Riemann-Hilbert approach to a generalised sine kernel.

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Abstract

We derive the large distance asymptotics of the Fredholm determinant of the so-called generalised sine kernel at the critical point. This kernel corresponds to a generalisation of the pure sine kernel arising in the theory of random matrices and has potential applications to the analysis of the large-distance asymptotic behaviour of the so-called emptiness formation probability for various quantum integrable models away from their free fermion point.

Introduction

Correlation functions in quantum integrable systems at their free fermion point are represented by Fredholm determinants (or minors thereof) of integrable integral operators [8]. In this setting, the kernels of these operators depend, in an oscillatory way, on the spacial (and/or temporal) separation between the various operators whose correlation function is being computed. Various physical reasons such as the testing of universality in the large-distance x (and/or temporal t) regime motivate the analysis of the large x (and/or t) behaviour of the underlying Fredholm determinants. The setting of a Riemann–Hilbert problem [8] associated with these operators along with the development of the Deift-Zhou non-linear steepest descent method [4] led to a complete understanding of the asymptotical regimes of correlation functions at the free fermion point. The situation changes drastically in the case of quantum integrable systems that are away from their free fermion point. Indeed, then, the representation for the correlation functions become much more complicated and take the form of so-called multidimensional Fredholm series [12, 14, 16]. These can be seen as certain very specific deformations of the Fredholm series for

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a Fredholm determinant which do preserve some of the features of the integrands arising in the Fredholm series. Although, the general theory of these entities has not yet been fully established, there has already emerged a certain number of efficient methods allowing one to study, under certain hypotheses, the asymptotic behaviour of certain examples of such series [12, 14, 16]. In particular, the method of multidimensional deformations of a Natte series allows one to prove the asymptotic expansion of the correlator *under* the assumption of the convergence of the multidimensional Natte series of interest. The proof of the convergence of such series is a hard open problem that is, however, disconnected from the asymptotic analysis *per se*. The scheme of this method goes as follows. One starts by identifying the integrable integral operator that is underlying to the multidimensional Fredholm series of interest. The next step consists in studying, with the help of Riemann–Hilbert problem-based methods, the asymptotic behaviour of the Fredholm determinant associated with this operator. Having obtained these, one should then raise them to the level of the multidimensional Fredholm series by using the techniques developed in [14, 16]. Of a particular interest is a specific correlation function, called the "emptiness formation probability", which arises in quantum integrable models and has a similar interpretation to the gap probability arising in matrix models. The multidimensional Fredholm series for this correlator has been obtained in [12]. The integrable kernel coming up in the aforesaid scheme of asymptotic analysis corresponds to the generalised sine kernel at the critical point. It is this very reason that triggered our interest in this kernel.

We shall now be more precise. By generalised sine kernel at the critical point, we mean the integral operator $I + V$ acting on $L^2([a; b])$, $a, b \in \mathbb{R}$, $a < b$, with the integral kernel

$$V(\lambda, \mu) = -\frac{e(\lambda)e^{-1}(\mu) - e(\mu)e^{-1}(\lambda)}{2i\pi(\lambda - \mu)} \quad \text{where} \quad e(\lambda) = e^{\frac{im}{2}p(\lambda) + \frac{g(\lambda)}{2}}. \quad (0.1)$$

In the following, we shall assume that

- $p' > 0$ on $[a; b]$;
- p and g are holomorphic on some open neighbourhood U of $[a; b]$.

Note that, in the large- m regime, one may absorb g in the definition of p : $p(\lambda) \hookrightarrow p_g(\lambda) = p(\lambda) - g(\lambda)/m$. However, treating g on separate grounds leads to a different representation for the asymptotic expansion of the solution to the associated Riemann–Hilbert problem and, hence, of the determinant. It is this alternative form of the asymptotic expansion that would be more useful from the point of view of the potential applications of our analysis to large-distance asymptotics of the emptiness formation probability.

As it has already been mentioned, our interest to the integrable integral operators with kernel (0.1) is primary motivated by the asymptotic problems related to the emptiness formation probability in the non-free fermion exactly solvable quantum models. Hence our motivation to devote the last section of the paper to one special variant of the sine kernel which appears in the description of the emptiness formation probability in the XXZ spin-1/2 Heisenberg chain. This generalisation can be, of course, treated as a particular case of the kernel (0.1); however, it is not directly in this class. Indeed, in that case, the integral operator acts on functions supported on an arc instead of an interval so that some modifications of the analysis are needed. This particular instance of the generalised sine kernel is also a deformation of the specific integrable kernel of sine-type considered in [6]. Moreover, this deformation does not affect much the analysis of [6], so that, after deriving the relevant differential identity for the determinant in question, we can just use, with a proper adjustment, the results of [6] and produce the asymptotics of this specific determinant. We hope that the results of this last section can be further used for the final evaluation of the asymptotics of the XXZ emptiness formation probability. For the reader's convenience, at the beginning of the last section, we discuss in more details the definition of the emptiness formation probability in the XXZ chain.

We start the paper by carrying out an asymptotic in m resolution of the Riemann–Hilbert problem associated with the kernel (0.1) in Section 1. Then, in Section 2 we build on the previous analysis so as to estimate the

Fredholm determinant $\det[I + V]$ asymptotically in m . Finally, in Section 3, we discuss the asymptotics of the Fredholm determinant directly connected with the XXZ chain.

1 Asymptotic resolution of the Riemann–Hilbert problem

1.1 The kernel and initial Riemann–Hilbert problem

The kernel (0.1) is of integrable type in that it admits the representation:

$$V(\lambda, \mu) = \frac{(E_L(\lambda), E_R(\mu))}{\lambda - \mu}, \quad (1.1)$$

with

$$E_L^T(\lambda) = (-e^{-1}(\lambda) \ e(\lambda)), \quad \text{and} \quad E_R(\lambda) = \frac{-1}{2i\pi} \begin{pmatrix} e(\lambda) \\ e^{-1}(\lambda) \end{pmatrix}. \quad (1.2)$$

Above, T in the exponent refers to the transposition.

This kernel is associated with the Riemann–Hilbert problem [8] for a 2×2 matrix $\chi(\lambda)$

- $\chi \in O(\mathbb{C} \setminus [a; b])$ and has continuous boundary values on $(a; b)$;
- $\chi(\lambda) = \ln|\lambda - a| \cdot \ln|\lambda - b| \cdot O\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ as $\lambda \rightarrow a$ or $\lambda \rightarrow b$;
- $\chi(\lambda) = I_2 + O(\lambda^{-1})$ when $\lambda \rightarrow \infty$;
- $\chi_-(\lambda) = \chi_+(\lambda) \cdot G_\chi(\lambda)$ for $\lambda \in (a; b)$, where

$$G_\chi(\lambda) = I_2 + 2i\pi E_R(\lambda) \cdot E_L^T(\lambda) \quad (1.3)$$

Here, $O(U)$, U open in $\mathbb{C}$, stands for the ring of holomorphic functions on U . I_2 is the identity matrix. Also, we should explain that relations of the type $M(z) = O(N(z))$ for two matrix functions M, N should be understood entrywise, *i.e.* $M_{jk}(z) = O(N_{jk}(z))$. Further, given a function f defined on $\mathbb{C} \setminus \gamma$, with γ an oriented curve in $\mathbb{C}$, we denote by $f_\pm(s)$ the boundary values of $f(z)$ on Γ when the argument z approaches the point $s \in \Gamma$ non-tangentially and from the right (+) or the left (−) side of the curve. Again, if one deals with matrix function, then this relation has to be understood entrywise.

The unique solution to the above Riemann–Hilbert problem takes the form

$$\chi(\lambda) = I_2 - \int_a^b \frac{F_R(\mu) \cdot E_L^T(\mu)}{\mu - \lambda} d\mu, \quad \text{and} \quad \chi^{-1}(\lambda) = I_2 + \int_a^b \frac{E_R(\mu) \cdot F_L^T(\mu)}{\mu - \lambda} d\mu, \quad (1.4)$$

where $F_R(\lambda)$ and $F_L(\lambda)$ correspond to the solutions to the below linear integral equations

$$F_R(\lambda) + \int_a^b V(\mu, \lambda) F_R(\mu) d\mu = E_R(\lambda), \quad \text{and} \quad F_L(\lambda) + \int_a^b V(\lambda, \mu) F_L(\mu) d\mu = E_L(\lambda). \quad (1.5)$$

The jump matrix for χ has a 0 in its lower diagonal:

$$G_\chi(\lambda) = \begin{pmatrix} 2 & -e^2(\lambda) \\ e^{-2}(\lambda) & 0 \end{pmatrix}. \quad (1.6)$$

1.2 The first transformation: h -function

The function

$$h(\lambda) = q(\lambda) \int_a^b \frac{-ip(s)}{q_+(s)(s-\lambda)} \frac{ds}{2i\pi} \quad \text{with} \quad q(\lambda) = (\lambda-a)^{\frac{1}{2}}(\lambda-b)^{\frac{1}{2}} \quad (1.7)$$

solves the Riemann–Hilbert problem

$$h \in O(\mathbb{C} \setminus [a; b]) \quad , \quad h_+(\lambda) + h_-(\lambda) = -ip(\lambda) \quad \text{for} \quad \lambda \in (a; b) \quad (1.8)$$

and h is bounded at infinity. Moreover

$$h(\lambda) \xrightarrow{\lambda \rightarrow \infty} h_\infty = \frac{1}{2\pi} \int_a^b \frac{p(s)}{q_+(s)} ds . \quad (1.9)$$

Let Γ be a counterclockwise loop around $[a; b]$ in U . If λ belongs to U and is located outside of the loop Γ , then it is readily seen that h admits the representation

$$h(\lambda) = q(\lambda) \oint_{\Gamma} \frac{p(s)}{s-\lambda} \frac{ds}{4\pi q(s)} . \quad (1.10)$$

Furthermore, if λ is located inside of Γ , then $f(\lambda)$ admits the alternative representation

$$h(\lambda) = q(\lambda) \oint_{\Gamma} \frac{p(s) - p(\lambda)}{s-\lambda} \frac{ds}{4\pi q(s)} - i \frac{p(\lambda)}{2} . \quad (1.11)$$

