GENERA OF INTEGER REPRESENTATIONS AND THE LYNDON-HOCHSCHILD-SERRE SPECTRAL SEQUENCE

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ABSTRACT

There has been in the past ten to fifteen years a surge of activity concerning the cohomology of semi-direct product groups of the form $\mathbb{Z}^n \rtimes G$ with G finite. A problem first stated by Adem-Ge-Pan-Petrosyan asks for suitable conditions for the Lyndon-Hochschild-Serre Spectral Sequence associated to this group extension to collapse at second page of the Lyndon-Hochschild-Serre spectral sequence. In this thesis we use facts from integer representation theory to reduce this problem to only considering representatives from each genus of representations, and establish techniques for constructing new examples in which the spectral sequence collapses.

1. INTRODUCTION

1.1 Setting, History, and the Problem

In 1993 Thomas Brady produced a paper, [1], in which he described a methodology to construct free resolutions for semi-direct product groups described by a short exact sequence:

$$0 \to \mathrm{L} \to \mathrm{L} \rtimes_{\phi} \mathrm{G} \to \mathrm{G} \to 0.$$

The methodology hinged on what has come to be called a "Compatible Action":

Definition 1.1. *Given a free resolution $\epsilon : F \to \mathbb{Z}$ over $\mathbb{Z}L$, we say that this resolution admits an action of G compatible with ϕ if for all $h \in G$ there is an augmentation-preserving chain map $\tau(h) : F \to F$ satisfying*

1. *$\tau(h)[k \cdot f] = k^h \cdot [\tau(h)f]$ for all $k \in L$ and $f \in F$, and*

2. *$\tau(h)\tau(h') = \tau(hh')$ for all $h, h' \in G$,*

where k^h means $\phi(h)k$.

This new development in the construction of free resolutions was initially utilized for constructing a free resolution of a dihedral group thought of as a split extension of $\mathbb{Z}_2$ by $\mathbb{Z}$ in [1] as proof of concept that compatible actions are useful. Thirteen years later, in [2], Adem and Pan used the notion of a compatible action to help understand the cohomology of split extensions of a finite group G by a free abelian group $\mathbb{Z}^n$. In that paper they proved a central theorem that would be used in several subsequent research papers ([3], [4], [5], [6], [7], [8], [9], for example). The theorem is:

Theorem 1.1. *Let $\epsilon : F \to \mathbb{Z}$ be a free resolution of $\mathbb{Z}$ over $\mathbb{Z}[\mathbb{Z}^n]$ with the special property that when $Hom_{\mathbb{Z}[\mathbb{Z}^n]}(- , \mathbb{Z})$ is applied to the resolution, the resultant cochain complex has trivial differentials. Suppose further that there is a compatible action of G on F. Then for all integers $k \geq 0$, we have an isomorphism:*

$$H^k(\mathbb{Z}^n \rtimes G, \mathbb{Z}) = \bigoplus_{i+j=k} H^i(G, H^j(\mathbb{Z}^n, \mathbb{Z})).$$

The requirement about the differentials becoming trivial was then shown to be satisfied by taking F to be the Koszul complex[1] associated to the free abelian group.

It has been noted that this theorem implies that the Lyndon-Hochschild-Serre (LHS) Spectral Sequence associated to the short exact sequence:

$$0 \to \mathbb{Z}^n \to \mathbb{Z}^n \rtimes \mathrm{G} \to \mathrm{G} \to 0$$

collapses at E_2 with no extension problems. This is meaningful because Totaro, in [11], showed that even in the case of an abelian normal subgroup the spectral sequence need not collapse, and indeed the differentials can become very complicated.

In a 2008 paper, [3], Adem et al. used this methodology of compatible actions to completely describe the situation for cyclic groups of prime order:

Theorem 1.2. *Let $G = \mathbb{Z}/p\mathbb{Z}$, where p is any prime. If L is any finitely generated $\mathbb{Z}G$-lattice[2], and $L \rtimes G$ is the associated semi-direct product group, then for each $k \geq 0$*

$$H^k(L \rtimes G, \mathbb{Z}) \cong \bigoplus_{\mathrm{i}+\mathrm{j}=k} H^{\mathrm{i}}(G, \wedge^{\mathrm{j}}(L^*)).$$

where $\wedge^{\mathrm{j}}(L^*)$ denoted the jth exterior power[3] of the dual module $L^* = Hom(L, \mathbb{Z})$.

