicit expression for G_χ, one has that

$$\begin{aligned}\chi_+(\lambda) \cdot G_\chi(\lambda) &= \chi_+(\lambda) + 2\mathrm{i}\pi \cdot \vec{\boldsymbol{E}}_R(\lambda) \otimes \left(\vec{\boldsymbol{E}}_L(\lambda)\right)^{\boldsymbol{T}} - 2\mathrm{i}\pi \int_a^b \vec{\boldsymbol{F}}_R(\mu) \cdot V_t(\mu, \lambda) \otimes \left(\vec{\boldsymbol{E}}_L(\lambda)\right)^{\boldsymbol{T}} \cdot \mathrm{d}\mu \\ &= \chi_+(\lambda) + 2\mathrm{i}\pi \cdot \vec{\boldsymbol{F}}_R(\lambda) \otimes \left(\vec{\boldsymbol{E}}_L(\lambda)\right)^{\boldsymbol{T}} .\end{aligned} \tag{2.34}$$

where, in the last equality, we have used the integral equation satisfied by $\vec{\boldsymbol{F}}_R(\lambda)$. By using the above two relations, one indeed obtains that χ satisfies to the jump conditions. Finally, it follows from the first equality in (2.34) that

$$\chi_+(\lambda) \cdot \vec{\boldsymbol{E}}_R(\lambda) \otimes \left(\vec{\boldsymbol{E}}_L(\lambda)\right)^{\boldsymbol{T}} = \vec{\boldsymbol{F}}_R(\lambda) \otimes \left(\vec{\boldsymbol{E}}_L(\lambda)\right)^{\boldsymbol{T}} . \tag{2.35}$$

Acting with both sides of this equality on a vector function $\vec{G}$ such that $(\vec{\boldsymbol{E}}_L(\lambda), \vec{G}) \neq 0$ for $\lambda \in [a\,;b]$, we obtain (2.18). The proofs of similar statements relative to χ^{-1} are left to the reader. ∎

We remind that it is a classical fact [7] that the resolvent operator $\mathcal{R}_t$ to $\mathcal{V}_t$ belongs to the class of integrable integral operator and that its integral kernel $R_t(\lambda, \mu)$ reads

$$R_t(\lambda, \mu) = \frac{\left(\vec{\boldsymbol{F}}_L(\lambda), \vec{\boldsymbol{F}}_R(\mu)\right)}{\lambda - \mu} . \tag{2.36}$$

3 Towards the implementation of the non-linear steepest descent method

3.1 Auxiliary operator-valued scalar Riemann–Hilbert problems

Let

$$\tau_1(\lambda) \;=\; -\frac{F(\lambda)}{1+F(\lambda)} \qquad \text{and} \qquad \tau_2(\lambda) \;=\; F(\lambda)\,. \tag{3.1}$$

In the present section, we investigate the solution of two operator-valued scalar Riemann–Hilbert problems that will become useful in our future handlings. Before stating the Riemann–Hilbert problems of interst, we however need to introduce a function that will become handy:

$$\nu(\lambda) \;=\; \frac{-1}{2\mathrm{i}\pi}\cdot \ln\left[1 \,+\, F(\lambda)\right] \qquad \text{and} \qquad \alpha(\lambda) \;=\; \exp\Big\{\int\limits_a^b \frac{\nu(\mu)}{\mu-\lambda}\cdot \mathrm{d}\mu\Big\} \tag{3.2}$$

The Riemann–Hilbert problem for $\beta_k = \mathrm{id} + \widehat{\beta}_k$ with $k = 1, 2$ reads:

- $\widehat{\beta}_k(\lambda)$ is a holomorphic in $\lambda \in \mathbb{C}\setminus[a\,;b]$ integral operator on $L^2(\mathbb{R}^+, \mathrm{d}s)$;
- $\widehat{\beta}_k(\lambda)$ admits continuous $\pm$-boundary values $\widehat{\beta}_{k;\pm}$ on $]a\,;b[$;
- uniformly in $(s,s')\in\mathbb{R}^+\times\mathbb{R}^+$ and for any compact K such that $\mathrm{Int}(K)\supset\{a,b\}$, there exist a constant $C>0$ such that
$$\big|\widehat{\beta}_k(\lambda \mid s,s')\big| \;\le\; \frac{C}{1+|\lambda|}\cdot \mathrm{e}^{-\frac{c}{4}(s+s')} \quad \text{for} \quad \mathbb{C}\setminus K\,. \tag{3.3}$$
- There exists a function $\boldsymbol{n}_{k;\varsigma}\in(L^1\cap L^\infty)(\mathbb{R}^+,\mathrm{d}s)$ and a neighbourhood U_ς of $\varsigma\in\{a,b\}$ such that for λ in
$$\widehat{\beta}_k(\lambda) \;=\; [w(\lambda)]^{-\epsilon_k \nu(\varsigma)}\cdot \boldsymbol{n}_{k;\varsigma}\otimes\boldsymbol{\kappa}_k(\varsigma) \;+\; \widehat{\beta}^{(\varsigma)}_{k;\mathrm{reg}}(\lambda) \tag{3.4}$$
where $w(\lambda)$ is as given in (2.13) while, for any $\lambda\in U_\varsigma$,
$$\big|\widehat{\beta}^{(\varsigma)}_{k;\mathrm{reg}}(\lambda \mid s,s')\big| \;\le\; C\mathrm{e}^{-\frac{c}{4}(s+s')}(s+1)(s'+1) \quad \text{for some} \quad C>0\,. \tag{3.5}$$
- the boundary values satisfy $\beta_{k;+}(\lambda)\cdot\Big(\mathrm{id} \,+\, \tau_k(\lambda)\cdot\boldsymbol{m}_k(\lambda)\otimes\boldsymbol{\kappa}_k(\lambda)\Big) \;=\; \beta_{k;-}(\lambda)$.

Proposition 3.1 *There exists $\delta>0$ small enough such that the Riemann–Hilbert problem for β_k admits a unique solution provided that $1+\tau_k(\lambda)\neq 0$ on $[a\,;b]$ and $|\Im(t)|<\delta$. Furthermore, the solution exists as soon as*

$$|\Im(t)|<\delta \quad \text{and} \quad \det_{\Gamma([a\,;b])}\left[\mathrm{id} \,+\, \mathcal{U}_{k;t}\right] \;\neq\; 0 \tag{3.6}$$

where the integral kernel $U_{k;t}(\lambda,\mu)$ of the integral operator $\mathcal{U}_{k;t}$ acting on $L^2(\Gamma([a\,;b]))$ reads

$$U_{k;t}(\lambda,\mu) \;=\; -t\frac{\alpha_k(\lambda)\cdot\alpha_k^{-1}(\mu+\mathrm{i}\epsilon_k c/t)}{2\mathrm{i}\pi\cdot[t(\mu-\lambda)+\mathrm{i}\epsilon_k c]} \qquad \text{with} \quad \epsilon_1=-1 \;\text{ and }\; \epsilon_2 = 1\,, \tag{3.7}$$

in which

$$\alpha_k(\lambda) \;=\; \exp\Big\{\int\limits_a^b \frac{\nu_k(\mu)}{\mu-\lambda}\cdot\mathrm{d}\mu\Big\} \qquad \text{with} \qquad \nu_k(\mu) \;=\; \frac{-1}{2\mathrm{i}\pi}\ln\left[1+\tau_k(\mu)\right]\,. \tag{3.8}$$

Note that one has

$$\nu_k(\lambda) \;=\; \epsilon_k\nu(\lambda) \qquad \text{and} \qquad \alpha_k(\lambda) \;=\; [\alpha(\lambda)]^{\epsilon_k}\,. \tag{3.9}$$

Proof —

• **Uniqueness**

For any $\lambda \in \mathbb{C} \setminus [a\,; b]$, in virtue of (3.3) the Fredholm determinant $\alpha_k(\lambda) = \det\left[\mathrm{id} + \widehat{\beta}_k(\lambda)\right]$ is well defined. It follows from the reasoning outlined previously that α_k is holomorphic on $\mathbb{C} \setminus [a\,; b]$ and that it admits continuous $\pm$-boundary values on $]a\,; b[$, which furthermore satisfy

$$\alpha_{k;\pm}(\lambda) \;=\; \det\left[\beta_{k;\pm}(\lambda)\right] \,. \tag{3.10}$$

Since the integral kernel of $\tau_k(\lambda) \cdot \boldsymbol{m}_k(\lambda) \otimes \boldsymbol{\kappa}_k(\lambda)$ is bounded by $C\mathrm{e}^{-\frac{c}{4}(s+s')}$, this independently of $\lambda \in [a\,; b]$, the multiplicative property of Fredholm determinants and the local structure of $\widehat{\beta}_k$ in some neighbourhood of the endpoints a, b ensure that α_k solves the scalar Riemann–Hilbert problem

- α_k is holomorphic on $\mathbb{C} \setminus [a\,; b]$;
- α_k admits continuous $\pm$ boundary values $\alpha_{k;\pm}$ on $]a\,; b[$ which satisfy $\alpha_{k;+}(\lambda) \cdot \Big(1 + \tau_k(\lambda)\Big) \;=\; \alpha_{k;-}(\lambda)$;
- $\alpha_k(\lambda) = \mathrm{O}\Big(|w(\lambda)^{-\epsilon_k \nu(\varsigma)}|\Big)$ when $\lambda \to \varsigma \in \{a, b\}$;
- $\alpha_k \;=\; 1 \,+\, \mathrm{O}(\lambda^{-1})$ when $\lambda \to \infty$.