Here, the regularising term could have been added since the corresponding integral is zero (the residue at infinity vanishes). It is then easy to see that, for $\lambda \in (a; b)$,

$$h_-(\lambda) - h_+(\lambda) = (\lambda-a)^{\frac{1}{2}} \cdot (b-\lambda)^{\frac{1}{2}} \int_a^b \frac{p(\lambda) - p(s)}{\pi(\lambda-s)} \frac{ds}{(s-a)^{\frac{1}{2}} \cdot (b-s)^{\frac{1}{2}}} > 0 , \quad (1.12)$$

where the positivity follows from

$$\int_a^b \frac{p(\lambda) - p(s)}{\pi(\lambda-s)} \frac{ds}{(s-a)^{\frac{1}{2}} \cdot (b-s)^{\frac{1}{2}}} > \inf_{s \in [a; b]} [p'(s)] \cdot \int_a^b \frac{ds}{\pi(s-a)^{\frac{1}{2}} \cdot (b-s)^{\frac{1}{2}}} > 0 . \quad (1.13)$$

We then set

$$\Xi(\lambda) = e^{mh_\infty \sigma_3} \chi(\lambda) e^{-mh(\lambda) \sigma_3} . \quad (1.14)$$

It is readily seen, that

- $\Xi \in O(\mathbb{C} \setminus [a; b])$ and has continuous boundary values on $(a; b)$;
- $\Xi(\lambda) = \ln |\lambda - a| \cdot \ln |\lambda - b| \cdot O \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ as $\lambda \rightarrow a$ or $\lambda \rightarrow b$;
- $\Xi(\lambda) = I_2 + O(\lambda^{-1})$ when $\lambda \rightarrow \infty$;
- $\Xi_-(\lambda) = \Xi_+(\lambda) \cdot G_\Xi(\lambda)$ for $\lambda \in (a; b)$ where

$$G_\Xi(\lambda) = \begin{pmatrix} 2e^{-m[h_-(\lambda) - h_+(\lambda)]} & -e^{g(\lambda)} \\ e^{-g(\lambda)} & 0 \end{pmatrix} . \quad (1.15)$$

1.3 The parametrix on $[a; b]$

The scheme for building the global parametrix on $[a; b]$ is standard [3, 19]. We set

$$D(\lambda) = \exp \left\{ q(\lambda) \int_a^b \frac{-g(s)}{q_+(s)} \frac{ds}{2i\pi(s-\lambda)} \right\} \quad \text{and} \quad D_\infty = \exp \left\{ \int_a^b \frac{g(s)}{2i\pi q_+(s)} ds \right\}. \quad (1.16)$$

This can be alternatively recast as

$$D(\lambda) = \exp \left\{ q(\lambda) \oint_{\Gamma} \frac{g(s)}{q(s)(s-\lambda)} \frac{ds}{4i\pi} - \frac{g(\lambda)}{2} \right\}. \quad (1.17)$$

In which Γ refers to a loop around $[a; b]$ in U that, furthermore, encircles λ . Hence, one has

$$D_+(\lambda)D_-(\lambda) = e^{-g(\lambda)}, \quad \lambda \in (a; b). \quad (1.18)$$

Agreeing upon

$$U = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \quad \text{and} \quad U^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}, \quad (1.19)$$

we get that the matrix

$$M(\lambda) = D_\infty^{-\sigma_3} \cdot U^{-1} \cdot \left(\frac{\lambda-a}{\lambda-b} \right)^{\frac{\sigma_3}{4}} \cdot U \cdot D^{\sigma_3}(\lambda), \quad (1.20)$$

solves the Riemann–Hilbert problem

- $M \in \mathcal{O}(\mathbb{C} \setminus [a; b])$ and has continuous boundary values on $(a; b)$;
- $M(\lambda) = |(\lambda-a)(\lambda-b)|^{-\frac{1}{4}} \cdot \mathcal{O} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ as $\lambda \rightarrow a$ or $\lambda \rightarrow b$;
- $M(\lambda) = I_2 + \mathcal{O}(\lambda^{-1})$ when $\lambda \rightarrow \infty$;
- $M_-(\lambda) = M_+(\lambda) \cdot G_M(\lambda)$ for $\lambda \in (a; b)$ where

$$G_M(\lambda) = \begin{pmatrix} 0 & -e^{g(\lambda)} \\ e^{-g(\lambda)} & 0 \end{pmatrix}. \quad (1.21)$$

1.4 Parametrix around a

Following [3], we construct the parametrix at a . We choose some $\delta > 0$ and define

$$E_a(\lambda) = \frac{\sqrt{\pi}e^{i\frac{\pi}{4}}}{2} M(\lambda) e^{\frac{g(\lambda)}{2}\sigma_3} \frac{1}{2} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} [-m^2 \zeta_a(\lambda)]^{-\frac{\sigma_3}{4}}, \quad (1.22)$$

where

$$\zeta_a(\lambda) = (\lambda-a)(b-\lambda) \cdot \left(\int_a^b \frac{p(\lambda)-p(s)}{\pi(\lambda-s)} \frac{ds}{(s-a)^{\frac{1}{2}} \cdot (b-s)^{\frac{1}{2}}} \right)^2. \quad (1.23)$$

In particular, one has that $\zeta'_a(a) > 0$, meaning that, at least for δ small enough

$$\zeta_a(\mathcal{D}_{a,\delta} \cap \mathbb{H}_\pm) \subset \mathbb{H}_\pm, \quad \text{and} \quad \inf_{\lambda \in \partial \mathcal{D}_{a,\delta}} |\zeta'_a(\lambda)| > 0. \quad (1.24)$$

There $\mathcal{D}_{a,\delta} = \{z \in \mathbb{C} : |z - a| < \delta\}$ is the disc of radius δ centred at a and $\partial \mathcal{D}_{a,\delta}$ refers to its canonically oriented boundary. It then follows from the jump conditions satisfied by M that E_a is continuous across $(a; a + \delta)$. Furthermore, the local behaviour of M at a ensures that the singularity of E_a at a is of removable type. As a consequence, we get that E_a is analytic on $\mathcal{D}_{a,\delta}$.

Next, we introduce a matrix valued function built out of Hankel functions $H_a^{(k)}$ whose definition is recalled in Appendix A

$$\mathcal{P}_a^{(0)}(\lambda) = \begin{pmatrix} \sqrt{-m^2 \zeta_a(\lambda)} [H_0^{(2)}]'(\sqrt{-m^2 \zeta_a(\lambda)}/2) & \sqrt{-m^2 \zeta_a(\lambda)} [H_0^{(1)}]'(\sqrt{-m^2 \zeta_a(\lambda)}/2) \\ H_0^{(2)}(\sqrt{-m^2 \zeta_a(\lambda)}/2) & H_0^{(1)}(\sqrt{-m^2 \zeta_a(\lambda)}/2) \end{pmatrix} e^{[i \sqrt{-m^2 \zeta_a(\lambda)} - g(\lambda)] \frac{\sigma_3}{2}}. \quad (1.25)$$

It follows from the asymptotic expansions of the Hankel functions (A.5)-(A.9) that, uniformly in $\lambda \in \partial \mathcal{D}_{a,\delta}$,

$$\mathcal{P}_a^{(0)}(\lambda) \simeq [-m^2 \zeta_a(\lambda)]^{\frac{\sigma_3}{4}} \frac{2e^{-i\frac{\pi}{4}}}{\sqrt{\pi}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \sum_{n \geq 0} (0, n) \left(\frac{i}{\sqrt{-m^2 \zeta_a(\lambda)}} \right)^n \begin{pmatrix} (-1)^n a_n & i b_n \\ -i(-1)^n b_n & a_n \end{pmatrix} e^{-g(\lambda) \frac{\sigma_3}{2}}, \quad (1.26)$$

where $\simeq$ indicates an equality in the sense of asymptotic expansions, while

$$a_n = \frac{1}{1-2n}, \quad b_n = \frac{2n}{1-2n} \quad (1.27)$$

and (ν, n) is as defined in (A.7). We are now in position to write down the parametrix at a :

$$\mathcal{P}_a(\lambda) = E_a(\lambda) \mathcal{P}_a^{(0)}(\lambda). \quad (1.28)$$

It satisfies, uniformly in $\lambda \in \partial \mathcal{D}_{a,\delta}$,

$$\mathcal{P}_a(\lambda) = M(\lambda) \cdot [I_2 + O(m^{-1})], \quad (1.29)$$

uniformly on $\partial \mathcal{D}_{a,\delta}$. Elementary properties satisfied by the Hankel functions (A.1)-(A.2) allow one to readily check that the parametrix $\mathcal{P}_a$ solves the Riemann–Hilbert problem

- $\mathcal{P}_a \in O(\mathcal{D}_{a,\delta} \setminus [a; a + \delta))$ and has continuous boundary values on $(a; a + \delta)$;
- $\mathcal{P}_a(\lambda) = \ln |\lambda - a| \cdot O \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ as $\lambda \rightarrow a$;
- $\mathcal{P}_a(\lambda) = M(\lambda) \cdot [I_2 + O(m^{-1})]$ uniformly in $\lambda \in \partial \mathcal{D}_{a,\delta}$;
- $[\mathcal{P}_a]_-(\lambda) = [\mathcal{P}_a]_+(\lambda) \cdot G_\Xi(z)$ for $\lambda \in (a; a + \delta)$.