The proof of this theorem together with Theorem 4.3 stated in the induction section of this thesis gives a corollary:

Corollary 1.1. Let G be a finite group with $\mathbb{Z}/p\mathbb{Z} \subset \mathrm{G}$. If L is a finitely generated $\mathbb{Z}$G-lattice such that for some $\mathbb{Z}[\mathbb{Z}/p\mathbb{Z}]$-lattice, M, we have $\mathrm{L} \cong \mathrm{Ind}^{G}_{\mathbb{Z}/p\mathbb{Z}}\mathrm{M}$, then we have for $k \geq 0$

$$\mathrm{H}^k(\mathrm{L} \rtimes \mathrm{G}, \mathbb{Z}) \cong \bigoplus_{\mathrm{i}+\mathrm{j}=k} \mathrm{H}^{\mathrm{i}}(\mathrm{G}, \wedge^{\mathrm{j}}(\mathrm{L}^*)).$$

Theorem 1.2 proved by noting a few things. First, the indecomposable integer representations of cyclic groups of prime order are well known (and can be found in 34B of [12]), and, more importantly, they form a finite set. Second, Charlap and Vasquez, in [13], proved

[1] ↑See [10] for details on the Koszul Complex.
[2] ↑A $\mathbb{Z}$G-lattice is a $\mathbb{Z}$G-module that is finitely generated and free as a $\mathbb{Z}$-module.
[3] ↑Note: $\wedge^{\mathrm{j}}(\mathrm{L}^*) \cong \mathrm{H}^{\mathrm{j}}(\mathrm{L}, \mathbb{Z})$.

Theorem 1.3. *Let* π_1 *and* π_2 *be* $\mathbb{Z}_p$*-Bieberbach groups*[4,5]*. Then we have the following exact sequences:*

$$0 \to M_{\text{i}} \to \pi_{\text{i}} \to \mathbb{Z}_p \to 0$$

for i = 1, 2), where the M_{i} *are* $\mathbb{Z}_p$*-modules. If* $M_1 \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \cong M_2 \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$[6]*, then there is a monomorphism* $\psi : \pi_1 \to \pi_2$ *such that* $\psi^* : H^*(\pi_2, \mathbb{Z}_{(p)}) \to H^*(\pi_1, \mathbb{Z}_{(p)})$ *isomorphism.*

This reduced the problem (in the setting of G = $\mathbb{Z}_p$ for some prime p) to showing that the G-modules $\mathbb{Z}$, $\mathbb{Z}$G, and IG, where IG is the augmentation ideal of $\mathbb{Z}$G, each admit a compatible action[7].

In this thesis, we prove an expansion on the result of Charlap and Vasquez, showing

Theorem 1.4. *Let G be any finite group. Suppose L and L' are* $\mathbb{Z}G$*-lattices and that* $L\otimes\mathbb{Z}_{(p)} \cong L'\otimes\mathbb{Z}_{(p)}$ *as G-modules for all primes p. Let* Γ *and* Γ*' be constructed such that so that the action of* $\Gamma/L \cong \Gamma'/L' \cong G$ *on L and L', respectively, is the given one. Then the integral cohomology of* Γ *and* Γ*' are isomorphic if the extensions*

$$0 \to L \to \Gamma \to G \to 0$$

and

$$0 \to L' \to \Gamma' \to G \to 0$$

are split.

---

[4]↑Here Bieberbach group means torsion-free crystallographic group, and $\mathbb{Z}_p$-Bieberbach group means a Bieberbach group with Holonomy equal to $\mathbb{Z}_p$, the cyclic group with p elements. The Holonomy of a crystallographic group is the quotient of the crystallographic group by its maximal free abelian normal subgroup, which is guaranteed to be finite by the first Bieberbach Theorem. See [14] and [15]

[5]↑Ludwig Bieberbach, the namesake of Bieberbach groups, was a terrible person and an enthusiastic Nazi who assisted the Gestapo in arresting his Jewish colleagues throughout the 1930's as well as attempting to get his Jewish colleagues thrown out of their academic positions (see [16], [17]). Attaching his name to these groups (and to the structure theorems about them) is an ethical dilemma because of the inseparable nature of celebrating a person (which, in this case, is abhorrent) and celebrating their work. I attach his name to the groups and to his theorems here simply because that is what they are commonly called in the literature and I want to be understood. However, I encourage the usage of "Torsion-free crystallographic groups" as a reasonable substitute for "Bieberbach groups".

[6]↑Here $\mathbb{Z}_{(p)}$ refers to the localization of $\mathbb{Z}$ to the ideal (p).

[7]↑We say L admits a compatible action if there exists a compatible G-action on the associated Koszul Complex of L. The focus on the Koszul Complex is due to Theorem 1.1 and the subsequent discussion.

I also prove a slighter stronger version that loosens the requirement that the extensions be split, but adds a requirement that G be a p-group. This is proven in section 3.1.