The hypothesis of the theorem ensure the unique solvability of this scalar problem, with its solution being given by (3.8). In particular, $\alpha_k(\lambda) \neq 0$ for $\lambda \in \mathbb{C} \setminus [a\,; b]$, just as $\alpha_{k;\pm}(\lambda) \neq 0$ for $\lambda \in]a\,; b[$. As a consequence, the operator $\beta_k(\lambda)$ is invertible for any $\lambda \in \mathbb{C} \setminus [a\,; b]$. Furthermore, its $\pm$-boundary values $\beta_{k;\pm}(\lambda)$ are also invertible for any $\lambda \in]a\,; b[$. Assume that $\beta_k^{(1)}$ and $\beta_k^{(2)}$ are two solutions to the Riemann–Hilbert problem in question. Observe that, due to

$$\det\Big[\mathrm{id} \,+\, \tau_k(\lambda) \cdot \boldsymbol{m}_k(\lambda) \otimes \boldsymbol{\kappa}_k(\lambda)\Big] \;=\; 1 + \tau_k(\lambda) \neq 0 \,, \tag{3.11}$$

the operator $\mathrm{id} \,+\, \tau_k(\lambda) \cdot \boldsymbol{m}_k(\lambda) \otimes \boldsymbol{\kappa}_k(\lambda)$ is invertible. As a consequence, $\gamma_k \;=\; \beta_k^{(1)} \cdot (\beta_k^{(2)})^{-1}$ solves a Riemann–Hilbert problem analogous to the one for β_k with the sole exception that now $\gamma_{k;+}(\lambda) \;=\; \gamma_{k;-}(\lambda)$ on $]a\,; b[$ and that $\widehat{\gamma}_k(\lambda, \mid s, s')$ exhibits at most $\mathrm{O}\Big(|w(\lambda)^{-2\epsilon_k \nu(\varsigma)}|\Big)$ singularities in λ when $\lambda \to \varsigma \in \{a, b\}$. This means that, for any $(s, s') \in \mathbb{R}^+ \times \mathbb{R}^+$, the holomorphic function $\lambda \mapsto \widehat{\gamma}(\lambda \mid s, s')$ is continuous across $]a\,; b[$ and has removable singularities at the endpoints. This function is thus entire and, being bounded by 0 at infinity, it is identically zero by Liouville's theorem, *viz.* $\gamma_k(\lambda) = \mathrm{id}$.

• **Existence**

Due to its unique solvability, the solution to the Riemann–Hilbert problem for β_k, if it exists, is in one-to-one correspondence with the solution to the singular integral equation [1]

$$\beta_{k;+}(\lambda) \;=\; \mathrm{id} \;-\; C_+[\beta_{k;+} \cdot \tau_k \cdot \boldsymbol{m}_k \otimes \boldsymbol{\kappa}_k](\lambda) \quad \text{where} \quad C[f](\lambda) \;=\; \int_a^b \frac{f(\mu)}{\mu - \lambda} \cdot \frac{\mathrm{d}\mu}{2\mathrm{i}\pi} \tag{3.12}$$

and $C_+[f](\lambda)$ stands for the + boundary value of $C[f](\lambda)$ on $]a\,; b[$. More precisely, the solution β_k can be represented as

$$\beta_k(\lambda) \;=\; \mathrm{id} \;-\; C[\beta_{k;+} \cdot \tau_k \cdot \boldsymbol{m}_k \otimes \boldsymbol{\kappa}_k](\lambda) \,. \tag{3.13}$$

We transform the singular integral equation for $\beta_{k;+}$ into one for the function

$$\rho_k(\lambda; s) \;=\; \big(\beta_{k;+}(\lambda) \cdot \boldsymbol{m}_k(\lambda)\big)(s)\;. \tag{3.14}$$

We obtain

$$\rho_k(\lambda; s) \;=\; h_k(\lambda; s) - C_+[\tau_k(*)\rho_k(*; s)](\lambda) \quad \text{where} \quad h_k(\lambda; s) \;=\; \sqrt{c}\mathrm{e}^{-\frac{cs}{2}-\epsilon_k \mathrm{i}ts\lambda} + \int_a^b \frac{t\tau_k(\mu)\rho_k(\mu; s)}{t(\mu-\lambda)+\mathrm{i}\epsilon_k c} \cdot \frac{\mathrm{d}\mu}{2\mathrm{i}\pi}\;. \tag{3.15}$$

Above the $*$ indicates the running variable of the function on which the Cauchy transform acts and we remind that $\epsilon_1 = -1$ while $\epsilon_2 = 1$. Equation (3.15) can be recast as a non-homogeneous Riemann–Hilbert problem for the function

$$\aleph(\lambda; s) \;=\; \int_a^b \frac{\tau_k(\mu)\rho_k(\mu; s)}{\mu-\lambda} \cdot \frac{\mathrm{d}\mu}{2\mathrm{i}\pi}\;. \tag{3.16}$$

Indeed, $\lambda \mapsto \aleph(\lambda \mid s)$ is holomorphic on $\mathbb{C} \setminus [a\,;b]$, decays as $\mathrm{O}(\lambda^{-1})$ and satisfies to the non-homogeneous jump conditions

$$\aleph_+(\lambda; s)\cdot(1+\tau_k(\lambda)) \;-\; \aleph_-(\lambda; s) \;=\; \tau_k(\lambda)\cdot h_k(\lambda; s)\;. \tag{3.17}$$

This non-homogeneous Riemann-Hilbert problem is readily solved by standard techniques [16] leading to

$$\aleph(\lambda; s) \;=\; \alpha_k(\lambda)\cdot \int_a^b \frac{\alpha_{k;-}^{-1}(\mu)}{\mu-\lambda} \cdot \tau_k(\mu) h_k(\mu; s) \cdot \frac{\mathrm{d}\mu}{2\mathrm{i}\pi}\;. \tag{3.18}$$

We do stress that the functions α_k are well defined as a consequence of our hypothesis on F. Making most of the expression for h_k, one gets

$$\aleph(\lambda; s) \;=\; h_k(\lambda; s) \;-\; \alpha_k(\lambda) \oint_{\Gamma([a\,;b])} \frac{\sqrt{c}\cdot \mathrm{e}^{-\frac{cs}{2}-\mathrm{i}\epsilon_k ts\mu}}{\alpha_k(\mu)\cdot(\mu-\lambda)} \cdot \frac{\mathrm{d}\mu}{2\mathrm{i}\pi} \;-\; \alpha_k(\lambda) \int_a^b \frac{t\cdot \alpha_k^{-1}(\mu+\mathrm{i}\epsilon_k c/t)}{t(\mu-\lambda)+\mathrm{i}\epsilon_k c} \cdot \tau_k(\mu)\rho_k(\mu; s) \cdot \frac{\mathrm{d}\mu}{2\mathrm{i}\pi} \tag{3.19}$$

for λ belonging to a small vicinity of $[a\,;b]$. We remind that, in (3.19), $\Gamma([a\,;b])$ stands for a small counterclockwise loop around the segment $[a\,;b]$ and the point λ. As a consequence, ρ_k solves the linear integral equation

$$\Big(\mathrm{id} \;+\; \mathcal{K}_{k;t}\Big)[\rho_k(*; s)](\lambda) \;=\; \alpha_{k;+}(\lambda) \oint_{\Gamma\big([a\,;b]\big)} \frac{\sqrt{c}\cdot \mathrm{e}^{-\frac{cs}{2}-\mathrm{i}\epsilon_k ts\mu}}{\alpha_k(\mu)\cdot(\mu-\lambda)} \cdot \frac{\mathrm{d}\mu}{2\mathrm{i}\pi} \tag{3.20}$$

where the integral kernel $K_{k;t}(\lambda,\mu)$ of the integral operator $\mathcal{K}_{k;t}$ on $L^2([a\,;b])$ reads

$$K_{k;t}(\lambda,\mu) \;=\; -t\frac{\alpha_{k;+}(\lambda)\cdot \alpha_k^{-1}(\mu+\mathrm{i}\epsilon_k c/t)}{2\mathrm{i}\pi\cdot(t(\mu-\lambda)+\mathrm{i}\epsilon_k c)} \cdot \tau_k(\mu)\;. \tag{3.21}$$

In fact, using the jump condition satisfied by α_k in the form $\alpha_{k;-} - \alpha_{k;+} \;=\; \alpha_{k;+}\tau_k$, one can recast the kernel as

$$K_{k;t}(\lambda,\mu) \;=\; -t\Big(\alpha_{k;-}(\mu) - \alpha_{k;+}(\mu)\Big) \cdot \frac{\alpha_{k;+}(\lambda)}{\alpha_{k;+}(\mu)} \cdot \frac{\alpha_k^{-1}(\mu+\mathrm{i}\epsilon_k c/t)}{2\mathrm{i}\pi\cdot(t(\mu-\lambda)+\mathrm{i}\epsilon_k c)}\;. \tag{3.22}$$

As a consequence, one gets that

$$\det_{[a\,;b]}\big[\mathrm{id} + \mathcal{K}_{k;t}\big] \;=\; \det_{\Gamma([a\,;b])}\big[\mathrm{id} + \mathcal{U}_{k;t}\big] \tag{3.23}$$

where the integral kernel $U_{k;t}(\lambda,\mu)$ of the integral operator $\mathcal{U}_{k;t}$ acting on $L^2(\Gamma([a\,;b]))$ is as defined in (3.7). The operator $\mathrm{id}+\mathcal{K}_{k,t}$ is thus invertible. Let $\boldsymbol{\rho}_k(\lambda)$ denote the function $\big(\boldsymbol{\rho}_k(\lambda)\big)(s) \;=\; \rho_k(\lambda;s)$ where $\rho_k(\lambda;s)$ is as defined by (3.20). As a consequence,

$$\beta_k(\lambda) \;=\; \mathrm{id} \;-\; \int\limits_a^b \frac{\tau_k(\mu)\,\boldsymbol{\rho}_k(\mu)\otimes\boldsymbol{\kappa}_k(\mu)}{\mu-\lambda}\cdot\frac{\mathrm{d}\mu}{2\mathrm{i}\pi} \tag{3.24}$$

is the good candidate for the unique solution to the Riemann–Hilbert problem for β_k. It is readily checked by repeating the arguments invoked in the proof of the existence of solutions to the Riemann–Hilbert problem for χ, that β_k as defined above does satisfy all the requirements stated in the Riemann–Hilbert problem for β_k. ■