Also, it is easy to check that

$$\mathcal{P}_a^{-1}(\lambda) \simeq \sum_{n \geq 0} (0, n) \left(\frac{i}{\sqrt{-m^2 \zeta_a(\lambda)}} \right)^n \begin{pmatrix} a_n & -i b_n e^{g(\lambda)} \\ i(-1)^n b_n e^{-g(\lambda)} & (-1)^n a_n \end{pmatrix} M^{-1}(\lambda). \quad (1.30)$$

1.5 Parametrix around b

The parametrix at b is defined in a similar way. We introduce

$$\zeta_b(\lambda) = (\lambda - a)(\lambda - b) \cdot \left(\int_a^b \frac{p(\lambda) - p(s)}{\pi(\lambda - s)} \frac{ds}{(s - a)^{\frac{1}{2}} \cdot (b - s)^{\frac{1}{2}}} \right)^2, \quad (1.31)$$

and then define

$$E_b(\lambda) = \frac{\sqrt{\pi} e^{i\frac{\pi}{4}}}{2} M(\lambda) e^{\frac{g(\lambda)}{2} \sigma_3} \frac{1}{2} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} [m^2 \zeta_b(\lambda)]^{\frac{\sigma_3}{4}}. \quad (1.32)$$

Again, one has that $\zeta'_b(b) > 0$, meaning that, at least for δ small enough $\zeta(\mathcal{D}_{b,\delta} \cap \mathbb{H}_{\pm}) \subset \mathbb{H}_{\pm}$. The same reasoning as before leads to the conclusion that $E_b \in \mathcal{O}(\mathcal{D}_{b,\delta})$. Further, the matrix

$$\mathcal{P}_b^{(0)}(\lambda) = \begin{pmatrix} H_0^{(1)}(\sqrt{m^2 \zeta_b(\lambda)}/2) & H_0^{(2)}(\sqrt{m^2 \zeta_b(\lambda)}/2) \\ \sqrt{m^2 \zeta_b(\lambda)} [H_0^{(2)}]'(\sqrt{m^2 \zeta_b(\lambda)}/2) & \sqrt{m^2 \zeta_b(\lambda)} [H_0^{(1)}]'(\sqrt{m^2 \zeta_b(\lambda)}/2) \end{pmatrix} e^{-[i\sqrt{m^2 \zeta_b(\lambda)} + g(\lambda)] \frac{\sigma_3}{2}}, \quad (1.33)$$

admits, uniformly in $\lambda \in \partial \mathcal{D}_{b,\delta}$, the asymptotic expansion

$$\mathcal{P}_b^{(0)}(\lambda) \simeq [m^2 \zeta_b(\lambda)]^{-\frac{\sigma_3}{4}} \frac{2e^{-i\frac{\pi}{4}}}{\sqrt{\pi}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \sum_{n \geq 0} (0, n) \left(\frac{i}{\sqrt{m^2 \zeta_b(\lambda)}} \right)^n \begin{pmatrix} a_n & -i(-1)^n b_n \\ ib_n & (-1)^n a_n \end{pmatrix} e^{-g(\lambda) \frac{\sigma_3}{2}}. \quad (1.34)$$

The full parametrix in the neighbourhood of b then reads

$$\mathcal{P}_b(\lambda) = E_b(\lambda) \mathcal{P}_b^{(0)}(\lambda). \quad (1.35)$$

The latter solves the Riemann–Hilbert problem

- $\mathcal{P}_b \in \mathcal{O}(\mathcal{D}_{b,\delta} \setminus (b - \delta; b])$ and has continuous boundary values on $(b - \delta; b)$;
- $\mathcal{P}_b(\lambda) = \ln |\lambda - b| \cdot \mathcal{O} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ as $\lambda \rightarrow b$;
- $\mathcal{P}_b(\lambda) = M(\lambda) \cdot [I_2 + \mathcal{O}(m^{-1})]$ uniformly in $z \in \partial \mathcal{D}_{b,\delta}$;
- $[\mathcal{P}_b]_-(\lambda) = [\mathcal{P}_b]_+(\lambda) \cdot G_{\Xi}(\lambda)$ for $\lambda \in (b - \delta; b)$.

Finally, one can check that

$$\mathcal{P}_b^{-1}(\lambda) \simeq \sum_{n \geq 0} (0, n) \left(\frac{i}{\sqrt{m^2 \zeta_b(\lambda)}} \right)^n \begin{pmatrix} (-1)^n a_n & i(-1)^n b_n e^{g(\lambda)} \\ -ib_n e^{-g(\lambda)} & a_n \end{pmatrix} M^{-1}(\lambda). \quad (1.36)$$

1.6 The last transformation: the matrix Υ

We define a piecewise analytic matrix Υ by

$$\Upsilon(\lambda) = \begin{cases} \Xi(\lambda) \cdot M^{-1}(\lambda), & \lambda \in \mathbb{C} \setminus \{\partial \mathcal{D}_{a,\delta} \cup \partial \mathcal{D}_{b,\delta} \cup [a + \delta; b - \delta]\}, \\ \Xi(\lambda) \cdot \mathcal{P}_a^{-1}(\lambda), & \lambda \in \mathcal{D}_{a,\delta}, \\ \Xi(\lambda) \cdot \mathcal{P}_b^{-1}(\lambda), & \lambda \in \mathcal{D}_{b,\delta}. \end{cases} \quad (1.37)$$

It is then readily checked that the matrix Υ has its jumps solely on the two disks $-\partial \mathcal{D}_{a,\delta}$ and $-\partial \mathcal{D}_{b,\delta}$ endowed with an opposite (in respect to the canonical one) orientation. Namely, the matrix Υ is the unique solution to the Riemann–Hilbert problem

- $\Upsilon \in \mathcal{O}(\mathbb{C} \setminus \partial\{\mathcal{D}_{a,\delta} \cup \mathcal{D}_{b,\delta}\})$ and has continuous boundary values on $-\partial\mathcal{D}_{a,\delta} \cup -\partial\mathcal{D}_{b,\delta}$;
- $\Upsilon(\lambda) = I_2 + \mathcal{O}(\lambda^{-1})$ when $\lambda \rightarrow \infty$;
- $\Upsilon_-(\lambda) = \Upsilon_+(\lambda) \cdot G_\Upsilon(\lambda)$ for $\lambda \in -\partial\mathcal{D}_{a,\delta} \cup -\partial\mathcal{D}_{b,\delta}$, where

$$G_\Upsilon(\lambda) = M(\lambda) \cdot \mathcal{P}_a^{-1}(\lambda) \quad \text{for } \lambda \in -\partial\mathcal{D}_{a,\delta} \quad \text{and} \quad M(\lambda) \cdot \mathcal{P}_b^{-1}(\lambda) \quad \text{for } \lambda \in -\partial\mathcal{D}_{b,\delta}. \quad (1.38)$$

1.6.1 Asymptotic expansion of the jump matrix G_Υ

In the neighbourhood of a , it is readily seen that the matrix $G_\Upsilon(\lambda)$ admits the asymptotic expansion

$$G_\Upsilon(\lambda) \simeq \sum_{n \geq 0} (0, n) \left(\frac{i}{\sqrt{-m^2 \zeta_a(\lambda)}} \right)^n M(\lambda) \begin{pmatrix} a_n & -ib_n e^{g(\lambda)} \\ i(-1)^n b_n e^{-g(\lambda)} & (-1)^n a_n \end{pmatrix} M^{-1}(\lambda). \quad (1.39)$$

We then introduce

$$\beta(\lambda) = e^{g(\lambda)} D^2(\lambda) \quad , \quad r(\lambda) = \frac{\beta(\lambda) + \beta^{-1}(\lambda)}{2} \quad \text{and} \quad t(\lambda) = \frac{\beta(\lambda) - \beta^{-1}(\lambda)}{2q(\lambda)}. \quad (1.40)$$

The functions r and t are holomorphic on $\mathcal{D}_{a,\delta} \cup \mathcal{D}_{b,\delta}$

Then, agreeing upon

$$u(\lambda) = \int_a^b \frac{p(\lambda) - p(s)}{\pi(\lambda - s)} \frac{ds}{(s - a)^{\frac{1}{2}}(b - s)^{\frac{1}{2}}}, \quad (1.41)$$

one has the identities

$$\frac{\beta(\lambda) - \beta^{-1}(\lambda)}{2\sqrt{-\zeta_a(\lambda)}} = \frac{t(\lambda)}{u(\lambda)} \quad \left(\frac{\lambda - a}{\lambda - b} \right)^{\frac{1}{2}} \cdot \frac{\beta(\lambda) - \beta^{-1}(\lambda)}{2} = t(\lambda)(\lambda - a) \quad \text{and} \quad \left(\frac{\lambda - b}{\lambda - a} \right)^{\frac{1}{2}} \cdot \frac{\beta(\lambda) - \beta^{-1}(\lambda)}{2} = t(\lambda)(\lambda - b), \quad (1.42)$$

as well as

$$\left(\frac{\lambda - a}{\lambda - b} \right)^{\frac{1}{2}} \cdot \frac{1}{\sqrt{-\zeta_a(\lambda)}} = \frac{1}{(\lambda - b)u(\lambda)} \quad \text{and} \quad \left(\frac{\lambda - b}{\lambda - a} \right)^{\frac{1}{2}} \cdot \frac{1}{\sqrt{-\zeta_a(\lambda)}} = \frac{1}{(\lambda - a)u(\lambda)}. \quad (1.43)$$

What leads, after a quick computation, to

$$\begin{aligned} UD_\infty^{\sigma_3} G_\Upsilon(\lambda) D_\infty^{-\sigma_3} U^{-1} &= I_2 + \sum_{p=1}^{+\infty} \frac{(0, 2p)}{m^{2p} \zeta_a^p(\lambda)} \begin{pmatrix} a_{2p} - b_{2p} r(\lambda) & -i(\lambda - a) b_{2p} t(\lambda) \\ -i(\lambda - b) b_{2p} t(\lambda) & a_{2p} + b_{2p} r(\lambda) \end{pmatrix} \\ &+ \sum_{p=0}^{+\infty} \frac{i(0, 2p+1)}{m^{2p+1} \zeta_a^p(\lambda)} \begin{pmatrix} -b_{2p+1} t(\lambda)/u(\lambda) & -i[a_{2p+1} + b_{2p+1} r(\lambda)]/[(\lambda - b)u(\lambda)] \\ i[a_{2p+1} - b_{2p+1} r(\lambda)]/[(\lambda - a)u(\lambda)] & b_{2p+1} t(\lambda)/u(\lambda) \end{pmatrix}. \end{aligned} \quad (1.44)$$

In particular, we get that, to the first order

$$G_\Upsilon(\lambda) \simeq I_2 + \frac{G_1^{(a)}(\lambda)}{m} + \mathcal{O}(m^{-2}), \quad (1.45)$$

with

$$G_1^{(a)}(\lambda) = -\frac{(0,1)(2r(\lambda)-1)}{(\lambda-a)u(\lambda)} D_\infty^{-\sigma_3} U^{-1} \sigma^- U D_\infty^{\sigma_3} + G_{1;\text{reg}}^{(a)}(\lambda). \quad (1.46)$$

There $G_{1;\text{reg}}^{(a)}(\lambda)$ is a holomorphic matrix in some sufficiently small neighbourhood of a .