The proof provided by Adem et al. of Theorem 1.2 actually showed that IG, the augmentation ideal for $\mathbb{Z}$G, admits a compatible action for G an arbitrary finite cyclic group. We also extended this result: we show, in section 2.2, that IG admits a compatible action for arbitrary finite G.

The 2008 theorem of Adem et. al also produced a far reaching corollary to Theorem 1.2:

Corollary 1.2. Let G denote a finite group of square-free order, and L any finitely generated $\mathbb{Z}$G-lattice. Then for all k≥0 we have

$$\mathrm{H}^k(\mathrm{L} \rtimes \mathrm{G}, \mathbb{Z}) \cong \bigoplus_{\mathrm{i+j}=k} \mathrm{H}^{\mathrm{i}}(\mathrm{G}, \wedge^{\mathrm{j}}(\mathrm{L}^*))$$

This work was preceded by the completion of Petrosyan's doctoral thesis, [7], the second half of which consists of direct computations of $\mathrm{H}^*(\mathrm{L} \rtimes \mathrm{G}, \mathbb{Z})$, where L is still a finitely generated $\mathbb{Z}$G-lattice, but now G is a cyclic group of order p^2. In his thesis, Petrosyan took p to be 2 in particular, and each direct computation for various choices of L gave the same conclusion: that the cohomology of the semidirect product was the direct sum listed above in the theorem and corollary.

Perhaps because of the consistency with which they were seeing this pattern, in [3] a conjecture was put forth:

Conjecture 1. Suppose that G is a finite cyclic group and L a finitely generated $\mathbb{Z}$G-lattice; then for any k≥0 we have

$$\mathrm{H}^k(\mathrm{L} \rtimes \mathrm{G}, \mathbb{Z}) \cong \bigoplus_{\mathrm{i+j}=k} \mathrm{H}^{\mathrm{i}}(\mathrm{G}, \wedge^{\mathrm{j}}(\mathrm{L}^*))$$

This conjecture was limited to G cyclic because Burt Totaro, in [11], discovered what has come to be known as Totaro's Examples. These are created in the context of $\mathrm{G} = (\mathbb{Z}/p\mathbb{Z})^2$ for some prime p, and are $\mathbb{Z}$G-lattices L such that the mod-p Lyndon-Hochschild-Serre spectral sequence associated to the semi-direct product $\mathrm{L} \rtimes \mathrm{G}$ has nonzero differentials. This eliminates the possibility for the conjecture to be true in this context.

The conjecture was given as part of a broader problem first suggested by Totaro that appears to be much more challenging:

Problem. Given a finite group G, find suitable conditions on a $\mathbb{Z}$G-lattice L so that the spectral sequence for L$\rtimes$G collapses at page 2 of the LHS spectral sequence.

In 2012 Martin Langer and Wolfgang Lück contributed with a paper, [5], containing positive results for the problem, but negative results for the conjecture. For the negative result, they developed a methodology through which they were able to construct a $\mathbb{Z}$G-lattice L for G $= \mathbb{Z}/4\mathbb{Z}$ that produces a nonzero differential on the E_2-page of the spectral sequence. For the positive result, they were able to show, using similar machinery, that if G is cyclic and the action of G on L is free outside of the origin, then the spectral sequence collapses at E_2. The methodology, particularly the use of free groups in this context, will see extensive use in Section 4.

The negative result pushed them to write a "very optimistic" reformulation of the original conjecture of Adem et. al:

Conjecture 2. Corollary 1 is true if the order of G is not divisible by 4.

This proved to be too optimistic, as Petrosyan and Putrycz, in [9], where able to find a group of the form $\mathbb{Z}^8 \rtimes \mathbb{Z}/9\mathbb{Z}$ for which the conjecture fails. This group was found by considering *crystallographic*[8] groups, the original topic of interest in the 2008 Adem et al. paper, and using the computer algebra system GAP in a clever way to compute their cohomology. Then these cohomology groups were compared with the summations on the other side of the equality in the conjecture.

[8]↑Crystallographic groups are discrete cocompact subgroups of Euclidean space. They are most famous for their relationship to Hilbert's 18th problem. The low dimensional versions have been studied for millennia as the "Wallpaper groups".

Petrosyan and Putrycz did not try to reformulate the conjecture after stating their counterexample. An optimist might follow in Langer and Lück's tradition and predict that the conjecture holds as long as G has order not divisible by 4 or 9. However, one might also formulate a new conjecture as:

Conjecture 3. Corollary 1 is true for any finitely generated $\mathbb{Z}$G-lattice if and only if the order of G is square-free.