3.2 A regularity lemma

In the analysis that will follow, there will arise the one-parameter λ integral operator on $L^2(\mathbb{R}^+,\mathrm{d}s)\oplus L^2(\mathbb{R}^+,\mathrm{d}s)$ defined as

$$\boldsymbol{O}(\lambda) \;=\; \begin{pmatrix} \beta_1(\lambda)\cdot\boldsymbol{m}_1(\lambda)\otimes\boldsymbol{\kappa}_1(\lambda)\cdot\beta_1^{-1}(\lambda) & \alpha^2(\lambda)\beta_1(\lambda)\cdot\boldsymbol{m}_1(\lambda)\otimes\boldsymbol{\kappa}_2(\lambda)\cdot\beta_2^{-1}(\lambda) \\ \alpha^{-2}(\lambda)\beta_2(\lambda)\cdot\boldsymbol{m}_2(\lambda)\otimes\boldsymbol{\kappa}_1(\lambda)\cdot\beta_1^{-1}(\lambda) & \beta_2(\lambda)\cdot\boldsymbol{m}_2(\lambda)\otimes\boldsymbol{\kappa}_2(\lambda)\cdot\beta_2^{-1}(\lambda) \end{pmatrix}. \tag{3.25}$$

The main point is that even though the individual operators appearing in its matrix elements have cuts, the operator, as a whole, is regular. More precisely, one has the

Lemma 3.1 *There exists an open neighbourhood V of the segment $[a\,;b]$ such that the integral operator $\boldsymbol{O}(\lambda)$ on $L^2(\mathbb{R}^+,\mathrm{d}s)\oplus L^2(\mathbb{R}^+,\mathrm{d}s)$ defined in* (3.25) *is holomorphic on V.*

Proof —
By composition of holomorphic operators, $\boldsymbol{O}$ is holomorphic in $V\setminus[a\,;b]$, with V a sufficiently small open neighbourhood of $[a\,;b]$. We thus need to show that it is continuous across $[a\,;b]$ and that it has removable singularities at a,b. For this purpose, observe that

$$\Big(\mathrm{id} \;+\; \tau_k(\lambda)\cdot\boldsymbol{m}_k(\lambda)\otimes\boldsymbol{\kappa}_k(\lambda)\Big)\cdot\Big(\mathrm{id} \;-\; \frac{\tau_k(\lambda)}{1+\tau_k(\lambda)}\cdot\boldsymbol{m}_k(\lambda)\otimes\boldsymbol{\kappa}_k(\lambda)\Big) \;=\; \mathrm{id}\,. \tag{3.26}$$

Hence, since $\beta_{k;\pm}(\lambda)$ are invertible for all $\lambda\in]a\,;b[$, one has

$$\beta_{k;-}^{-1}(\lambda) \;=\; \Big(\mathrm{id} \;-\; \frac{\tau_k(\lambda)}{1+\tau_k(\lambda)}\boldsymbol{m}_k(\lambda)\otimes\boldsymbol{\kappa}_k(\lambda)\Big)\cdot\beta_{k;+}^{-1}(\lambda)\,. \tag{3.27}$$

As a consequence, one obtains the jump conditions

$$\beta_{k;+}(\lambda)\cdot\boldsymbol{m}_k(\lambda) \;=\; \beta_{k;-}(\lambda)\cdot\boldsymbol{m}_k(\lambda)\cdot\frac{1}{1+\tau_k(\lambda)} \qquad\text{and}\qquad \boldsymbol{\kappa}_k(\lambda)\cdot\beta_{k;+}^{-1}(\lambda) \;=\; \boldsymbol{\kappa}_k(\lambda)\cdot\beta_{k;-}^{-1}(\lambda)\cdot(1+\tau_k(\lambda)) \tag{3.28}$$

These are enough so as to conclude that $\boldsymbol{O}(\lambda\mid s,s')$ is continuous across $]a\,;b[$. It also has removable singularities at the endpoints as readily inferred from the local behaviour of α and of the operators β_k around a or b. It thus extends to a holomorphic function in some open neighbourhood of $[a\,;b]$. ■

3.3 Asymptotic resolution of the Riemann–Hilbert problem for χ

Observe that one has the factorisation

$$G_{\chi}(\lambda) = \begin{pmatrix} \mathrm{id} & \dfrac{F(\lambda)\mathrm{e}^{\mathrm{i}xp(\lambda)}}{1+F(\lambda)} \boldsymbol{m}_1(\lambda)\otimes\boldsymbol{\kappa}_2(\lambda) \\ 0 & \mathrm{id} \end{pmatrix} \cdot \begin{pmatrix} \mathrm{id} - \dfrac{F(\lambda)}{1+F(\lambda)}\cdot \boldsymbol{m}_1(\lambda)\otimes\boldsymbol{\kappa}_1(\lambda) & 0 \\ 0 & \mathrm{id} + F(\lambda)\boldsymbol{m}_2(\lambda)\otimes\boldsymbol{\kappa}_2(\lambda) \end{pmatrix}$$
$$\times \begin{pmatrix} \mathrm{id} & 0 \\ -\dfrac{F(\lambda)\mathrm{e}^{-\mathrm{i}xp(\lambda)}}{1+F(\lambda)} \boldsymbol{m}_2(\lambda)\otimes\boldsymbol{\kappa}_1(\lambda) & \mathrm{id} \end{pmatrix} . \qquad (3.29)$$

One can factor the diagonal operator valued matrix appearing in the centre by using the solutions of the operator valued scalar Riemann–Hilbert problems considered in Section 3.1. This allows one to factorise the jump matrix G_{χ} as

$$G_{\chi}(\lambda) = \begin{pmatrix} \beta_{1;+}^{-1}(\lambda) & 0 \\ 0 & \beta_{2;+}^{-1}(\lambda) \end{pmatrix} \cdot M_{\uparrow;+}(\lambda)\cdot M_{\downarrow;-}(\lambda)\cdot \begin{pmatrix} \beta_{1;-}(\lambda) & 0 \\ 0 & \beta_{2;-}(\lambda) \end{pmatrix} \qquad (3.30)$$

where the matrices $M_{\uparrow/\downarrow}$ read

$$M_{\uparrow}(\lambda) = \begin{pmatrix} \mathrm{id} & \boldsymbol{P}(\lambda)\mathrm{e}^{\mathrm{i}xp(\lambda)} \\ 0 & \mathrm{id} \end{pmatrix} \qquad \text{and} \qquad M_{\downarrow}(\lambda) = \begin{pmatrix} \mathrm{id} & 0 \\ \boldsymbol{Q}(\lambda)\mathrm{e}^{-\mathrm{i}xp(\lambda)} & \mathrm{id} \end{pmatrix} \qquad (3.31)$$

in which

$$\boldsymbol{P}(\lambda) = \frac{F(\lambda)}{1+F(\lambda)}\beta_1(\lambda)\cdot\boldsymbol{m}_1(\lambda)\otimes\boldsymbol{\kappa}_2(\lambda)\cdot\beta_2^{-1}(\lambda) \qquad \text{and} \qquad \boldsymbol{Q}(\lambda) = -\frac{F(\lambda)}{1+F(\lambda)}\beta_2(\lambda)\cdot\boldsymbol{m}_2(\lambda)\otimes\boldsymbol{\kappa}_1(\lambda)\cdot\beta_1^{-1}(\lambda) . \qquad (3.32)$$

Note that the operators $\boldsymbol{P}$ and $\boldsymbol{Q}$ can be recast as

$$\boldsymbol{P}(\lambda) = -2\mathrm{i}\mathrm{e}^{\mathrm{i}\pi\nu(\lambda)}\frac{\sin[\pi\nu(\lambda)]}{\alpha^2(\lambda)}\cdot\boldsymbol{O}_{12}(\lambda) \qquad \text{and} \qquad \boldsymbol{Q}(\lambda) = 2\mathrm{i}\mathrm{e}^{\mathrm{i}\pi\nu(\lambda)}\sin[\pi\nu(\lambda)]\alpha^2(\lambda)\cdot\boldsymbol{O}_{21}(\lambda) \qquad (3.33)$$

where $\boldsymbol{O}(\lambda)$ is as defined by (3.25).