Likewise, in the neighbourhood of b one has the asymptotic expansion

$$\begin{aligned} U D_\infty^{\sigma_3} G_Y(\lambda) D_\infty^{-\sigma_3} U^{-1} &\simeq I_2 + \sum_{p=1}^{+\infty} \frac{1}{m^{2p}(\lambda-b)^p} \frac{(-1)^p(0,2p)}{(\lambda-a)^p u^{2p}(\lambda)} \begin{pmatrix} a_{2p} + b_{2p}r(\lambda) & i(\lambda-a)b_{2p}t(\lambda) \\ i(\lambda-b)b_{2p}t(\lambda) & a_{2p} - b_{2p}r(\lambda) \end{pmatrix} \\ &+ \sum_{p=0}^{+\infty} \frac{1}{m^{2p+1}(\lambda-b)^p} \frac{i(-1)^p(0,2p+1)}{(\lambda-a)^p u^{2p+1}(\lambda)} \begin{pmatrix} -b_{2p+1}t(\lambda) & i[a_{2p+1} - b_{2p+1}r(\lambda)] \cdot (\lambda-b)^{-1} \\ -i[a_{2p+1} + b_{2p+1}r(\lambda)] \cdot (\lambda-a)^{-1} & b_{2p+1}t(\lambda) \end{pmatrix}. \end{aligned} \quad (1.47)$$

Similarly to (1.45), one has

$$G_Y(\lambda) = I_2 + \frac{G_1^{(b)}(\lambda)}{m} + O(m^{-2}), \quad (1.48)$$

with

$$G_1^{(b)}(\lambda) = -(0,1) \frac{(2r(\lambda)-1)}{(\lambda-b)u(\lambda)} D_\infty^{-\sigma_3} U^{-1} \sigma^+ U D_\infty^{\sigma_3} + G_{1;\text{reg}}^{(b)}(\lambda). \quad (1.49)$$

Here, again, $G_{1;\text{reg}}^{(b)}(\lambda)$ is a holomorphic matrix on some sufficiently small neighbourhood of b .

1.6.2 Asymptotic resolution of the Riemann–Hilbert problem for Υ

The Riemann–Hilbert problem for Υ is equivalent to the singular integral equation [1]

$$\Upsilon_+(\lambda) = I_2 + \int_{\Sigma_Y} \Upsilon_+(s) \frac{G_Y(s) - I_2}{(\lambda_+ - s)} \cdot \frac{ds}{2i\pi}, \quad \Sigma_Y = \{-\partial \mathcal{D}_{a,\delta}\} \cup \{-\partial \mathcal{D}_{b,\delta}\}, \quad (1.50)$$

which, due to the bound $G_Y - I_2 = O(m^{-1})$ which holds in $(L^1 \cap L^\infty)(\Sigma_Y)$, can be solved in terms of a Neumann series [2]. Recalling that the solution to the Riemann–Hilbert problem stated in Subsection 1.6 admits the representation

$$\Upsilon(\lambda) = I_2 + \int_{\Sigma_Y} \Upsilon_+(s) \frac{G_Y(s) - I_2}{(\lambda - s)} \cdot \frac{ds}{2i\pi}, \quad (1.51)$$

it follows that, uniformly away from $\mathcal{D}_{a,\delta} \cup \mathcal{D}_{b,\delta}$,

$$\Upsilon(\lambda) = I_2 + \frac{(0,1)}{m} D_\infty^{-\sigma_3} U^{-1} \left\{ \frac{\sigma^-}{(\lambda-a)u(a)} + \frac{\sigma^+}{(\lambda-b)u(b)} \right\} U D_\infty^{\sigma_3} + O(m^{-2}), \quad (1.52)$$

with a remainder that holds in the $L^\infty(\mathbb{C} \setminus \overline{\mathcal{D}_{a,2\delta}} \cup \overline{\mathcal{D}_{b,2\delta}})$ sense since G_Y is holomorphic in the neighbourhood of Σ_Y .

In view of the definition (1.37), it holds

$$\Xi(\lambda) = \left(I_2 + \frac{(0, 1)}{m} D_\infty^{-\sigma_3} U^{-1} \left\{ \frac{\sigma^-}{(\lambda - a)u(a)} + \frac{\sigma^+}{(\lambda - b)u(b)} \right\} U D_\infty^{\sigma_3} + O(m^{-2}) \right) \cdot M(\lambda), \quad (1.53)$$

where the $O(m^{-2})$ remainder holds in the $L^\infty(\mathbb{C} \setminus \overline{\mathcal{D}}_{a,2\delta} \cup \overline{\mathcal{D}}_{b,2\delta})$ sense.

Thus, one is led to

$$\chi(\lambda) = e^{-mh_\infty \sigma_3} \left\{ I_2 + \frac{(0, 1)}{m} D_\infty^{-\sigma_3} U^{-1} \left\{ \frac{\sigma^-}{(\lambda - a)u(a)} + \frac{\sigma^+}{(\lambda - b)u(b)} \right\} U D_\infty^{\sigma_3} + O(m^{-2}) \right\} \cdot M(\lambda) \cdot e^{mh(\lambda) \sigma_3}, \quad (1.54)$$

for $\lambda \in \mathbb{C} \setminus \overline{\mathcal{D}}_{a,2\delta} \cup \overline{\mathcal{D}}_{b,2\delta}$.

2 The determinant

In the present section, we build on the asymptotic expansion obtained in the previous section so as to obtain the leading asymptotics of $\det[I + V]$. Our proof builds on certain differential identities that have been first established in [13].

2.1 Differential identities for the determinant

Let the function p arising in the definition of the kernel (0.1) depend smoothly on an auxiliary variable t : $p(\lambda) \hookrightarrow p(\lambda, t)$, this in such a way that $\lambda \mapsto \partial_t p(\lambda, t) \in O(U)$ uniformly in $t \in [0; 1]$. Let V_t denote the associated kernel and $\chi(\lambda, t)$ the solution to the corresponding Riemann–Hilbert problem.

Then, by repeating the handlings developed in [13], one obtains the representation

$$\partial_t \ln \det [I + V_t] = m \oint_{\Gamma} \operatorname{tr} \left[\partial_\lambda \chi(\lambda, t) \cdot \sigma_3 \cdot \chi^{-1}(\lambda, t) \right] \cdot \frac{\partial_t p(\lambda, t)}{4\pi} \cdot d\lambda. \quad (2.1)$$

Above Γ refers to a small counterclockwise loop around $[a; b]$ that lies in U .

2.2 Asymptotic behaviour of $\det [I + V]$

Proposition 2.1 *The large- m asymptotics of the Fredholm determinant associated with the integral operator $I + V$ takes the form*

$$\begin{aligned} \ln \det [I + V] = m^2 \int_a^b \frac{2\lambda\mu + 2ab - (a+b)(\lambda+\mu)}{\sqrt{(\lambda-a)(b-\lambda)(\mu-a)(b-\mu)}} \cdot \left(\frac{p(\lambda) - p(\mu) - i[g(\lambda) - g(\mu)]/m}{\lambda - \mu} \right)^2 \cdot \frac{d\lambda d\mu}{(4\pi)^2} \\ - \frac{1}{4} \ln m + \frac{1}{8} \ln \left(\frac{16u(a)u(b)}{(b-a)^2} \right) + \frac{\ln 2}{12} + 3\zeta'(-1) + O(m^{-1}). \end{aligned} \quad (2.2)$$

Proof —

In order to get the leading asymptotic behaviour of the determinant one can, for the purpose of the intermediate calculations¹ set $g = 0$. In such a setting, one has $D = 0$. The whole dependence on g in the leading asymptotics can then be recovered from the replacement $p \hookrightarrow p - ig/m$. We shall build on the representation (2.1) so as to

¹It is of course not a problem to keep $g \neq 0$ throughout all of the below calculation.

interpolate, in the large- m regime, between the large- m asymptotics of a known case -the pure sine kernel which correspond to $p(\lambda) = \lambda$ - and our case of general holomorphic p . Thus, we introduce

$$\lambda \mapsto p(\lambda, t) = tp(\lambda) + (1-t)\lambda, \quad (2.3)$$

This function is holomorphic in the open neighbourhood U of $[a; b]$ and it further satisfies

$$\partial_\lambda p(\lambda, t) > \min(1, \inf_{[a; b]} \partial_\lambda p(\lambda)), \quad (2.4)$$

all this uniformly in $t \in [0; 1]$. As a consequence, the previous analysis can be carried out for any value of $t \in [0; 1]$. Furthermore, all the remainders will be uniform in $0 \leq t \leq 1$. Note that dealing with a t -dependent function p will generate an additional t -dependence in the auxiliary functions h and u , viz.:

$$h(\lambda, t) = q(\lambda) \int_a^b \frac{-ip(\lambda, t)}{q_+(s)(s-\lambda)} \cdot \frac{ds}{2i\pi} \quad \text{and} \quad u(\lambda, t) = \int_a^b \frac{p(\lambda, t) - p(s, t)}{\pi(s-\lambda)} \cdot \frac{ds}{\sqrt{(s-a)(b-s)}}. \quad (2.5)$$

In order to calculate *rhs* of (2.1) asymptotically in m , we first recast

$$\text{tr}[\partial_\lambda \chi(\lambda, t) \cdot \sigma_3 \cdot \chi^{-1}(\lambda, t)] \quad (2.6)$$

in terms of the objects arising in the various steps of the transformations carried out on the initial RHP for χ .