In this thesis, the free group methodology developed by Langer and Lück is utilized to show that under certain circumstances, if the Koszul complex associated to L, an H-module with H$\subset$G, admits a compatible action then the Koszul Complex associated to $\mathrm{Ind}_H^G \mathrm{L} = \mathbb{Z}\mathrm{G} \otimes_{\mathbb{Z}H} \mathrm{L}$ also admits a compatible action. We also establish analogous results for variants of the induction operator.

1.2 Motivation

I am including this subsection primarily out of respect for the tradition of including it. It seems difficult to imagine spending enough time and energy to read or write a document like this and not at least largely agree with the central theme in Hardy's apology and therefore not have any requirement of applicability in your pure mathematics. On the other hand, I have met few mathematicians who, upon encountering something truly out there, do not scratch their heads and ask "why bother?"[9]

And so the applicability of this research is as follows:

First, and perhaps foremost, the three (and sometimes four) dimensional crystallographic groups that make up a class of virtually free abelian groups are, in chemistry, referred to as Space Groups and have important applications to understanding the structure of molecules with a crystalline structure. The name of the field in which this is used is called Crystallography and my chemist friends tell me it is interesting. I'll have to take their word on it.

[9]↑A reasonable consideration then is whether group cohomology, integer representation theory, or virtually free abelian groups are "truly out there". I asked my mom (an industrial engineer by training). She says they are.

Second, each satellite NASA and the ESA has put up with instruments to measure the universe's curvature have given data that tells us that the universe is, up to the resolution of the instruments, flat. And if the universe is a flat manifold or flat orbifold, then the fundamental group of the universe is a crystallographic group. See [18] for details.

Third, back in the realm of mathematics we see at the end of [3] that crystallographic groups are of deep importance to the study of Calabi-Yau Orbifolds that are important the superstring theory of physics.

Fourth, in [6], the sequel to [5], the authors compute the K-theory of the group C*-algebra of a semi-direct product group $\mathbb{Z}^n \rtimes_\rho \mathrm{G}$ with G finite cyclic. Likely the interest in these groups within the context of K-theory is similar to my own interest within the context of group cohomology, which is:

Fifth, group cohomology is an interesting piece of machinery and virtually free abelian groups are an interesting class of groups (being infinite groups that are simultaneously not trivial to sort out but also manageable).[10] In fact, virtually free abelian groups are a little bit too interesting and contain some exotica that can be difficult to deal with at first blush. And so it is reasonable to want to take the cohomology of a restricted subset of virtually free abelian groups that are nice and somewhat well understood.

1.3 Organization

We begin in Section 2 with a description of when a compatible action can be found on the Koszul complex for a $\mathbb{Z}$G-lattice without any requirements on G other than it being finite. The main result of this section is proving that for an arbitrary finite group, G, the augmentation ideal, IG, admits a compatible action. This provides some general progress towards the problem put forward by Totaro and discussed in Section 1.1. In particular,

[10]↑And sometimes when two interesting things are put together, interesting events take place.

we show that many standard $\mathbb{Z}$G-lattices produce Koszul complexes that admit compatible actions.

Next, in Section 3, we generalize Theorem 1.3. The primary usage of Theorem 1.3 that I have seen is to allow for the study of $\mathbb{Z}_p$-crystallographic groups without needing to consider different $\mathbb{Z}_p$-lattices within the same genus[11]. This greatly simplifies the theory for $\mathbb{Z}_p$-crystallographic groups, as it allows one to essentially ignore the class field theory that is usually involved in the classification of indecomposable integer representations of finite cyclic groups. In section 3 we extend this result from $\mathbb{Z}_p$ to arbitrary finite groups G with the restriction that the extension needs to be split:

$$0 \to \mathrm{L} \to p = \mathrm{L} \rtimes \mathrm{G} \to \mathrm{G} \to 0.$$

This drastically improves the quality of life for the problem, removing for instance the class field theory necessary to distinguish $\mathbb{Z}_{p^2}$-lattices within the same genus.

It is also useful outside of the case where G is cyclic of prime order or prime squared order. However, the Jordan-Zassenhaus theorem implies that there will be finitely many isomorphism classes of lattices within each genus. Therefore, for a group of infinite integer representation type[12], there will be infinitely many indecomposable genera. This is unfortunate because Heller, Reiner (in [19]), and Jones (in [20]) have shown that a group will be of finite integer representation type if and only if the group's p-Sylow subgroups are cyclic of at most order p^2 for each prime p. And so the class of groups with any realistic hope of complete classification of integer representations is somewhat small. As mentioned, this extension of Charlap and Vasquez's theorem is limited to virtually free abelian groups that are a semi-direct product of a free abelian group with a finite group. However, I then prove a corollary that works for arbitrary group extensions so long as the finite group is a p-group, for some prime p.