Thus, agreeing to denote

$$\Xi(\lambda) = \chi(\lambda)\cdot\begin{pmatrix} \beta_1^{-1}(\lambda) & 0 \\ 0 & \beta_2^{-1}(\lambda) \end{pmatrix} \qquad (3.34)$$

and then defining the matrix Υ and the contour Σ_{Υ} according to Fig. 1 one gets, upon repeating the steps already explained previously, that $\Upsilon(\lambda) = I_2\otimes\mathrm{id} + \widehat{\Upsilon}(\lambda)$ solves the Riemann–Hilbert problem

- $\widehat{\Upsilon}(\lambda)$ is a holomorphic in $\lambda\in\mathbb{C}\setminus\Sigma_{\Upsilon}$ integral operator on $L^2(\mathbb{R}^+,\mathrm{d}s)\oplus L^2(\mathbb{R}^+,\mathrm{d}s)$;
- $\widehat{\Upsilon}(\lambda)$ admits continuous $\pm$-boundary values $\widehat{\Upsilon}_{\pm}(\lambda)$ on $\Sigma_{\Upsilon}\setminus\{a,b\}$;
- uniformly in $(s,s')\in\mathbb{R}^+\times\mathbb{R}^+$ and for any compact K such that $\mathrm{Int}(K)\supset\{a,b\}$, there exist a constant $C>0$ such that

$$\left|\widehat{\Upsilon}(\lambda\mid s,s')\right| \le \frac{C}{1+|\lambda|}\cdot\mathrm{e}^{-\frac{c}{4}(s+s')} \quad \text{for} \quad \mathbb{C}\setminus K . \qquad (3.35)$$

- there exists an open neighbourhood U_ς of $\varsigma \in \{a, b\}$, vector valued functions $\vec{\boldsymbol{N}}_\varsigma$ as well as functions $\widetilde{\boldsymbol{n}}_{k;\varsigma}$, $k = 1, 2$, all belonging to $(L^1 \cap L^\infty)(\mathbb{R}^+, \mathrm{d}s)$ such that, for $\lambda \in U_\varsigma \cap H_{III}$ one has $\Upsilon(\lambda) = \Upsilon_{H_{III}}(\lambda)$ where

$$\Upsilon_{H_{III}}(\lambda) = \Big(I_2 \otimes \mathrm{id} + \ln[w(\lambda)] \cdot \vec{\boldsymbol{N}}_\varsigma \otimes (\vec{\boldsymbol{E}}_L(\varsigma))^{\boldsymbol{T}} + \widehat{R}^{(\varsigma)}_{\Upsilon}(\lambda) \Big) \\ \times \begin{pmatrix} \mathrm{id} + [w(\lambda)]^{\nu_1(\lambda)} \widetilde{\boldsymbol{n}}_{1;\varsigma} \otimes \boldsymbol{\kappa}_1(\varsigma) + r^{(\varsigma)}_{1;\Upsilon}(\lambda) & 0 \\ 0 & \mathrm{id} + [w(\lambda)]^{\nu_2(\lambda)} \widetilde{\boldsymbol{n}}_{2;\varsigma} \otimes \boldsymbol{\kappa}_1(\varsigma) + r^{(\varsigma)}_{2;\Upsilon}(\lambda) \end{pmatrix},$$

$w(\lambda)$ is as defined in (2.13), and $\widehat{R}^{(\varsigma)}_{\Upsilon}(\lambda)$, resp. $r^{(\varsigma)}_{k;\Upsilon}$, is an integral operator on $L^2(\mathbb{R}^+, \mathrm{d}s) \oplus L^2(\mathbb{R}^+, \mathrm{d}s)$, resp. $L^2(\mathbb{R}^+, \mathrm{d}s)$, such that for any $\lambda \in U_\varsigma$

$$\left\| \widehat{R}^{(\varsigma)}_{\Upsilon}(\lambda \mid s, s') \right\| \leq C \mathrm{e}^{-\frac{\mathfrak{c}}{4}(s+s')}(s+1)(s'+1) \qquad \text{resp.} \qquad \left| r^{(\varsigma)}_{k;\Upsilon}(\lambda \mid s, s') \right| \leq C \mathrm{e}^{-\frac{\mathfrak{c}}{4}(s+s')}(s+1)(s'+1) \tag{3.36}$$

for some constant $C > 0$. Furthermore, one has that

$$\Upsilon(\lambda) = \Upsilon_{H_{III}}(\lambda) \cdot \begin{pmatrix} \mathrm{id} & [w(\lambda)]^{-2\nu(\lambda)} \boldsymbol{P}_{\mathrm{reg}}(\lambda) \\ 0 & \mathrm{id} \end{pmatrix} \quad \text{where} \quad \lambda \to \varsigma \in \{a, b\} \quad \text{with} \quad \lambda \in U_\varsigma \cap H_I$$
$$\Upsilon(\lambda) = \Upsilon_{H_{III}}(\lambda) \cdot \begin{pmatrix} \mathrm{id} & 0 \\ [w(\lambda)]^{2\nu(\lambda)} \boldsymbol{Q}_{\mathrm{reg}}(\lambda) & \mathrm{id} \end{pmatrix} \quad \text{where} \quad \lambda \to \varsigma \in \{a, b\} \quad \text{with} \quad \lambda \in U_\varsigma \cap \in H_{II}$$

where $\boldsymbol{P}_{\mathrm{reg}}(\lambda)$ and $\boldsymbol{Q}_{\mathrm{reg}}(\lambda)$ are integral operators on $L^2(\mathbb{R}^+, \mathrm{d}s)$ such that,

$$\left| \boldsymbol{P}_{\mathrm{reg}}(\lambda \mid s, s') \right| \leq C \mathrm{e}^{-\frac{\mathfrak{c}}{4}(s+s')}(s+1)(s'+1) \qquad \text{and} \qquad \left| \boldsymbol{Q}_{\mathrm{reg}}(\lambda \mid s, s') \right| \leq C \mathrm{e}^{-\frac{\mathfrak{c}}{4}(s+s')}(s+1)(s'+1) \tag{3.37}$$

for some constant $C > 0$ and any $\lambda \in U_\varsigma$.

- the boundary values satisfy $\Upsilon_+(\lambda) G_\Upsilon(\lambda) = \Upsilon_-(\lambda)$ where the jump matrix reads

$$G_\Upsilon(\lambda) = M_\uparrow(\lambda) \quad \text{for} \quad \lambda \in \Gamma_\uparrow \qquad \text{and} \qquad G_\Upsilon(\lambda) = M_\downarrow^{-1}(\lambda) \quad \text{for} \quad \lambda \in \Gamma_\downarrow . \tag{3.38}$$

Again, this Riemann–Hilbert problem is uniquely solvable and hence, its solution is in one-to-one correspondence with the one to the Riemann–Hilbert problem for χ. The fact that the operators $\boldsymbol{P}_{\mathrm{reg}}(\lambda)$ and $\boldsymbol{Q}_{\mathrm{reg}}(\lambda)$ satisfy (3.37) follows from (3.33), Lemma 3.1 as well as from the local behaviour of α around $\lambda = \varsigma \in \{a, b\}$. Finally, the local behaviour of Υ around $\varsigma \in \{a, b\}$ is inferred from the one of χ, *cf.* Fig. 1.

4 The parametrices

4.1 Parametrix around a

The local parametrix $\mathcal{P}_a = \mathrm{id} + \widehat{\mathcal{P}_a}$ on a small disk $\mathcal{D}_{a,\delta} \subset U$ of radius δ and centred at a, is an exact solution of the RHP:

- $\widehat{\mathcal{P}_a}(\lambda)$ is a holomorphic in $\lambda \in \mathcal{D}_{a,\delta} \setminus \{\Gamma_\uparrow \cup \Gamma_\downarrow\}$ integral operator on $L^2(\mathbb{R}^+, \mathrm{d}s) \oplus L^2(\mathbb{R}^+, \mathrm{d}s)$
- $\widehat{\mathcal{P}_a}(\lambda)$ admits continuous $\pm$-boundary values $(\widehat{\mathcal{P}_a})_\pm(\lambda)$ on $\{\Gamma_\uparrow \cup \Gamma_\downarrow \setminus \{a\}\} \cap \mathcal{D}_{a,\delta}$;
- $\widehat{\mathcal{P}_a}(\lambda)$ has the same singular structure as Υ around $\lambda = a$;

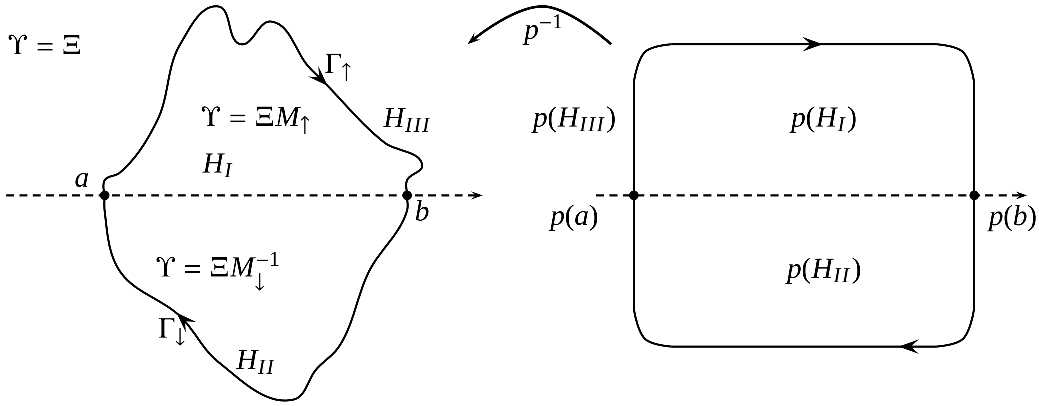


Figure 1: Contours $\Gamma_\uparrow$ and $\Gamma_\downarrow$ associated with the RHP for Υ. The second figure depicts how p maps the contours $\Gamma_\downarrow$ and $\Gamma_\uparrow$.

- uniformly in $(s, s') \in \mathbb{R}^+ \times \mathbb{R}^+$ and $\lambda \in \partial\mathcal{D}_{a,\delta}$, one has

$$\left\|\widehat{\mathcal{P}}_a(\lambda \mid s, s')\right\| \;\leq\; \frac{C}{x^{1-\varepsilon}} \cdot \mathrm{e}^{-\frac{c}{4}(s+s')} \quad \text{for some} \quad C > 0\,; \tag{4.1}$$

- $\begin{cases} \mathcal{P}_{a;+}(\lambda) \cdot M_\uparrow(\lambda) \;=\; \mathcal{P}_{a;-}(\lambda) & \text{for } \lambda \in \Gamma_\uparrow \cap \mathcal{D}_{a,\delta}, \\ \mathcal{P}_{a;+}(\lambda) \cdot M_\downarrow^{-1}(\lambda) = \mathcal{P}_{a;-}(\lambda) & \text{for } \lambda \in \Gamma_\downarrow \cap \mathcal{D}_{a,\delta}. \end{cases}$

Here $\varepsilon_a = 2 \sup\limits_{\lambda \in \partial\mathcal{D}_{a,\delta}} \left|\Re(\nu(\lambda))\right| < 1$. The canonically oriented contour $\partial\mathcal{D}_{a,\delta}$ is depicted in Fig. 2.