Using that, uniformly away from the endpoints a, b , one has

$$\chi(\lambda, t) = e^{-mh_\infty(t)\sigma_3} \cdot \Upsilon(\lambda, t) \cdot M(\lambda) \cdot e^{mh(\lambda, t)\sigma_3}, \quad (2.7)$$

we are led to

$$\begin{aligned} \text{tr}[\partial_\lambda \chi(\lambda, t) \cdot \sigma_3 \cdot \chi^{-1}(\lambda, t)] &= 2m\partial_\lambda h(\lambda, t) + \text{tr}[\partial_\lambda M(\lambda) \cdot \sigma_3 \cdot M^{-1}(\lambda)] \\ &\quad + \text{tr}[(\partial_\lambda \Upsilon(\lambda, t)) \cdot M(\lambda) \cdot \sigma_3 \cdot M^{-1}(\lambda) \cdot \Upsilon(\lambda, t)]. \end{aligned} \quad (2.8)$$

Furthermore, as a consequence of

$$U\sigma_3 U^{-1} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \quad (2.9)$$

one gets that

$$\text{tr}[\partial_\lambda M(\lambda) \cdot \sigma_3 \cdot M^{-1}(\lambda)] = \frac{1}{4} \left\{ \frac{1}{\lambda-a} - \frac{1}{\lambda-b} \right\} \cdot \text{tr}[U^{-1}\sigma_3 U\sigma_3] = 0. \quad (2.10)$$

Finally, the leading in m asymptotics of Υ

$$\Upsilon(\lambda, t) = I_2 + \frac{(0, 1)}{m} U^{-1} \left\{ \frac{\sigma^-}{(\lambda-a)u(a, t)} + \frac{\sigma^+}{(\lambda-b)u(b, t)} \right\} U + O(m^{-2}), \quad (2.11)$$

yield

$$\begin{aligned} \text{tr}[(\partial_\lambda \Upsilon(\lambda, t)) \cdot M(\lambda) \cdot \sigma_3 \cdot M^{-1}(\lambda) \cdot \Upsilon(\lambda, t)] \\ = -\frac{i(0, 1)}{m} \cdot \left[\frac{1}{(\lambda-b)^{\frac{3}{2}}(\lambda-a)^{\frac{1}{2}}u(b, t)} - \frac{1}{(\lambda-a)^{\frac{3}{2}}(\lambda-b)^{\frac{1}{2}}u(a, t)} \right] + O(m^{-2}). \end{aligned} \quad (2.12)$$

As a consequence, we are led to the representation

$$\partial_t \ln \det[I + V_t] = \frac{2m^2}{4\pi} \oint_{\Gamma} \partial_t p(\lambda, t) \cdot \partial_\lambda h(\lambda, t) \cdot d\lambda + C_t + O\left(\frac{1}{m}\right), \quad (2.13)$$

where the remainder is uniform in $t \in [0; 1]$ and the constant C_t reads

$$C_t = \frac{i(0, 1)}{4\pi} \oint_{\Gamma} \left\{ \frac{\partial_t p(\lambda, t)}{u(a, t)(\lambda - a)^{\frac{3}{2}}(\lambda - b)^{\frac{1}{2}}} - \frac{\partial_t p(\lambda, t)}{u(b, t)(\lambda - b)^{\frac{3}{2}}(\lambda - a)^{\frac{1}{2}}} \right\} \cdot d\lambda. \quad (2.14)$$

The constant C_t can be estimated as follows. Since the residue at ∞ does not contribute, one can regularise the behaviour of the integrand in the vicinities of a and b leading to

$$\begin{aligned} C_t &= -\frac{i(0, 1)}{2\pi u(a, t)} \int_a^b \frac{\partial_t p(\lambda, t) - \partial_t p(a, t)}{(\lambda - a)^{\frac{3}{2}}(\lambda - b)^{\frac{1}{2}}} \cdot d\lambda + \frac{i(0, 1)}{2\pi u(b, t)} \int_a^b \frac{\partial_t p(\lambda, t) - \partial_t p(b, t)}{(\lambda - a)^{\frac{1}{2}}(\lambda - b)^{\frac{3}{2}}} \cdot d\lambda \\ &= \frac{1}{8u(a, t)} \partial_t u(a, t) + \frac{1}{8u(b, t)} \partial_t u(b, t) = \frac{1}{8} \partial_t \ln [u(a, t)u(b, t)]. \end{aligned} \quad (2.15)$$

Thus, it solely remains to recast the first term in (2.13). Starting with the integral representation (1.10) for the function h and upon taking the derivatives explicitly, we get

$$\oint_{\Gamma} \partial_t p(\lambda, t) \cdot \partial_\lambda h(\lambda, t) \cdot \frac{d\lambda}{2\pi} = \frac{1}{16\pi^2} \oint_{\Gamma} d\lambda \oint_{\Gamma'} ds \frac{\partial_t p(\lambda, t) \cdot p(s, t)}{(s - \lambda)^2 \cdot q(\lambda) \cdot q(s)} \cdot [2\lambda s + 2ab - (a + b)(\lambda + s)], \quad (2.16)$$

where we agree that Γ and Γ' are two loops around $[a; b]$ such that $\Gamma' \subset \Gamma$. We could have of course started with the representation (1.11) for h . Then, the part involving $-ip(\lambda, t)/2$ produces a vanishing contribution by squeezing the contour Γ to 0. Further, one may get rid of the contribution to $h(\lambda, t)$ involving $p(\lambda, t)$ under the integral sign by deforming the s -contour in (1.11) to infinity. Then, one is led to exactly the same integral representation as above, with the sole difference that the contours Γ and Γ' are interchanged, *i.e.*

$$\oint_{\Gamma} \partial_t p(\lambda, t) \cdot \partial_\lambda h(\lambda, t) \cdot \frac{d\lambda}{2\pi} = \frac{1}{16\pi^2} \oint_{\Gamma'} d\lambda \oint_{\Gamma} ds \frac{\partial_t p(\lambda, t) \cdot p(s, t)}{(s - \lambda)^2 \cdot q(\lambda) \cdot q(s)} \cdot [2\lambda s + 2ab - (a + b)(\lambda + s)]. \quad (2.17)$$

Hence, we get that

$$\begin{aligned} \oint_{\Gamma} \partial_t p(\lambda, t) \cdot \partial_\lambda h(\lambda, t) \cdot \frac{d\lambda}{2\pi} &= \frac{1}{32\pi^2} \frac{\partial}{\partial t} \oint_{\Gamma} d\lambda \oint_{\Gamma'} ds \frac{p(\lambda, t) \cdot p(s, t)}{(s - \lambda)^2 \cdot q(\lambda) \cdot q(s)} \cdot [2\lambda s + 2ab - (a + b)(\lambda + s)] \\ &= -\frac{1}{64\pi^2} \frac{\partial}{\partial t} \oint_{\Gamma} d\lambda \oint_{\Gamma'} ds \frac{2\lambda s + 2ab - (a + b)(\lambda + s)}{q(\lambda) \cdot q(s)} \cdot \left\{ \left(\frac{p(s, t) - p(\lambda, t)}{s - \lambda} \right)^2 - \frac{p^2(s, t) + p^2(\lambda, t)}{(s - \lambda)^2} \right\} \\ &= -\frac{1}{16\pi^2} \frac{\partial}{\partial t} \int_a^b \frac{2\lambda s + 2ab - (a + b)(\lambda + s)}{q_+(\lambda) \cdot q_+(s)} \cdot \left(\frac{p(s, t) - p(\lambda, t)}{s - \lambda} \right)^2 \cdot d\lambda ds + G, \end{aligned} \quad (2.18)$$

The first part is already of the desired form, whereas G is given by

$$G = \frac{1}{64\pi^2} \frac{\partial}{\partial t} \oint_{\Gamma} d\lambda \oint_{\Gamma'} ds \frac{2\lambda s + 2ab - (a+b)(\lambda+s)}{q(\lambda) \cdot q(s) \cdot (s-\lambda)^2} [p^2(s, t) + p^2(\lambda, t)] . \quad (2.19)$$

Due to the inclusion of contours $\Gamma' \supset \Gamma$, the integral involving $p^2(\lambda, t)$ yields 0 by taking the residue in the s -integral at infinity. Now, in order to estimate the integral involving $p^2(s, t)$, we deform the contour Γ up to ∞ . Again the residue at ∞ does not contribute, but we have to take into account the residue at $\lambda = s$. All of these manipulations recast G as

$$G = \frac{-i}{32\pi} \frac{\partial}{\partial t} \oint_{\Gamma} ds \left\{ 2s - a - b - \frac{1}{2} \left(\frac{1}{s-a} + \frac{1}{s-b} \right) 2(s^2 - s(a+b) + ab) \right\} \frac{p^2(s, t)}{q^2(s)} = 0 . \quad (2.20)$$

All in all, by putting all the pieces of the analysis together, we are led to the representation

$$\begin{aligned} \partial_t \ln \det[I + V_t] &= \frac{m^2}{16\pi^2} \frac{\partial}{\partial t} \int_a^b \frac{2\lambda s + 2ab - (a+b)(\lambda+s)}{\sqrt{(\lambda-a)(b-\lambda)(s-a)(b-s)}} \cdot \left(\frac{p(s, t) - p(\lambda, t)}{s - \lambda} \right)^2 \cdot d\lambda ds \\ &\quad + \frac{1}{8} \frac{\partial}{\partial t} \ln [u(a, t)u(b, t)] + O(m^{-1}) . \end{aligned} \quad (2.21)$$

Thus integrating from $t = 0$ up to $t = 1$, inserting the large- m asymptotic behaviour of the pure sine kernel [6, 7, 17]

$$\ln \det[I + V_0] = -\frac{(b-a)^2}{32} m^2 - \frac{1}{4} \ln \left[\frac{b-a}{4} \cdot m \right] + \frac{\ln 2}{12} + 3\zeta'(-1) + O(m^{-1}) , \quad (2.22)$$

and using that

$$\int_a^b \frac{ds}{\pi \sqrt{(s-a)(b-s)}} = 1 \quad \text{along with} \quad \int_a^b \frac{2\lambda s + 2ab - (a+b)(\lambda+s)}{\sqrt{(\lambda-a)(b-\lambda)(s-a)(b-s)}} \cdot d\lambda ds = -\frac{\pi^2}{2} (b-a)^2 , \quad (2.23)$$

we are led to the claim. ■

3 Emptiness formation probability in the XXZ-spin 1/2 Heisenberg chain.

In this section, as it has already been indicated in the introduction, we will consider one special example of the generalised sine kernel which has a particular importance in the theory of the XXZ spin-1/2 Heisenberg chain.