11↑See Appendix on Integer Representation Theory

12↑A group G is of infinite integer representation type if there are infinitely many pairwise non-equivalent indecomposable integer representations of G.

Next, following a common motif in modern mathematics, I show that there are ways to take knowledge about smaller, simpler groups and use that knowledge to inform about larger, more complicated groups. This is done through the standard group theoretic technique of induction, as well as some less common techniques that, lacking a name in the literature, I have denoted “Higher induction”. Among these higher inductions there is the somewhat well-known Tensor induction. I have also included, for completeness, a somewhat recently developed notion of Cocycle induction which attempts to generalize ordinary induction. For each of these types of induction I prove various statements, the main one being that they preserve the property that the associated Koszul complex will admit a compatible action.

Next, I give a lengthy exposition on practical computations within the setting of the cohomology of virtually free abelian groups. These include standard methods of computing group cohomology in conjunction with knowledge that the spectral sequence collapses. Due to the restrictions required for a group to be of finite integer representation type, the focus of these computations are on $\mathrm{G} = \mathbb{Z}/\mathrm{p}^2\mathbb{Z}$.

2. COMPATIBLE ACTIONS

For certain constructions of $\mathbb{Z}$G-lattices, it can be shown that regardless of the choice of G, a finite group, the lattice will have a projective resolution that admits a compatible action. In this section we will explore this premise. Due to the wide generality afforded here in terms of choice of G, the choice of $\mathbb{Z}$G-lattices is severely restricted.

Because of the fundamental lemma of homological algebra, it is not important which resolution we decide to use, the resultant cohomology groups will be isomorphic. However, this does not mean that all resolutions are created equal. Some are computationally nice, others computationally burdensome. For our work we will need a particular resolution, described here: We will be working with the Koszul complex of the free abelian group L, as described in [10]. If $l_1, \ldots, l_n \in$L form a basis, then $l_1 - 1, \ldots, l_n - 1 \in \mathbb{Z}[L]$ is a regular sequence and we denote the first few terms of the associated Koszul complex as

(i) $K_0 = \mathbb{Z}L$,

(ii) $K_1 = \mathbb{Z}[L]\langle a_h \rangle_{i=1,\ldots,n}$, the free $\mathbb{Z}[L]$ module with generators a_h for i = 1, . . . , n.

In addition, the first differential of the above Koszul complex $d : K_1 \to K_0$ is defined by $d(a_{l_i}) = l_i - 1$.

We will be using the Koszul complex for two reasons: first, because it is has been studied extensively, has had many of its algebraic properties determined, and because it is a general construction applicable to any $\mathbb{Z}$G-lattice. Second, because in [3] the Adem, Pan, Ge, and Petrosyan were able to prove a theorem giving the Koszul complex a great deal of importance in the study of compatible actions:

Theorem 2.1. *If G acts on the lattice L of $\mathbb{Z}$-rank n, let $K_* = K(l_1 - 1, \ldots, l_n - 1)$ denote the special[1] free resolution of R[2] over R[L] defined using the Koszul complex associated to*

1↑"Special" here means a resolution with trivial differentials after applying the Hom functor in the usual way of computing group cohomology.
2↑R here is a commutative ring with unit.

the elements l_1 - 1, ..., l_n - 1, where $\{l_1, \ldots, l_n\}$ form a basis for L. Suppose that there is a homomorphism $\tau : G \to Aut(K_1)$ such that for every $g \in G$ and $a \in K_1$ it satisfies

$$d\tau(g)(a) = g \cdot d(a)$$

where $d : K_1 \to K_0 = R[L]$ is the usual Koszul differential. Then τ extends to K_* using its DGA structure and so defines a compatible G-action on K_*.

This provides a massive improvement on the quality of life of the problem of constructing compatible actions because it reduces the work an investigator has to do from constructing an action that works on every entry in a resolution (possibly infinitely many) as well as their (possibly infinitely many) differentials to only constructing a compatible action on the first two modules of a free resolution and the one differential between them. This result is used extensively in the following section.