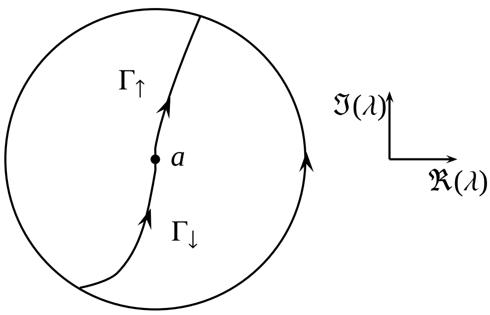


Figure 2: Contours in the RHP for $\mathcal{P}_a$.

Let $\zeta_a(\lambda) = x(p(\lambda) - p(a))$ with $\arg[\zeta_a(\lambda)] \in]-\pi\,;\pi[$ for $\lambda \in \mathcal{D}_{a,\delta} \setminus]a - \delta\,; a]$ and set

$$\mathcal{P}_a(\lambda) \;=\; \Psi_a(\lambda) \cdot [\zeta_a(\lambda)]^{-\nu(\lambda)\sigma_3} \cdot \mathrm{e}^{\frac{\mathrm{i}\pi\nu(\lambda)}{2}} \cdot L_a(\lambda) \;+\; \begin{pmatrix} \mathrm{id} - \boldsymbol{O}_{11}(\lambda) & 0 \\ 0 & \mathrm{id} - \boldsymbol{O}_{22}(\lambda) \end{pmatrix}. \tag{4.2}$$

Above, we agree upon

$$\Psi_a(\lambda) = \begin{pmatrix} \Psi(-\nu(\lambda),1;\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}}\zeta_a(\lambda))\cdot \boldsymbol{O}_{11}(\lambda) & \mathrm{i}b_{12}(\lambda)\cdot\Psi(1+\nu(\lambda),1;\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}\zeta_a(\lambda))\cdot\boldsymbol{O}_{12}(\lambda) \\ -\mathrm{i}b_{21}(\lambda)\cdot\Psi(1-\nu(\lambda),1;\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}}\zeta_a(\lambda))\cdot\boldsymbol{O}_{21}(\lambda) & \Psi(\nu(\lambda),1;\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}\zeta_a(\lambda))\cdot\boldsymbol{O}_{22}(\lambda)\end{pmatrix}, \tag{4.3}$$

with

$$b_{12}(\lambda) \;=\; -\mathrm{i}\frac{\sin\left[\pi\nu(\lambda)\right]\cdot\Gamma^2(1+\nu(\lambda))}{\pi\alpha_0^2(\lambda)\cdot\left[\zeta_a(\lambda)\right]^{2\nu(\lambda)}\mathrm{e}^{-2\mathrm{i}\pi\nu(\lambda)}}\cdot\mathrm{e}^{ixp(a)}, \tag{4.4}$$

$$b_{21}(\lambda) \;=\; -\mathrm{i}\frac{\pi\alpha_0^2(\lambda)\cdot\left[\zeta_a(\lambda)\right]^{2\nu(\lambda)}\mathrm{e}^{-2\mathrm{i}\pi\nu(\lambda)}}{\sin\left[\pi\nu(\lambda)\right]\cdot\Gamma^2(\nu(\lambda))}\cdot\mathrm{e}^{-\mathrm{i}xp(a)}. \tag{4.5}$$

In (4.3), $\Psi(a,c;z)$ denotes the Tricomi confluent hypergeometric function (CHF) of the second kind (see Appendix A) with the convention of choosing the cut along $\mathbb{R}^-$. The function $\Psi(a,c;z)$ admits an analytical continuation on the universal covering of $\mathbb{C}\setminus\{0\}$ and satisfies there monodromy relations (A.2) - (A.3) together with the asymptotic property (A.4). Also, we have introduced the new function α_0 by the equations,

$$\alpha_0(\lambda) \;=\; \alpha(\lambda)\begin{cases} 1 & \text{for } \lambda\in\mathcal{D}_{a,\delta}, \quad \Im\lambda>0 \\ \mathrm{e}^{2\mathrm{i}\pi\nu(\lambda)} & \text{for } \lambda\in\mathcal{D}_{a,\delta}, \quad \Im\lambda<0. \end{cases} \tag{4.6}$$

which is a holomorphic function on $\mathcal{D}_{a,\delta}\setminus]a-\delta\,;a[$. Finally, the expression for the piecewise holomorphic constant matrix $L_a(\lambda)$ depends on the region of the complex plane. Namely,

$$L_a(\lambda) \;=\; \begin{cases} I_2\otimes\mathrm{id} & -\pi/2<\arg[p(\lambda)-p(a)]<\pi/2, \\ \begin{pmatrix} \mathrm{id} & -\mathrm{e}^{-2\mathrm{i}\pi\nu(\lambda)}\boldsymbol{P}(\lambda)\mathrm{e}^{ixp(\lambda)} \\ 0 & \mathrm{id}\end{pmatrix} & \pi/2<\arg[p(\lambda)-p(a)]<\pi, \\ \begin{pmatrix} \mathrm{id} & 0 \\ -\boldsymbol{Q}(\lambda)\mathrm{e}^{-ixp(\lambda)} & \mathrm{id}\end{pmatrix} & -\pi<\arg[p(\lambda)-p(a)]<-\pi/2. \end{cases} \tag{4.7}$$

Using (A.4), (A.2) and (A.3) together with the relations,

$$\boldsymbol{O}_{jl}(\lambda)\cdot\boldsymbol{O}_{lk}(\lambda) \;=\; \boldsymbol{O}_{jk}(\lambda)\,. \tag{4.8}$$

one checks that our choice of the matrix L_a implies that $\mathcal{P}_a$ has the desired form of its asymptotic behaviour on the boundary $\partial\mathcal{D}_{a,\delta}$ while the desired jump conditions are satisfied automatically. Furthermore, referring again to (A.4), (A.2), one can see that the function $\mathcal{P}_a$ is continuous across the cut $]a-\delta\,;a[$ and thus indeed equation (4.2) determines a parametrix around $\lambda=a$.

4.2 Parametrix around b

The RHP for the parametrix $\mathcal{P}_b = I_2\otimes\mathrm{id} \;+\; \widehat{\mathcal{P}_b}$ around b reads

- $\widehat{\mathcal{P}_b}(\lambda)$ is a holomorphic in $\lambda\in\mathcal{D}_{b,\delta}\setminus\{\Gamma_\uparrow\cup\Gamma_\downarrow\}$ integral operator on $L^2(\mathbb{R}^+,\mathrm{d}s)\oplus L^2(\mathbb{R}^+,\mathrm{d}s)$;
- $\widehat{\mathcal{P}_b}(\lambda)$ admits L^2 $\pm$-boundary values $(\widehat{\mathcal{P}_b})_\pm(\lambda)$ on $\{\Gamma_\uparrow\cup\Gamma_\downarrow\setminus\{b\}\}\cap\mathcal{D}_{b,\delta}$;
- $\widehat{\mathcal{P}_b}(\lambda)$ has the same singular structure as Υ around $\lambda=b$;

- uniformly in $(s, s') \in \mathbb{R}^+ \times \mathbb{R}^+$ and $\lambda \in \partial\mathcal{D}_{b,\delta}$, one has

$$\left\|\widehat{\mathcal{P}}_b(\lambda \mid s, s')\right\| \;\leq\; \frac{C}{x^{1-\varepsilon_b}} \cdot \mathrm{e}^{-\frac{c}{4}(s+s')} \quad \text{for some} \quad C > 0; \tag{4.9}$$

- $\begin{cases} \mathcal{P}_{b;+}(\lambda) \cdot M_{\uparrow}(\lambda) \;=\; \mathcal{P}_{b;-}(\lambda) & \text{for } \lambda \in \Gamma_{\uparrow} \cap \mathcal{D}_{b,\delta}\,, \\ \mathcal{P}_{b;+}(\lambda) \cdot M_{\downarrow}^{-1}(\lambda) = \mathcal{P}_{b;-}(\lambda) & \text{for } \lambda \in \Gamma_{\downarrow} \cap \mathcal{D}_{b,\delta} \end{cases} \quad ;$

and $\varepsilon_b = 2\sup_{\lambda\in\partial D_{b,\delta}}\left|\Re(\nu(\lambda))\right| < 1$.

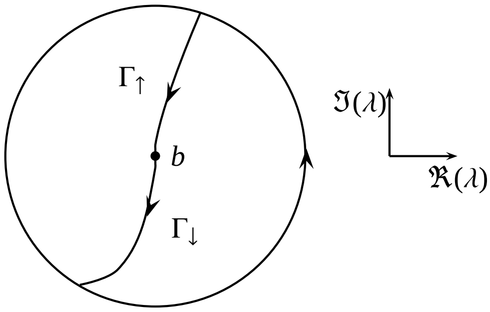


Figure 3: Contours in the RHP for $\mathcal{P}_b$.