Let V be the trace class integral operator acting on $L^2(\Gamma_\alpha)$, where Γ_α is the arc,

$$|\lambda| = 1, \quad -\alpha < \arg \lambda < \alpha \quad 0 < \alpha < \pi,$$

traversed counterclockwise. The operator's kernel is given by

$$V(\lambda, \mu) = -\frac{1}{2\pi i(\lambda - \mu)} \left(\lambda^{m/2} \mu^{-m/2} e^{t(\frac{\phi(\lambda) - \phi(\mu)}{2})} - \lambda^{-m/2} \mu^{m/2} e^{t(\frac{\phi(\mu) - \phi(\lambda)}{2})} \right) \quad (3.1)$$

where, as before, m is a positive integer, t is a real parameter, and the function $\phi(\lambda)$ is assumed to be analytic in the neighbourhood of the arc Γ_α . In the following, we explain the connection of this kernel to the XXZ spin-1/2 Heisenberg chain.

The XXZ spin-1/2 Heisenberg chain of size N is determined by the Hamiltonian,

$$H_{XXZ} = \sum_{n=1}^N \left(\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \Delta (\sigma_n^z \sigma_{n+1}^z - 1) - h \sigma_n^z \right), \quad (3.2)$$

where the periodic boundary conditions are assumed. In (3.2), $\sigma^x, \sigma^y, \sigma^z$ are Pauli matrices, $h, 0 \leq h < 4(1 + \Delta)$, is an external (moderate) magnetic field, and Δ is the anisotropy parameter which takes the values $-1 < \Delta < 1$. At the point $\Delta = 0$, the model becomes the free fermionic XX0 spin chain.

One of the principal objects of the analysis of the XXZ model is the *emptiness formation probability* (EFP) which is defined at zero temperature as the correlation function,

$$P^{(N)}(m) = \langle \psi_g, \prod_{j=1}^m \frac{\sigma_j^z + 1}{2} \psi_g \rangle. \quad (3.3)$$

The physical meaning of $P^{(N)}(m)$ is the probability of finding a string of m adjacent parallel spins up (i.e., a piece of the *ferromagnetic* state) in the antiferromagnetic ground state ψ_g for a given value of the magnetic field h . We shall denote,

$$P(m) := \lim_{N \rightarrow \infty} P^{(N)}(m), \quad (3.4)$$

the emptiness formation probability in the thermodynamic limit. The existence of this limit follows from the works [11, 15]. The principal analytical question is the large m behaviour of $P(m)$.

At the free fermionic case, when $\Delta = 0$, the EFP is given by the explicit determinant formula involving the integral operator (3.1). Indeed, one has that

$$P(m)_{|\Delta=0} = \det \left[I + V \right] \Big|_{l=0}. \quad (3.5)$$

The Fredholm determinant in the right hand side of this formula can also be expressed as a Toeplitz determinant whose symbol is the characteristic function of the complimentary arc, $C \setminus \Gamma_\alpha$. The large m asymptotics of this determinant was obtained in the classical work by Widom [18] and it reads (see also [6, 17] for the error estimate),

$$P(m) = m^2 \ln \cos \frac{\alpha}{2} - \frac{1}{4} \ln \left(m \sin \frac{\alpha}{2} \right) + c_0 + O\left(\frac{1}{m}\right), \quad m \rightarrow \infty, \quad (3.6)$$

where the constant c_0 is the famous Widom's constant

$$c_0 = \frac{1}{12} \ln 2 + 3\zeta'(-1).$$

There exists a Fredholm determinant representation for the EFP in the general XXZ case as well. A remarkable fact is that this representation also involves the operator V but this time with $t \neq 0$. The exact formula relating $P(m)$ and V for $\Delta \neq 0$ was extracted by N. Slavnov by using the results obtained in [12]. The function $\phi(\lambda)$ in Slavnov's formula, however, is not a scalar function, but is in fact a *dual quantum field* acting in an auxiliary bosonic Fock space with vacuum $|0\rangle$. Indeed, Slavnov's representation takes the form

$$P(m) = \langle 0 | C(\varphi) \cdot \frac{\det[I + V]}{\det[I + \frac{1}{2\pi}K]} | 0 \rangle, \quad (3.7)$$

where the integral operator K and the quantity $C(\varphi)$, which is also expressed in terms of certain Fredholm determinants, do not depend on m . The constant $C(\varphi)$ as well as the kernel $V(\lambda, \mu)$ depend on the dual fields $\varphi(\lambda)$ and $\phi(\lambda)$. The dual fields commute for all values of spectral parameter λ . Their contribution to the expectation value (3.7) is obtained through the averaging procedure which suggests the decomposition of the dual fields on the relevant creation and annihilation parts and then moving all exponentials of annihilation operators to the right, picking up contributions whenever passing by a creation operator.

The general strategy of using Slavnov's formula (3.7) can be formulated as the following two step procedure. The first step consists in finding the large m asymptotics of $\det[I + V]$ when treating $\phi(\lambda)$ as a usual function. The second step would consist in averaging of the asymptotic formulae obtained in the first step over the dual field vacuum². In this section of the current work we will pass through the first step.

The kernel (3.1) is of the type (0.1) with

$$e(\lambda) = \lambda^{\frac{m}{2}} e^{\frac{t\phi(\lambda)}{2}},$$

and with the contour of integration being the arc Γ_α instead of the interval $[a; b]$. Formally, the results of the previous sections are not directly applicable to kernel (3.1). However, as we will see in Section 3.3.1 (see Remark 3.1), one can map this kernel to a kernel having exactly the structure given in (0.1). Hence, in principle, it is possible to use the general asymptotic results of Section 1.

At the same time, the kernel (3.1) is very close to the integrable kernel studied in [6]. Indeed, the latter is the particular case of the former corresponding $\phi(\lambda) \equiv 0$ (or, $t = 0$). Moreover, as we will see below, most of the results and the constructions of [6], after some minimal modification, can be used in the case $\phi(\lambda) \neq 0$ ($t > 0$). This observation allows us to simplify greatly the evaluation of the large m asymptotics of $\det[I + V]$. Basically, the only additional analytical ingredient which is needed is the relevant differential identity for $\det[I + V]$ and some modifications in constructing global and local parametrices of the solution of the corresponding Riemann-Hilbert problem. The mentioned differential identity and parametrix constructions can be extracted from the general analysis of Section 1.

3.1 The χ - RH problem

Although not exactly of the form (0.1), the kernel (3.1) is still of integrable type. Therefore the arguments of Section 1.1 are applicable and we can associate with this kernel the Riemann-Hilbert problem which consists in finding the 2×2 matrix valued function $\chi(\lambda)$ satisfying the following properties:

- $\chi(\lambda) \in O(\mathbb{C} \setminus \Gamma_\alpha)$ and has continuous boundary values $\chi_\pm(\lambda)$ on $\Gamma_\alpha \setminus \{e^{\pm i\alpha}\}$;
- $\chi_-(\lambda) = \chi_+(\lambda)G_\chi(\lambda)$ for $\lambda \in \Gamma_\alpha \setminus \{e^{\pm i\alpha}\}$, where

$$G_\chi(\lambda) = I_2 + 2i\pi \mathbf{E}_R(\lambda) \cdot \mathbf{E}_L^T(\lambda) = \begin{pmatrix} 2 & -\lambda^m e^{t\phi(\lambda)} \\ \lambda^{-m} e^{-t\phi(\lambda)} & 0 \end{pmatrix}; \quad (3.8)$$

- $\chi(\lambda) = \log |\lambda - e^{\pm i\alpha}| \cdot O \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ as $\lambda \rightarrow e^{\pm i\alpha}$;
- $\chi(\infty) = I_2$.

²This strategy had already been used in the two point correlation function in the case of the 1D Bose gas at the finite coupling [9, 20] - another fundamental non-free fermion model.

As in Section 1.1 (cf. (1.4)), the unique solution of this Riemann-Hilbert problem, which we will from now on call χ - RH problem, admits the Cauchy representations,

$$\chi(\lambda) = I_2 - \int_{\Gamma_\alpha} \frac{F_R(\mu)E_L^T(\mu)}{\mu - \lambda} d\mu, \quad \text{and} \quad \chi^{-1}(\lambda) = I_2 + \int_{\Gamma_\alpha} \frac{E_R(\mu)F_L^T(\mu)}{\mu - \lambda} d\mu, \quad (3.9)$$

in terms of the solutions, $F_R(\lambda)$ and $F_L(\lambda)$ of the linear integral equations

$$F_R(\lambda) + \int_{\Gamma_\alpha} V(\mu, \lambda) F_R(\mu) d\mu = E_R(\lambda), \quad \text{and} \quad F_L(\lambda) + \int_{\Gamma_\alpha} V(\lambda, \mu) F_L(\mu) d\mu = E_L(\lambda).$$

Conversely, the vector functions $F_R(\lambda)$ and $F_L(\lambda)$ are given in terms of $\chi(\lambda)$ by the equations,

$$F_R(\lambda) = \chi(\lambda)E_R(\lambda), \quad \text{and} \quad F_L^T(\lambda) = E_L^T(\lambda)\chi^{-1}(\lambda). \quad (3.10)$$

3.2 The differential identities for the determinant

It has already been noticed that the kernel (3.1) is not exactly of the form (0.1). However, one can repeat the arguments of [13] and arrive at an analogous formula to (2.1) for the logarithmic derivative of the determinant $\det[I + V]$. Notice that in the case of kernel (3.1) we have

$$p(\lambda, t) = -i \ln \lambda - \frac{it}{m} \phi(\lambda),$$

and therefore the general identity (2.1) is specified to the identity

$$\partial_t \ln \det[I + V] = \frac{1}{4\pi i} \int_C \phi(\lambda) \text{tr}[\partial_\lambda \chi(\lambda, t) \cdot \sigma_3 \cdot \chi^{-1}(\lambda, t)] d\lambda. \quad (3.11)$$

Here, C is a small counterclockwise loop around the arc Γ_α .

Formula (3.11) reduces the asymptotic evaluation of $\det[I + V]$ to the evaluation of the uniform in t asymptotics of the solution of the χ - RH problem and the calculation of $\det[I + V]|_{t=0}$ which was achieved in [6].