2.1 Trivial and Permutation Lattices

The ring $\mathbb{Z}$ can always be considered a $\mathbb{Z}$G-lattice because the G-action can always be made trivial:

$$\gamma \cdot \mathrm{z} = \mathrm{z}, \forall \gamma \in \mathbb{Z}\mathrm{G}, \forall \mathrm{z} \in \mathbb{Z}.$$

Similarly, the symmetric group on n letters, S_n can act on $\mathbb{Z}^n$ by permuting its basis elements. This makes finding a compatible action of S_n on the Koszul complex associated to $\mathbb{Z}^n \cong$ L a simple matter. Let $a_1, \ldots, a_n$ be the generators for the Koszul complex corresponding to the basis $l_1, \ldots, l_n$ in L. Then we can define τ as:

$$\tau(\sigma)(\mathrm{a_i}) = \mathrm{a}_{\sigma(\mathrm{i})}.$$

Compatibility follows from the fact that for all a_i and all σ_g we have:

$$\mathrm{d}\tau(\sigma)(\mathrm{a_i}) = \mathrm{d}(\mathrm{a}_{\sigma(\mathrm{i})}) = \mathrm{l}_{\sigma(\mathrm{i})} - 1 = \sigma \cdot (\mathrm{l_i} - 1)$$

and we have a compatible action. This is quite nice because of a lemma stated in [2]:

Lemma 2.1. If L is a G_1-module, $\phi : G_2 \to G_1$ is a group homomorphism, and $\epsilon : F \to \mathbb{Z}$ is a L-resolution of $\mathbb{Z}$ such that G_1 acts compatibly on it by τ_1, then G_2 also acts compatibly on it via $\tau_2(g)f = \tau_1(\phi(g))f$ for any $g \in G_2$.

This lemma and the argument above show that if that action of G on L can be factored through a symmetric group then that choice of action of G on L produces a compatible action. This was summarized in a theorem proved in [3]:

Theorem 2.1. *Let G act on $\mathbb{Z}^n$ through a homomorphism $\phi : G \to S_n$. Then the Koszul complex K_* associated to L admits a compatible G-action.*

Proof. The result is given by the preceding argument. □

Constructions of compatible actions by hand are extremely difficult to develop and in the next section we construct our only example of a compatible action constructed by hand. All other examples are constructed indirectly using higher level methodology.

2.2 Augmentation Ideal

Let G be a finite group. By IG we mean the ideal of $\mathbb{Z}$G generated by elements of the form g - 1. The augmentation ideal has considerable importance in many fields of algebra and beyond. Some of the importance can be seen already because the augmentation map ϵ : $\mathbb{Z}G \to \mathbb{Z}$ will have IG as its kernel.

The action of G on $\mathbb{Z}$G restricts to an action on IG, given by

$$g \cdot (h - 1) = gh - g = gh - 1 - g + 1 = (gh - 1) - (g - 1).$$

We will now extend the action of G on IG to an action of G on $\mathbb{Z}$[IG] through unital ring homomorphisms. And we can see the G-module structure by defining a function $z : IG \to \mathbb{Z}[IG]$ as $z(\gamma) = 1_{\mathbb{Z}}(\gamma)$ for all $\gamma \in IG$[3]. We also require $x_g := z(g - 1)$ and write IG additively

[3]↑For a standard exercise in the notation of this subject, prove to yourself that $z(0) = 1_{\mathbb{Z}[IG]}$.

so that $z(\gamma)z(\eta) = z(\gamma + \eta)$ for γ, $\eta \in$IG. In light of this requirement, we see that $z(-\gamma) = (z(\gamma))^{-1}$ The action of G on $\mathbb{Z}$[IG] is then given by

$$g \cdot (x_h) = g \cdot z(h-1) = z(g(h-1)) \tag{2.1}$$
$$= z(gh - g) \tag{2.2}$$
$$= z(gh - 1 - g + 1) \tag{2.3}$$
$$= z((gh-1) - (g-1)) \tag{2.4}$$
$$= z(gh-1)z(-(g-1)) \tag{2.5}$$
$$= x_{gh}x_g^{-1}. \tag{2.6}$$

This action extends to a unital ring homomorphism $\mathbb{Z}[IG] \to \mathbb{Z}[IG]$ because $\mathbb{Z}$[IG] is isomorphic to the ring of Laurent polynomials on the x_g's. This definition makes $d(a_h) = x_h - 1$, where d is standard differential for the Koszul Complex.

Proposition 2.1. *Suppose G is a finite group. The Koszul complex associated to IG admits a compatible G-action.*

Proof. We will define maps $\tau(g) : K_1 \to K_1$ for each g$\in$G. There are two properties that these maps must satisfy.

(i) We need

$$d(\tau(g)(a_h)) = g \cdot (d(a_h))$$

for all g, h$\in$G, so that τ(g) satisfies Theorem 2.1.

(ii) We need property 2 in Brady's definition for a compatible action. This will identify τ(g) as an action of G.

Once we have shown these things, we will extend this function additively and assume property 1 in Brady's definition. Then we will have our compatible action.