Note that the solution to the RHP for the parametrix $\mathcal{P}_b$ around b can be formally obtained from the one at a through the transformation $b \to a$ and $\nu \to -\nu$ on the solution to the RHP for $\mathcal{P}_a$. Indeed, the two RHP are identical modulo this negation. Just as for the parametrix around a, we focus on the solution

$$\mathcal{P}_b(\lambda) \;=\; \Psi_b(\lambda) \cdot [\zeta_b(\lambda)]^{\nu(\lambda)\sigma_3} \mathrm{e}^{-\frac{\mathrm{i}\pi\nu(\lambda)}{2}} \cdot L_b(\lambda) \cdot [\zeta_b(\lambda)]^{\nu(\lambda)\sigma_3} \;+\; \begin{pmatrix} \mathrm{id} - \boldsymbol{O}_{11}(\lambda) & 0 \\ 0 & \mathrm{id} - \boldsymbol{O}_{22}(\lambda) \end{pmatrix} \tag{4.10}$$

where $\zeta_b(\lambda) = x[p(\lambda) - p(b)]$ with $\arg[\zeta_b(\lambda)] \in]-\pi\,;\pi[$ for $\lambda \in \mathcal{D}_{b,\delta} \setminus]b - \delta\,; f]$, and

$$\Psi(\lambda) = \begin{pmatrix} \Psi(\nu(\lambda), 1; \mathrm{e}^{-\mathrm{i}\frac{\pi}{2}}\zeta_b(\lambda)) \cdot \boldsymbol{O}_{11}(\lambda) & \mathrm{i}\tilde{b}_{12}(\lambda) \cdot \Psi(1 - \nu(\lambda), 1; \mathrm{e}^{\mathrm{i}\frac{\pi}{2}}\zeta_b(\lambda)) \cdot \boldsymbol{O}_{12}(\lambda) \\ -\mathrm{i}\tilde{b}_{21}(\lambda) \cdot \Psi(1 + \nu(\lambda), 1; \mathrm{e}^{-\mathrm{i}\frac{\pi}{2}}\zeta_b(\lambda)) \cdot \boldsymbol{O}_{21}(\lambda) & \Psi(-\nu(\lambda), 1; \mathrm{e}^{-\mathrm{i}\frac{\pi}{2}}\zeta_b(\lambda)) \cdot \boldsymbol{O}_{22}(\lambda) \end{pmatrix}, \tag{4.11}$$

with

$$\tilde{b}_{12}(\lambda) \;=\; \mathrm{i}\frac{\sin[\pi\nu(\lambda)]\Gamma^2(1 - \nu(\lambda))}{\pi\alpha^2(\lambda)} \cdot [\zeta_b(\lambda)]^{2\nu(\lambda)} \cdot \mathrm{e}^{ixp(b)}\,, \tag{4.12}$$

$$\tilde{b}_{21}(\lambda) \;=\; \mathrm{i}\frac{\pi\alpha^2(\lambda) \cdot \mathrm{e}^{-\mathrm{i}xp(b)}}{\sin[\pi\nu(\lambda)]\Gamma^2(-\nu(\lambda)) \cdot [\zeta_b(\lambda)]^{2\nu(\lambda)}}\,. \tag{4.13}$$

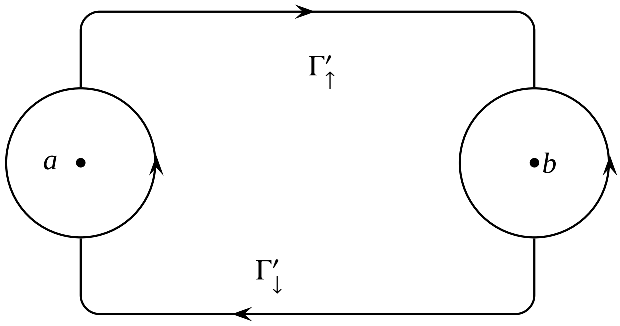
Finally, the parametrix $L_b(\lambda)$ reads

$$L_b(\lambda) \;=\; \begin{cases} I_2 \otimes \mathrm{id} & -\pi/2 < \arg[p(\lambda) - p(b)] < \pi/2, \\[1em] \begin{pmatrix} \mathrm{id} & -\boldsymbol{P}(\lambda)\mathrm{e}^{ixp(\lambda)} \\ 0 & \mathrm{id} \end{pmatrix} & \pi/2 < \arg[p(\lambda) - p(b)] < \pi, \\[1em] \begin{pmatrix} \mathrm{id} & 0 \\ -\boldsymbol{Q}(\lambda)\mathrm{e}^{-ixp(\lambda)} & \mathrm{id} \end{pmatrix} & -\pi < \arg[p(\lambda) - p(b)] < -\pi/2. \end{cases} \tag{4.14}$$

4.3 The last transformation

We define the integral operator $\Pi(\lambda) = \mathrm{id} \otimes I_2 \; + \; \widehat{\Pi}(\lambda)$ as

$$\Pi(\lambda) \; = \; \begin{cases} \Upsilon(\lambda) \cdot \mathcal{P}_b^{-1}(\lambda) & \text{for } \lambda \in \mathcal{D}_{b,\delta} \, , \\ \Upsilon(\lambda) \cdot \mathcal{P}_a^{-1}(\lambda) & \text{for } \lambda \in \mathcal{D}_{a,\delta} \, , \\ \Upsilon(\lambda) & \text{for } \lambda \in \mathbb{C} \setminus \{ \overline{\mathcal{D}}_{a,\delta} \cup \overline{\mathcal{D}}_{b,\delta} \} \, . \end{cases} \tag{4.15}$$



$$\Sigma_{\Pi} = \Gamma'_{\downarrow} \cup \Gamma'_{\uparrow} \cup \partial\mathcal{D}_{a,\delta} \cup \partial\mathcal{D}_{b,\delta}$$

Figure 4: Contour Σ_{Π} appearing in the RHP for Π.

It is readily checked that $\Pi = I_2 \otimes \mathrm{id} \; + \; \widehat{\Pi}$ satisfies the Riemann–Hilbert problem

- $\widehat{\Pi}(\lambda)$ is a holomorphic in $\lambda \in \mathbb{C} \setminus \Sigma_{\Pi}$ integral operator on $L^2(\mathbb{R}^+, \mathrm{d}s) \oplus L^2(\mathbb{R}^+, \mathrm{d}s)$;
- $\widehat{\Pi}(\lambda)$ admits continuous $\pm$-boundary values $(\widehat{\Pi})_{\pm}(\lambda)$ on Σ_{Π};
- uniformly in $(s, s') \in \mathbb{R}^+ \times \mathbb{R}^+$ and for any compact K such that $\Sigma_{\Pi} \subset \mathrm{Int}(K)$, one has

$$\left\| \widehat{\Pi}(\lambda \mid s, s') \right\| \; \leq \; \frac{C}{x^{1-\varepsilon}} \cdot \mathrm{e}^{-\frac{\varsigma}{4}(s+s')} \qquad \begin{array}{l} \text{for some} \quad C > 0 \, , \quad \text{any } \lambda \in \mathbb{C} \setminus K \\ \text{and for } \varepsilon = \max\{\varepsilon_a, \varepsilon_b\} \end{array} \qquad ; \tag{4.16}$$

- $\Pi_+(\lambda) \cdot G_{\Pi}(\lambda) \; = \; \Pi_-(\lambda) \qquad \text{where} \qquad G_{\Pi}(\lambda) \; = \; \begin{cases} M_{\uparrow}(\lambda) \;\; \text{for} \;\; \lambda \in \Gamma'_{\uparrow} \, ; \\ M_{\downarrow}^{-1}(\lambda) \;\; \text{for} \;\; \lambda \in \Gamma'_{\downarrow} \, ; \\ \mathcal{P}_{\varsigma}(\lambda) \text{ for } \lambda \in \partial\mathcal{D}_{\varsigma,\delta} \;\; \text{with } \varsigma \in \{a, b\} \, . \end{cases}$

Proposition 4.1 *The solution to the Riemann–Hilbert problem for* Π *exists and is unique, provided that* x *is large enough and* $|\Im(t)| < \delta$*, with* $\delta > 0$ *but small enough.*

Proof —

The unique solvability of the Riemann–Hilbert problem for Π is established along the lines already discussed. We hence solely focus on the existence of solutions. Introduce the following operator

$$C_{\Sigma_\Pi}[M](\lambda) = \int_{\Sigma_\Pi} \frac{M(\mu)}{\mu-\lambda}\cdot\frac{\mathrm{d}\mu}{2\mathrm{i}\pi} \quad \text{for } \lambda \in \mathbb{C}\setminus\Sigma_\Pi \quad \text{and} \quad M \in \mathcal{M}_2\Big(L^2(\Sigma_\Pi\times\mathbb{R}^+\times\mathbb{R}^+)\Big)\,. \tag{4.17}$$

Then, we consider the below singular integral equation for the unknown matrix $\widehat{\Pi}_+ \in \mathcal{M}_2\Big(L^2(\Sigma_\Pi\times\mathbb{R}^+\times\mathbb{R}^+)\Big)$:

$$\widehat{\Pi}_+(\lambda) + C_{\Sigma_\Pi;+}[\widehat{\Pi}_+\widehat{G}_\Pi](\lambda) = -C_{\Sigma_\Pi;+}[\widehat{G}_\Pi](\lambda) \qquad \text{where} \quad G_\Pi(\lambda) = \mathrm{id}\otimes I_2 + \widehat{G}_\Pi(\lambda)\,. \tag{4.18}$$

It follows from

$$\widehat{G}_\Pi \in \mathcal{M}_2(L^2\cap L^\infty(\Sigma_\Pi\times\mathbb{R}^+\times\mathbb{R}^+)) \quad \text{with} \quad \|\widehat{G}_\Pi\|_{\mathcal{M}_2\left(L^2\cap L^\infty(\Sigma_\Pi\times\mathbb{R}^+\times\mathbb{R}^+)\right)} \le \frac{C}{x^{1-\varepsilon}}\,, \tag{4.19}$$

that, for any $M \in \mathcal{M}_2(L^2\cap L^\infty(\Sigma_\Pi\times\mathbb{R}^+\times\mathbb{R}^+))$, one has $M\widehat{G}_\Pi \in \mathcal{M}_2(L^2(\Sigma_\Pi\times\mathbb{R}^+\times\mathbb{R}^+))$ Furthermore, one has that $\lambda\mapsto(M\widehat{G}_\Pi)(\lambda\mid s,s')$ belongs to $\mathcal{M}_2(L^2(\Sigma_\Pi))$ almost everywhere in $(s,s')\in\mathbb{R}^+\times\mathbb{R}^+$. Therefore, using Fubbini's theroem and the continuity of the $+$ boundary value of the Cauchy operator on Σ_Π in respect to the $L^2(\Sigma_\Pi)$ norm, we get

$$\begin{aligned} \Big\|C_{\Sigma_\Pi;+}[M\widehat{G}_\Pi]\Big\|^2_{\mathcal{M}_2\left(L^2(\Sigma_\Pi\times\mathbb{R}^+\times\mathbb{R}^+)\right)} &= \int_{\mathbb{R}^+\times\mathbb{R}^+} \mathrm{d}s\mathrm{d}s'\Big\{\Big\|C_{\Sigma_\Pi;+}[(M\widehat{G}_\Pi)(*\mid s,s')]\Big\|^2_{\mathcal{M}_2\left(L^2(\Sigma_\Pi)\right)}\Big\} \\ &\le c_\Pi\cdot\Big\|M\widehat{G}_\Pi\Big\|^2_{\mathcal{M}_2\left(L^2(\Sigma_\Pi\times\mathbb{R}^+\times\mathbb{R}^+)\right)} \le \frac{C_\Pi}{x^{1-\varepsilon}}\Big\|M\Big\|^2_{\mathcal{M}_2\left(L^2(\Sigma_\Pi\times\mathbb{R}^+\times\mathbb{R}^+)\right)}\,. \end{aligned} \tag{4.20}$$