3.3 The Riemann-Hilbert analysis

The goal of this section is to produce the asymptotic solution of the χ - RH problem. This Riemann-Hilbert problem is very close to the Riemann-Hilbert problem that was studied in [6]. In fact, if we put $\phi(\lambda) \equiv 0$, then $\chi(\lambda)$ will be the solution of the Riemann-Hilbert problem whose asymptotics has been obtained in [6] (the m - RH problem of [6]). It turns out that the presence of the nontrivial phase function $\phi(\lambda)$ does not affect the analysis of [6] much, so that we will be able to use most of the results obtained in the case $\phi(\lambda) \equiv 0$ and to shorten our analysis considerably. In the rest of this section we follow the steps used in [6] with proper technical modifications which we will handle with the help of the general analysis of Section 1.

3.3.1 Mapping onto a fixed interval

Similar to Section 3.1 of [6], we define the linear fractional transformation $z(\lambda)$ by the formulae

$$z(\lambda) = -i \cot \frac{\alpha}{2} \frac{\lambda - 1}{\lambda + 1}, \quad \text{viz.} \quad \lambda(z) = \frac{1 + iz \tan \frac{\alpha}{2}}{1 - iz \tan \frac{\alpha}{2}}. \quad (3.12)$$

This change of variable transforms the χ -RH problem to the following Riemann-Hilbert problem which we call the Y -RH problem posed on the interval $(-1; 1)$, traversed from -1 to 1 :

- $Y(z) \in \mathcal{O}(\mathbb{C} \setminus [-1; 1])$;
- $Y_-(z) = Y_+(z)G_Y(z)$, $z \in (-1; 1)$, where

$$G_Y(z) = \begin{pmatrix} 2 & -\left(\frac{1+iz \tan(\frac{\alpha}{2})}{1-iz \tan(\frac{\alpha}{2})}\right)^m e^{t\phi(\lambda(z))} \\ \left(\frac{1+iz \tan(\frac{\alpha}{2})}{1-iz \tan(\frac{\alpha}{2})}\right)^{-m} e^{-t\phi(\lambda(z))} & 0 \end{pmatrix}; \quad (3.13)$$

- $Y(\lambda) = \log |z \mp 1| \cdot \mathcal{O} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, as $z \rightarrow \pm 1$;
- $Y(\infty) = I_2$.

Once we have the solution $Y(z; m, t)$ of the Y -RH problem, we can find the solution $\chi(\lambda; m, t)$ of the χ -RH problem according to the equation

$$\chi(\lambda; m, t) = \left(Y(-i \cot \frac{\alpha}{2}; m, t) \right)^{-1} Y(z(\lambda); m, t) \quad (3.14)$$

Remark 3.1 One can notice that up to the replacement, $z \rightarrow \lambda$, the Y - problem is a particular case of the generalized sine kernel χ - problem (1.3) with the following specifications,

$$p(z) = -i \ln \frac{1 + iz \tan \frac{\alpha}{2}}{1 - iz \tan \frac{\alpha}{2}}, \quad \text{and} \quad g(z) = t\phi(\lambda(z)).$$

3.3.2 r - function transformation

Following [6] again, we introduce the r -function (this is the g - function of [6] - see equation (24) there),

$$r(z) := \frac{1 + i \sqrt{z^2 - 1} \sin(\alpha/2)}{1 + iz \tan(\alpha/2)}. \quad (3.15)$$

The branch of the square root is fixed by the condition

$$\sqrt{z^2 - 1} \sim z, \quad z \rightarrow \infty.$$

Let us list the key properties of the r -function (cf. Section 3.2 of [6]) :

- (i) $r(z)$ is holomorphic for all $z \notin [-1; 1]$.
- (ii) $r(z) \neq 0$ for all $z \notin [-1; 1]$. At the points $z = -i \cot(\alpha/2)$ (or $z = \infty$) and $z = i \cot(\alpha/2)$ (or $z = 0$) the values of the function $r(z)$ are :

$$r(-i \cot(\alpha/2)) = 1, \quad \text{and} \quad r(i \cot(\alpha/2)) = \cos^2(\alpha/2) \equiv \kappa. \quad (3.16)$$

- (iii) The boundary values $r_{\pm}(z)$, $z \in [-1; 1]$ satisfy the following equations :

$$r_+(z)r_-(z) = \kappa \frac{1 - iz \tan(\alpha/2)}{1 + iz \tan(\alpha/2)},$$

and

$$\frac{r_+(z)}{r_-(z)} = \frac{1 - \sqrt{1 - z^2} \sin(\alpha/2)}{1 + \sqrt{1 - z^2} \sin(\alpha/2)}.$$

Here,

$$0 < \sqrt{1 - z^2} \equiv -i \lim_{\epsilon \rightarrow 0^+} \sqrt{(z + i\epsilon)^2 - 1}, \quad z \in (-1; 1),$$

is the "+" boundary value of the function $\sqrt{z^2 - 1}$ on the segment $(-1; 1)$, oriented from left to right. This, in particular, means that for any fixed $0 < \delta < 1$, the following inequality holds

$$\left| \frac{r_+}{r_-} \right| \leq \varepsilon_0 < 1, \quad z \in [-1 + \delta, 1 - \delta] \quad (3.17)$$

for some $\varepsilon_0 = \varepsilon_0(\delta) > 0$.

(iv) The behaviour of $r(z)$ as $z \rightarrow \infty$ is described by the asymptotic relation

$$r(z) = \cos(\alpha/2) + O\left(\frac{1}{z}\right).$$

These properties suggest to transform the original Riemann-Hilbert problem by the formula,

$$Y(z) \rightarrow \Phi(z) \equiv Y(z) r^{-m\sigma_3}(z) \kappa^{\frac{m}{2}\sigma_3}. \quad (3.18)$$

The matrix valued function $\Phi(z) \equiv \Phi(z; m, t)$ is the solution of the following Riemann-Hilbert problem, which we call the Φ -RH problem :

- $\Phi(z) \in O(\mathbb{C} \setminus [-1; 1])$;
- $\Phi_-(z) = \Phi_+(z) G_\Phi(z)$, $z \in (-1; 1)$, where

$$G_\Phi(z) = \begin{pmatrix} 2(r_+(z)/r_-(z))^m & -e^{t\phi(\lambda(z))} \\ e^{-t\phi(\lambda(z))} & 0 \end{pmatrix}; \quad (3.19)$$

- $\Phi(z) = O(\log |z \mp 1|)$, as $z \rightarrow \pm 1$;
- $\Phi(\infty) = I_2$.

Our original problem is now reduced to the asymptotic solution of the Φ -RH problem.

Remark 3.2 One can notice that the Φ -RH problem is a special case of the Ξ -RH problem (1.15) considered in the main body of the paper, and that the transition of the Y -RH problem (3.13) to the Φ -RH problem (3.19) which we performed following [6], is a particular case of the transition of the general χ -RH problem (1.3) to the Ξ -RH problem (1.15) done in Section 1.2. One can also check that the function $h(\lambda)$ associated with the problem (3.13) is related to the r -function (3.15) by the equation,

$$h(z) = \ln r(z) - \ln \cos \frac{\alpha}{2} \equiv \ln \frac{1 + i \sqrt{z^2 - 1} \sin(\alpha/2)}{1 + iz \tan(\alpha/2)} - \ln \cos \frac{\alpha}{2}.$$

3.3.3 Global parametrix

In virtue of estimate (3.17), we have that, for every $z \in (-1; 1)$,

$$G_\Phi(z) \rightarrow \begin{pmatrix} 0 & -e^{t\phi(\lambda(z))} \\ e^{-t\phi(\lambda(z))} & 0 \end{pmatrix},$$

as $m \rightarrow \infty$. Hence, similar to the general generalised sine-kernel χ -RH problem, one expects that $\Phi(z)$ is approximated by its global parametrix, $M(z)$, which is the solution of the following RHP:

- $M \in O(\mathbb{C} \setminus [-1; 1])$ and has continuous boundary values on $(-1; 1)$;
- $M(z) = |(z^2 - 1)|^{-\frac{1}{4}} \cdot O \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, as $z \rightarrow \pm 1$;
- $M(z) = I_2 + O(z^{-1})$, as $z \rightarrow \infty$;
- $M_-(z) = M_+(z) \cdot G_M(z)$ for $z \in (-1; 1)$ where

$$G_M(z) = \begin{pmatrix} 0 & -e^{i\phi(\lambda(z))} \\ e^{-i\phi(\lambda(z))} & 0 \end{pmatrix}.$$

This is, of course, exactly the M - RH problem (1.21) from Section 1.3 with the specifications $b = -a = 1$, and whose solution is given by the equation (cf. (1.20)),

$$M(z) = D_\infty^{-\sigma_3} \cdot N(z) \cdot D^{\sigma_3}(z), \quad (3.20)$$

where, given U as in (1.19),

$$N(z) = U^{-1} \cdot \left(\frac{z+1}{z-1} \right)^{\frac{\sigma_3}{4}} \cdot U, \quad (3.21)$$

and (cf. (1.16)),

$$D(z) = e^{-i\eta(z)}, \quad \text{and} \quad D_\infty = e^{-i\eta_\infty}, \quad (3.22)$$

with

$$\eta(z) = -\frac{\sqrt{z^2 - 1}}{2\pi} \int_{-1}^1 \frac{\phi(\lambda(s))}{\sqrt{1 - s^2}(s - z)} ds, \quad (3.23)$$

and

$$\eta_\infty = \frac{1}{2\pi} \int_{-1}^1 \frac{\phi(\lambda(z))}{\sqrt{1 - z^2}} dz. \quad (3.24)$$

The relation of the parametrix $M(z)$ to the exact solution of the Φ - RH problem is presented in the following theorem.

Theorem 3.1 *Let δ be a positive number less than $\frac{1}{4}$. Introduce the domain*

$$\Omega^{(\delta)} = \mathbb{C} \setminus \overline{\mathcal{D}_{1,\delta}} \cup \overline{\mathcal{D}_{-1,\delta}},$$

where $\mathcal{D}_{1,\delta}$ and $\mathcal{D}_{-1,\delta}$ are the open balls with radius δ centered at $z = 1$ and $z = -1$ respectively. Then for any $t_0 \geq 0$ and δ sufficiently small, there exist s_0 such that for all $m \geq s_0$, the solution of the Φ -RH problem uniquely exists and satisfies the estimate

$$\Phi(z; m, \alpha) = \left(I + O \left(\frac{1}{m(1 + |z|)} \right) \right) M(z), \quad \rho \rightarrow \infty, \quad (3.25)$$

uniformly for $z \in \Omega^{(\delta)}$ and $0 \leq t \leq t_0$.