(i) - Define τ(g)(a$_h$) = x$_g^{-1}$(a$_{gh}$ - a$_g$). Then we have

$$d(\tau(g)(a_h)) = x_g^{-1}d(a_{gh} - a_g) \tag{2.7}$$
$$= x_g^{-1}(x_{gh} - 1 - x_g + 1) \tag{2.8}$$
$$= x_g^{-1}(x_{gh} - x_g). \tag{2.9}$$
$$= x_g^{-1}x_{gh} - 1 \tag{2.10}$$

Meanwhile, we have

$$g \cdot d(a_h)) = g \cdot (x_h - 1) \tag{2.11}$$
$$= x_{gh}x_g^{-1} - 1 \tag{2.12}$$

Which gives us

$$d(\tau(g)(a_h)) = g \cdot (d(a_h)). \tag{2.13}$$

This shows that we satisfy property (i).

(ii) - Let g, g'∈G. We can see this property through a series of equalities:

$$\begin{aligned}
\tau(g')\tau(g)(a_h) &= \tau(g')(\tau(g)(a_h)) && (2.14)\\
&= \tau(g')(x_g^{-1}(a_{gh} - a_g)) && (2.15)\\
\text{(Brady's Property (i))} &= g' \cdot (x_g^{-1})(\tau(g')(a_{gh} - a_g)) && (2.16)\\
&= g' \cdot z(1-g)(\tau(g')(a_{gh} - a_g)) && (2.17)\\
&= z(g' - g'g)(\tau(g')(a_{gh} - a_g)) && (2.18)\\
&= z(g' - 1 - g'g + 1)(\tau(g')(a_{gh} - a_g)) && (2.19)\\
&= z((g'-1) - (g'g - 1))(\tau(g')(a_{gh} - a_g)) && (2.20)\\
&= z(g'-1)z(-(g'g-1))(\tau(g')(a_{gh} - a_g)) && (2.21)\\
&= (x_{g'})(x_{g'g}^{-1})(\tau(g')(a_{gh} - a_g)) && (2.22)\\
\text{(Comm. of Mult. in } \mathbb{Z}[\text{IG}]) &= (x_{g'g}^{-1})(x_{g'})(\tau(g')(a_{gh} - a_g)) && (2.23)\\
\text{(Assuming Linearity)} &= (x_{g'g}^{-1})(x_{g'})(\tau(g')(a_{gh}) - \tau(g')(a_g)) && (2.24)\\
&= (x_{g'g}^{-1})(x_{g'})(x_{g'}^{-1}(a_{g'gh} - a_{g'}) - x_{g'}^{-1}(a_{g'g} - a_{g'})) && (2.25)\\
&= (x_{g'g}^{-1})(x_{g'})(x_{g'}^{-1})(a_{g'gh} - a_{g'} - a_{g'g} + a_{g'})) && (2.26)\\
&= (x_{g'g}^{-1})(a_{g'gh} - a_{g'g}) && (2.27)\\
&= \tau(g'g)(a_h) && (2.28)
\end{aligned}$$

This shows we satisfy property (ii).

Now, extending linearly and assuming the first property of Brady's definition, we see that we meet the requirements for Theorem 2.1, and we see that we have a compatible G-action on the Koszul complex associated to the augmentation ideal, IG. □

2.3 Combining Lattices Compatibly

A common motif in modern mathematics is the idea of building new things out of old things. Take two interesting ideas and smash them together to form a new interesting idea. In the setting of this problem, we continue this tradition by showing how two or more lattices that admit compatible actions may be put together to form a new lattice with a compatible action.

First, we state a lemma of Adem and Ge, found in [2]:

Lemma 2.2. If $\epsilon_{\mathrm{i}} : \mathrm{F}_{\mathrm{i}} \to \mathbb{Z}$ is a projective $\mathbb{Z}[\mathrm{L}_{\mathrm{i}}]$-resolution of $\mathbb{Z}$ for i = 1, 2, then $\epsilon_1 \otimes \epsilon_2 : \mathrm{F}_1 \otimes_{\mathbb{Z}} \mathrm{F}_2 \to \mathbb{Z}$ is a projective $\mathbb{Z}[\mathrm{L}_1 \times \mathrm{L}_2]$-resolution of $\mathbb{Z}$. Furthermore, if G acts compatibly on F_{i} by τ_{i} for i = 1, 2, then a compatible action of G on $\epsilon_1 \otimes \epsilon_2 : \mathrm{F}_1 \otimes_{\mathbb{Z}} \mathrm{F}_2 \to \mathbb{Z}$ is given by $\tau(\mathrm{g})(\mathrm{f}_1 \otimes \mathrm{f}_2) = \tau_1(\mathrm{g})(\mathrm{f}_1) \otimes \tau_2(\mathrm{g})(\mathrm{f}_2)$.