This guarantees the invertibility of the operator $\mathrm{id}\otimes I_2 + C_{\Sigma_\Pi;+}[\,\cdot\,\widehat{G}_\Pi]$ on $\mathcal{M}_2(L^2(\Sigma_\Pi\times\mathbb{R}^+\times\mathbb{R}^+))$. Since $C_{\Sigma_\Pi;+}[\widehat{G}_\Pi]\in\mathcal{M}_2(L^2(\Sigma_\Pi\times\mathbb{R}^+\times\mathbb{R}^+))$, it follows that $\widehat{\Pi}_+$ exists and that, furthermore,

$$\Big\|\widehat{\Pi}_+\Big\|^2_{\mathcal{M}_2\left(L^2(\Sigma_\Pi\times\mathbb{R}^+\times\mathbb{R}^+)\right)} \le \frac{C}{x^{1-\varepsilon}} \quad \text{for some} \quad C>0\,. \tag{4.21}$$

We then define

$$\Pi(\lambda) = \mathrm{id}\otimes I_2 - C_{\Sigma_\Pi}[\widehat{G}_\Pi](\lambda) - C_{\Sigma_\Pi}[\widehat{\Pi}_+\widehat{G}_\Pi](\lambda)\,. \tag{4.22}$$

It is then straightforward, by using the bounds on $\widehat{\Pi}_+$ and $\widehat{G}_\Pi$, to deduce that $\Pi(\lambda)$ as defined through (4.22) does satisfy the Riemann–Hilbert problem stated above, with the sole difference that it admits L^2 $\pm$-boundary values on Σ_Π. However, $\widehat{G}_\Pi$ being a holomorphic integral operator on $L^2(\mathbb{R}^+,\mathrm{d}s)\oplus L^2(\mathbb{R}^+,\mathrm{d}s)$ in some open neighbourhood of Σ_Π it is readily seen that $\widehat{\Pi}_\pm$ admits a holomorphic continuation to some open neighbourhood of Σ_Π located on its $\mp$-side. In particular, this ensures that $\widehat{\Pi}$ does admit, in fact, continuous $\pm$ boundary values on Σ_Π. ■

5 The asymptotic behaviour of the determinant

5.1 A determinant identity

Lemma 5.1 *The following holds*

$$\partial_t \ln\det\left[I+V_t\right] = \oint_{\Gamma([a\,;b])} z\cdot\mathrm{tr}\Big[\partial_z\chi(z)\cdot\sigma_3\cdot\mathfrak{s}\cdot\chi^{-1}(z)\Big]\cdot\frac{\mathrm{d}z}{2\pi} \qquad \text{where} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5.1}$$

and $\mathfrak{s}$ *is the operator of multiplication by* s*, viz.* $(\mathfrak{s}\cdot f)(s) = sf(s)$*. Note that* tr *appearing above refers to the matrix and operator trace.*

Note that the trace used above is well defined due to (2.12) and the fact that $\widehat{\chi}(\lambda \mid s, s')$ is smooth in all its variables for λ uniformly away from $[a\,;b]$.

Proof —

Starting from the identity

$$\partial_t \ln \det[I + V_t] \;=\; \int_a^b [\partial_t V_t \cdot (I - R_t)](\lambda, \lambda) \cdot \mathrm{d}\lambda\;, \tag{5.2}$$

along with

$$\partial_t V_t(\lambda, \mu) \;=\; -\oint_{\Gamma([a\,;b])} \frac{\mathrm{d}z}{2\pi} \cdot \frac{z}{(z-\lambda)(z-\mu)} \cdot \Big(\vec{\boldsymbol{E}}_L(\lambda), \mathfrak{s}\sigma_3 \vec{\boldsymbol{E}}_R(\mu)\Big)\;, \tag{5.3}$$

as well as invoking the representation of the resolvent R_t in terms of $\vec{\boldsymbol{F}}_L$ and $\vec{\boldsymbol{F}}_R$, we get

$$\begin{aligned}\partial_t \ln \det[I + V_t] \;=\; & -\oint_{\Gamma([a\,;b])} \frac{\mathrm{d}z}{2\pi}\, z \int_a^b \mathrm{d}\lambda\, \frac{\Big(\vec{\boldsymbol{E}}_L(\lambda), \mathfrak{s}\sigma_3 \vec{\boldsymbol{E}}_R(\lambda)\Big)}{(z-\lambda)^2} \\ & + \mathrm{tr}\bigg\{ \oint_{\Gamma([a\,;b])} \frac{\mathrm{d}z}{4\pi}\, z \int_a^b \mathrm{d}\lambda \mathrm{d}\mu\, \vec{\boldsymbol{F}}_R(\lambda) \otimes \Big(\boldsymbol{E}_L(\lambda)\Big)^{\boldsymbol{T}} \Big\{ \frac{1}{\lambda - z} - \frac{1}{\lambda - \mu} \Big\} \frac{\mathfrak{s}\sigma_3}{(z-\mu)^2} \vec{\boldsymbol{E}}_R(\mu) \otimes \Big(\vec{\boldsymbol{F}}_L(\mu)\Big)^{\boldsymbol{T}} \bigg\}\;.\end{aligned} \tag{5.4}$$

By using the integral representation for χ , we obtain

$$\begin{aligned}\partial_t \ln \det[I + V_t] \;=\; & -\oint_{\Gamma([a\,;b])} \frac{\mathrm{d}z}{2\pi}\, z \int_a^b \mathrm{d}\lambda\, \frac{\Big(\vec{\boldsymbol{E}}_L(\lambda), \mathfrak{s}\sigma_3 \vec{\boldsymbol{E}}_R(\lambda)\Big)}{(z-\lambda)^2} \\ & + \oint_{\Gamma([a\,;b])} \frac{\mathrm{d}z}{2\pi}\, z \int_a^b \mathrm{d}\mu\, \mathrm{tr}\bigg\{ \Big[\chi(\mu) - \chi(z)\Big] \cdot \frac{\mathfrak{s}\sigma_3}{(z-\mu)^2} \vec{\boldsymbol{E}}_R(\mu) \otimes \Big(\vec{\boldsymbol{F}}_L(\mu)\Big)^{\boldsymbol{T}} \bigg\}\;.\end{aligned} \tag{5.5}$$

Finally, recalling the integral representation for $\chi^{-1}(\lambda)$, one gets

$$\partial_t \ln \det[I + V_t] \;=\; -\oint_{\Gamma([a\,;b])} z \cdot \mathrm{tr}\Big\{ \chi(z) \cdot \sigma_3 \mathfrak{s} \cdot \partial_z \chi^{-1}(z) \Big\} \cdot \frac{\mathrm{d}z}{2\pi}\;. \tag{5.6}$$

It solely remains to invoke that $\partial_z\big(\chi^{-1}(z)\big) \;=\; -\chi^{-1}(z) \cdot \partial_z \chi(z) \cdot \chi^{-1}(z)$ and the cyclic property of the trace. ■

5.2 The asymptotic evaluation of the determinant

Proposition 5.1 *The following representation holds for the ratio of determinants*

$$\frac{\det[I + V_1]}{\det[I + V_0]} \;=\; \det[I + \mathcal{U}_{1;t=1}] \cdot \det[I + \mathcal{U}_{2;t=1}] \cdot \Big(1 \;+\; \mathrm{o}(1)\Big)\;. \tag{5.7}$$

Proof —

Let t be such that

$$\det_{\Gamma([a\,;b])}\big[\mathrm{id}+\mathcal{U}_{k;t}\big] \;\neq\; 0 \qquad \text{for } k=1,2\;. \tag{5.8}$$

Then, the Riemann–Hilbert analysis ensures that, uniformly away from $[a\,;b]$, the solution χ can be represented as

$$\chi(\lambda) \;=\; \Big(I_2\otimes\mathrm{id} \;+\; \widehat{\Pi}\Big)\cdot\begin{pmatrix}\beta_1(\lambda) & 0\\ 0 & \beta_2(\lambda)\end{pmatrix}\;. \tag{5.9}$$

where $\widehat{\Pi}$ is an integral operator on $L^2(\mathbb{R}^+,\mathrm{d}s)\oplus L^2(\mathbb{R}^+,\mathrm{d}s)$ that, furthermore, satisfies to the bounds

$$\Big(\widehat{\Pi}_{ak}\cdot\beta_k\Big)(\lambda\mid s,s') \;\leq\; \frac{C\mathrm{e}^{-\frac{c}{4}(s+s')}}{x^{1-\varepsilon}(1+|\lambda|)} \qquad \text{with} \quad \varepsilon \;=\; \max\{\varepsilon_a,\varepsilon_b\} \tag{5.10}$$

for $\lambda\in\mathbb{C}\setminus K$, with K a small compact such that $\mathrm{Int}(K)\supset\Sigma_\Pi$, and any $s,s'\in\mathbb{R}^+$.