The statement of this theorem is just a particular case of the asymptotic formula (1.53) which was proven in the main body of the paper for the solution of the generalised sine - kernel Riemann-Hilbert problem (1.3). Alternatively, the proof can be produced by literally repeating the proof given in [6] for the case $\phi(\lambda) \equiv 0$. One would only need to use the more general local parametrices at the points $z = 1$ and $z = -1$ which in turn can be taken as special cases of the parametrices constructed above in Sections 1.4 and 1.5.

3.4 Asymptotics of the determinant

We shall use the differential identity (3.11) and Theorem 3.1. Technically, this is simpler than to try to extract the result from our general formula (2.2) of Proposition 2.1. Tracing back the chain of transformations that led us from the original function $\chi(\lambda)$ to the function $\Phi(z)$ we have that

$$\chi(\lambda) = \kappa^{\frac{m}{2}\sigma_3} \Phi^{-1}\left(-i \cot \frac{\alpha}{2}\right) \Phi(z(\lambda)) r^{m\sigma_3}(z(\lambda)) \kappa^{-\frac{m}{2}\sigma_3}. \quad (3.26)$$

This would yield the following expression for the product $\chi^{-1}(\lambda) \partial_\lambda \chi(\lambda)$ which is involved in the integral in the right hand side of (3.11),

$$\chi^{-1}(\lambda) \partial_\lambda \chi(\lambda) = m \partial_\lambda r(z(\lambda)) r^{-1}(z(\lambda)) \sigma_3 + \kappa^{\frac{m}{2}\sigma_3} r^{-m\sigma_3}(z(\lambda)) \Phi^{-1}(z(\lambda)) \partial_\lambda \Phi(z(\lambda)) r^{m\sigma_3}(z(\lambda)) \kappa^{-\frac{m}{2}\sigma_3},$$

and, in turn,

$$\text{tr}[\partial_\lambda \chi(\lambda, t) \cdot \sigma_3 \cdot \chi^{-1}(\lambda, t)] = 2m \partial_\lambda r(z(\lambda)) r^{-1}(z(\lambda)) + \text{tr}[\sigma_3 \Phi^{-1}(z(\lambda)) \partial_\lambda \Phi(z(\lambda))]. \quad (3.27)$$

As it follows from Theorem 3.1, on the loop C the function $\Phi(z(\lambda))$ can be approximated by the the global parametrix $M(z(\lambda))$. Indeed, from (3.25) we have that

$$\Phi(z(\lambda)) = \left(I_2 + O\left(\frac{1}{m}\right)\right) M(z(\lambda)), \quad m \rightarrow \infty, \quad (3.28)$$

where the estimate holds uniformly for $\lambda \in C$, and are differentiable with respect to λ . Combining (3.28) and (3.27) we conclude that

$$\text{tr}[\partial_\lambda \chi(\lambda, t) \cdot \sigma_3 \cdot \chi^{-1}(\lambda, t)] = 2m \partial_\lambda r(z(\lambda)) r^{-1}(z(\lambda)) + \text{tr}[\sigma_3 M^{-1}(z(\lambda)) \partial_\lambda M(z(\lambda))] + O\left(\frac{1}{m}\right). \quad (3.29)$$

Using the definition (3.20) of the global parametrix $M(z(\lambda))$, we derive from (3.29) the following asymptotic formula for the integrand (3.11) expressed in terms of the known objects,

$$\text{tr}[\partial_\lambda \chi(\lambda, t) \cdot \sigma_3 \cdot \chi^{-1}(\lambda, t)] = 2m \partial_\lambda r(z(\lambda)) r^{-1}(z(\lambda)) - 2t \partial_\lambda \eta(z(\lambda)) + \text{tr}[\sigma_3 N^{-1}(z(\lambda)) \partial_\lambda N(z(\lambda))] + O\left(\frac{1}{m}\right), \quad (3.30)$$

where the functions $\eta(z)$ and $N(z)$ are given by the equations (3.23) and (3.21), respectively. Observe that

$$\text{tr}[\sigma_3 N^{-1}(z(\lambda)) \partial_\lambda N(z(\lambda))] = \frac{1}{4} \partial_\lambda \ln \beta(\lambda) \text{tr}[\sigma_3 U^{-1} \sigma_3 U] = 0, \quad (3.31)$$

where

$$\beta(\lambda) = \frac{z(\lambda) + 1}{z(\lambda) - 1}.$$

Therefore, the asymptotic formulae (3.30) reduces to the relation

$$\text{tr}[\partial_\lambda \chi(\lambda, t) \cdot \sigma_3 \cdot \chi^{-1}(\lambda, t)] = 2m \partial_\lambda r(z(\lambda)) r^{-1}(z(\lambda)) - 2t \partial_\lambda \eta(z(\lambda)) + O\left(\frac{1}{m}\right), \quad (3.32)$$

as $m \rightarrow \infty$, uniformly for $\lambda \in C$.

Substituting the estimate (3.32) into the right hand side of (3.11) and changing the variable of integration, $\lambda \rightarrow z$, we obtain that

$$\partial_t \ln \det [I + V] = \frac{m}{2\pi i} \int_{\mathcal{L}} \phi(\lambda(z)) \partial_z \ln r(z) dz - \frac{t}{2\pi i} \int_{\mathcal{L}} \phi(\lambda(z)) \partial_z \eta(z) dz + O\left(\frac{1}{m}\right), \quad m \rightarrow \infty,$$

where $\mathcal{L}$ is a small loop around the interval $[-1, 1]$ and the estimate is uniform with respect to t . Integrating this estimate, we arrive at the following asymptotics for the determinant,

$$\ln \det [I + V] = \ln \det [I + V] \Big|_{t=0} + \frac{mt}{2\pi i} \int_{\mathcal{L}} \phi(\lambda(z)) \partial_z \ln r(z) dz - \frac{t^2}{4\pi i} \int_{\mathcal{L}} \phi(\lambda(z)) \partial_z \eta(z) dz + O\left(\frac{1}{m}\right), \quad m \rightarrow \infty, \quad (3.33)$$

Using the known [18] (see also [6]) large m asymptotics of the $\ln \det [I + V] \Big|_{t=0}$, we transform (3.33) into our final asymptotic result,

$$\begin{aligned} \ln \det [I + V] = m^2 \ln \cos \frac{\alpha}{2} + m \frac{t}{2\pi i} \int_{\mathcal{L}} \phi(\lambda(z)) \partial_z \ln r(z) dz \\ - \frac{1}{4} \ln \left(m \sin \frac{\alpha}{2} \right) - \frac{t^2}{4\pi i} \int_{\mathcal{L}} \phi(\lambda(z)) \partial_z \eta(z) dz + c_0 + O\left(\frac{1}{m}\right), \quad m \rightarrow \infty, \end{aligned} \quad (3.34)$$

where c_0 is the famous Widom's constant:

$$c_0 = \frac{1}{12} \ln 2 + 3\zeta'(-1).$$

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A Special functions

We gather in the appendix some elementary information about Hankel functions. The Hankel functions satisfy to the addition formulae

$$H_0^{(1)}(e^{-i\pi}z) = 2H_0^{(1)}(z) + H_0^{(2)}(z) \quad \text{and} \quad H_0^{(2)}(e^{-i\pi}z) = -H_0^{(1)}(z), \quad (A.1)$$

from which follow formulae for additions of derivatives

$$[H_0^{(1)}]'(e^{-i\pi}z) = -2[H_0^{(1)}]'(z) - [H_0^{(2)}]'(z) \quad \text{and} \quad [H_0^{(2)}]'(e^{-i\pi}z) = [H_0^{(1)}]'(z). \quad (A.2)$$

Similar results follow for the rotations by $e^{i\pi}$.

$$H_0^{(2)}(e^{i\pi}z) = 2H_0^{(2)}(z) + H_0^{(1)}(z) \quad \text{and} \quad H_0^{(1)}(e^{i\pi}z) = -H_0^{(2)}(z), \quad (A.3)$$

$$[H_0^{(2)}]'(e^{i\pi}z) = -2[H_0^{(2)}]'(z) - [H_0^{(1)}]'(z) \quad \text{and} \quad [H_0^{(1)}]'(e^{i\pi}z) = [H_0^{(2)}]'(z). \quad (A.4)$$

These function exhibit the local behavior at $z = 0$: $H_0^{(a)}(z) = O(\ln z)$ with $a = 1, 2$.

The Hankel functions admit the $z \rightarrow \infty$ asymptotic expansions

$$H_\nu^{(1)}(z) \simeq -i \left(\frac{2i}{\pi z} \right)^{\frac{1}{2}} e^{iz} e^{-i\frac{\pi\nu}{2}} \sum_{n=0}^{+\infty} \left(\frac{i}{2z} \right)^n (\nu, n), \quad (A.5)$$

$$H_\nu^{(2)}(z) \simeq i \left(\frac{-2i}{\pi z} \right)^{\frac{1}{2}} e^{-iz} e^{i\frac{\pi\nu}{2}} \sum_{n=0}^{+\infty} \left(\frac{-i}{2z} \right)^n (\nu, n), \quad (A.6)$$

where we agree upon

$$(v, m) = \Gamma \left(\begin{matrix} v + 1/2 + m \\ v + 1/2 - m, m + 1 \end{matrix} \right). \quad (\text{A.7})$$

These asymptotic expansions being differentiable, we infer that

$$[H_0^{(1)}]'(z) \simeq \left(\frac{2i}{\pi z} \right)^{\frac{1}{2}} e^{iz} \sum_{n=0}^{+\infty} \left(\frac{i}{2z} \right)^n (0, n) \frac{1+2n}{1-2n} \quad (\text{A.8})$$

$$[H_0^{(2)}]'(z) \simeq \left(\frac{-2i}{\pi z} \right)^{\frac{1}{2}} e^{-iz} \sum_{n=0}^{+\infty} \left(\frac{-i}{2z} \right)^n (0, n) \frac{1+2n}{1-2n}. \quad (\text{A.9})$$

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