As a consequence, one gets that

$$\partial_t\ln\det\big[I+V_t\big] \;=\; \oint\limits_{\Gamma(\Sigma_\Pi)} \mathrm{tr}\Big[\partial_z\beta_1(z)\cdot\mathfrak{s}\cdot\beta_1^{-1}(z) \;-\; \partial_z\beta_2(z)\cdot\mathfrak{s}\cdot\beta_2^{-1}(z)\Big]\cdot z\frac{\mathrm{d}z}{2\pi} \;+\; \mathrm{O}\Big(\frac{1}{x^{1-\varepsilon}}\Big) \tag{5.11}$$

where the remainder $\mathrm{O}(x^{\varepsilon-1})$ is in respect to the $x\to+\infty$ limit. Thus, by using the representation

$$\beta_k(\lambda) \;=\; \mathrm{id} \;-\; C\Big[\tau_k\boldsymbol{\rho}_k\otimes\boldsymbol{\kappa}_k\Big](\lambda) \qquad \text{where} \quad \boldsymbol{\rho_k(\mu)}(s) \;=\; \rho_k(\mu;s) \tag{5.12}$$

we are led to:

$$\partial_t\ln\det\big[I+V_t\big] \;=\; \sum_{k=1}^{2}\epsilon_k\int\limits_a^b\frac{\mathrm{d}\mu}{2\pi}\tau_k(\mu)\cdot\boldsymbol{\kappa}_k(\mu)[\mathfrak{s}\boldsymbol{\rho}_k(\mu)] \;+\; \mathrm{O}(x^{\varepsilon-1}) \tag{5.13}$$

where we remind that the function $\rho_k(\mu;s)$ is defined by

$$\rho_k(\mu;s) \;=\; \Big(I+\mathcal{K}_{k;t}\Big)^{-1}[w_k(*;s)](\mu) \qquad \text{where} \qquad w_k(\lambda;s) \;=\; \alpha_{k;+}(\lambda)\oint\limits_{\Gamma([a\,;b])}\frac{\sqrt{c}\cdot\mathrm{e}^{-\frac{c}{2}s-\mathrm{i}t\epsilon_k\mu s}}{\alpha_k(\mu)\cdot(\mu-\lambda)}\cdot\frac{\mathrm{d}\mu}{2\mathrm{i}\pi}\;. \tag{5.14}$$

Note that, above, the operator $\Big(I+\mathcal{K}_{k;t}\Big)^{-1}$ acts on the $*$ variable of its argument. As a consequence,

$$\partial_t\ln\det\big[I+V_t\big] \;=\; -\sum_{k=1}^{2}\epsilon_k\int\limits_a^b\tau_k(\mu)\cdot\Big(I+\mathcal{K}_{k;t}\Big)^{-1}\Big[\boldsymbol{\kappa}_k(\mu)[\mathfrak{s}w_k(*;\bullet)]\Big](\mu)\cdot\frac{\mathrm{d}\mu}{2\pi} \;+\; \mathrm{O}(x^{\varepsilon-1})\;. \tag{5.15}$$

where the $*$ indicates the variable on which $\Big(I+\mathcal{K}_{k;t}\Big)^{-1}$ acts whereas the $\bullet$ variable refers to the one on which the one-form $\boldsymbol{\kappa}_k(\mu)$ acts. Observe that

$$\epsilon_k\tau_k(\mu)\boldsymbol{\kappa}_k(\mu)[\mathfrak{s}w_k(\nu;\bullet)] \; = \; c\epsilon_k\tau_k(\mu)\alpha_{k;+}(\nu) \oint\limits_{\Gamma([a\,;b])} \frac{\mathrm{d}\lambda}{2\mathrm{i}\pi}\frac{\alpha_k^{-1}(\lambda)}{\lambda-\nu}\int\limits_0^{+\infty}\mathrm{d}s s \mathrm{e}^{-cs+\mathrm{i}t\epsilon_k(\mu-\lambda)s}$$

$$= \; c\epsilon_k\tau_k(\mu)\alpha_{k;+}(\nu)\frac{\partial}{\partial t}\bigg\{ \oint\limits_{\Gamma([a\,;b])}\frac{\mathrm{d}\lambda}{2\mathrm{i}^2\pi}\frac{\alpha_k^{-1}(\lambda)}{(\lambda-\nu)\epsilon_k(\mu-\lambda)}\int\limits_0^{+\infty}\mathrm{d}s\mathrm{e}^{-cs+\mathrm{i}t\epsilon_k(\mu-\lambda)s}\bigg\}$$

$$= \; c\epsilon_k\tau_k(\mu)\alpha_{k;+}(\nu)\frac{\partial}{\partial t}\bigg\{ \oint\limits_{\Gamma([a\,;b])}\frac{\alpha_k^{-1}(\lambda)}{(\lambda-\nu)(\mu-\lambda)(t(\mu-\lambda)+\mathrm{i}\epsilon_k c)}\cdot\frac{\mathrm{d}\lambda}{2\mathrm{i}\pi}\bigg\}$$

$$= \; -\frac{\partial}{\partial t}\bigg\{\frac{\alpha_{k;+}(\nu)}{\alpha_{k;+}(\mu)}\cdot\frac{\alpha_{k;-}(\mu)-\alpha_{k;+}(\mu)}{\alpha_k(\mu+\mathrm{i}\epsilon_k c/t)}\cdot\frac{t}{\mathrm{i}(t(\mu-\nu)\,+\,\mathrm{i}\epsilon_k c)}\bigg\} \; = \; -2\pi\cdot\partial_t\Big(K_{k;t}(\nu,\mu)\Big)\,. \quad (5.16)$$

Therefore, we get that

$$\partial_t \ln\det\,[I+V_t] \; = \; -\sum_{k=1}^{2}\int\limits_a^b\Big((I+\mathcal{K}_{k;t})^{-1}\cdot\partial_t\mathcal{K}_{k;t}\Big)(\mu,\mu)\cdot\mathrm{d}\mu \; + \; \mathrm{O}(x^{\varepsilon-1})$$

$$= \; -\frac{\partial}{\partial t}\ln\Big\{\det\,[I+\mathcal{K}_{1;t}]\cdot\det\,[I+\mathcal{K}_{2;t}]\Big\} \; + \; \mathrm{O}(x^{\varepsilon-1})$$

$$= \; -\frac{\partial}{\partial t}\ln\Big\{\det\nolimits_{\Gamma([a\,;b])}\,[I+\mathcal{U}_{1;t}]\cdot\det\nolimits_{\Gamma([a\,;b])}\,[I+\mathcal{U}_{2;t}]\Big\} \; + \; \mathrm{O}(x^{\varepsilon-1})\,. \quad (5.17)$$

Now, observe that there exists $\delta>0$ such that

$$t\mapsto\det\nolimits_{\Gamma([a\,;b])}\,[I+\mathcal{U}_{k;t}] \quad k=1,2\,, \qquad (5.18)$$

are holomorphic functions on $\{t\in\mathbb{C}\;:\;|\Re(t)|<2 \text{ and } |\Im(t)|<\delta\}$ that furthermore do not vanish at $t=0$ and $t=1$. As a consequence, it has a finite amount of zeroes located in $\{t\in\mathbb{C}\;:\;|\Re(t)|<1.5 \text{ and } |\Im(t)|<\delta/2\}$. Thus, there exists a smooth curve $\mathscr{C}$ joining 0 to 1, located in the region $|\Im(t)|<\delta/2$ and such that

$$\det\nolimits_{\Gamma([a\,;b])}\,[I+\mathcal{U}_{k;t}]\neq 0 \quad \text{for any } t\in\mathscr{C} \text{ and } \; k=1,2\,. \qquad (5.19)$$

As a consequence, the formula (5.17) holds for any $t\in\mathscr{C}$. Thence, integrating both sides of (5.17) along $\mathscr{C}$ leads to the claim upon taking the exponent. Note that different choices of the curve $\mathscr{C}$ could lead to different values of the integral. However, any two such integrals will differ by integer multiples of $2\mathrm{i}\pi$, hence leading to the same value of the exponents. ■

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A Some properties of confluent hypergeometric function

For generic parameters (a, c) the Tricomi confluent hypergeometric function $\Psi(a, c; z)$ is one of the solutions to the differential equation

$$zy'' + (c - z)\, y' - ay = 0\ . \tag{A.1}$$

It enjoys the monodromy properties

$$\Psi(a, 1; z\mathrm{e}^{2\mathrm{i}\pi}) \ = \ \Psi(a, 1; z)\mathrm{e}^{-2\mathrm{i}\pi a} \ + \ \frac{2\pi \mathrm{i}\mathrm{e}^{-\mathrm{i}\pi a + z}}{\Gamma^2(a)}\Psi(1 - a, 1; \mathrm{e}^{\mathrm{i}\pi} z)\,, \tag{A.2}$$

$$\Psi(a, 1; ze^{-2i\pi}) = \Psi(a, 1; z)e^{2i\pi a} \ - \ \frac{2\pi \mathrm{i}\mathrm{e}^{\mathrm{i}\pi a + z}}{\Gamma^2(a)}\Psi(1 - a, 1; \mathrm{e}^{-\mathrm{i}\pi} z)\,, \tag{A.3}$$

and has the asymptotic expansion:

$$\Psi(a, c; z) \sim \sum_{n=0}^{\infty} (-1)^n \frac{(a)_n (a - c + 1)_n}{n!} z^{-a-n}, \quad z \to \infty, \quad -\frac{3\pi}{2} < \arg(z) < \frac{3\pi}{2}, \tag{A.4}$$

with $(a)_n \ = \ \Gamma(a + n)/\Gamma(a)$.

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