The authors in [3] cited a previous paper, [2], as having a proof for this lemma. However, in that paper we are told that the proof is straightforward and nothing else.

Most of the proof is, but it requires some familiarity with homological algebra and a bit more thought than I would think to apply the adjective straightforward. The second statement about compatible actions is obvious.

Proof. The 'projectiveness' of the chain complex is requires a little bit of thought. First, we want to use the definition of a projective module that a R-module P is projective if there exists a free R-module F and an R-module P' such that P⊕P' ≅ F. In our setting this corresponds to showing, after recognizing the structure of the terms in the $\mathrm{F}_1 \otimes_{\mathbb{Z}} \mathrm{F}_2$ resolution and that Hom and direct sum commute, that there exists a free $\mathbb{Z}[\mathrm{L}_1 \otimes_{\mathbb{Z}} \mathrm{L}_2]$-module F and a $\mathbb{Z}[\mathrm{L}_1 \otimes_{\mathbb{Z}} \mathrm{L}_2]$-module P such that $((\mathrm{F}_1)_{\mathrm{i}} \otimes_{\mathbb{Z}} (\mathrm{F}_2)_{\mathrm{j}}) \oplus \mathrm{P} \cong \mathrm{F}$.

To show this, we first make use of the fact that a group ring constructed out of a direct product of groups is isomorphic to the tensor product of the group rings constructed out of the factor groups. In our setting this means:

$$\mathbb{Z}[\mathrm{L}_1 \times \mathrm{L}_2] \cong \mathbb{Z}[\mathrm{L}_1] \otimes_{\mathbb{Z}} \mathbb{Z}[\mathrm{L}_2].$$

And so we want to show that there exist a free $\mathbb{Z}[\mathrm{L}_1] \otimes_{\mathbb{Z}} \mathbb{Z}[\mathrm{L}_2]$-module F and a $\mathbb{Z}[\mathrm{L}_1] \otimes_{\mathbb{Z}} \mathbb{Z}[\mathrm{L}_2]$-module P such that $((\mathrm{F}_1)_{\mathrm{i}} \otimes_{\mathbb{Z}} (\mathrm{F}_2)_{\mathrm{j}}) \oplus \mathrm{P} \cong \mathrm{F}$. To show this, we use the fact that $(\mathrm{F}_k)_{\mathrm{i}}$ is

$\mathbb{Z}[L_k]$-projective for k = 1, 2. And so there exist $\mathbb{Z}[L_k]$-modules P_k for k = 1, 2 such that $(F_1)_i \oplus P_1 \cong \mathbb{Z}[L_1]^{m_1}$ and $(F_2)_j \oplus P_2 \cong \mathbb{Z}[L_2]^{m_2}$. This becomes useful because if we set

$$P \cong ((F_1)_i \otimes_{\mathbb{Z}} P_2) \oplus (P_1 \otimes_{\mathbb{Z}} (F_2)_j) \oplus (P_1 \otimes_{\mathbb{Z}} P_2)$$

then we see:

$$((F_1)_i \otimes_{\mathbb{Z}} (F_2)_j) \oplus P \cong ((F_1)_i \otimes_{\mathbb{Z}} (F_2)_j) \oplus ((F_1)_i \otimes_{\mathbb{Z}} P_2) \oplus (P_1 \otimes_{\mathbb{Z}} (F_2)_j) \oplus (P_1 \otimes_{\mathbb{Z}} P_2) \tag{2.29}$$

$$\cong ((F_1)_i \oplus P_1) \otimes_{\mathbb{Z}} ((F_2)_j \oplus P_2) \tag{2.30}$$

$$\cong \mathbb{Z}[L_1]^{m_1} \otimes_{\mathbb{Z}} \mathbb{Z}[L_2]^{m_2} \tag{2.31}$$

$$\cong (\mathbb{Z}[L_1] \otimes_{\mathbb{Z}} \mathbb{Z}[L_2])^{m_1 m_2}. \tag{2.32}$$

And we see this last item is a free $\mathbb{Z}[L_1] \otimes_{\mathbb{Z}} \mathbb{Z}[L_2]$-module, and so we can set F equal to this. And with that we have shown that this chain complex consists of projective modules. None of the individual components of this proof required very much heavy lifting and involved only material covered in an introductory course into homological algebra, but all of it taken together makes for a proof that I would be hesitant to label as straightforward. Plus we aren't done:

The next tricky portion of the lemma is showing that this chain complex is a resolution. This is tricky because in general the tensor product of two projective resolutions need not be exact. In fact, the homology of the tensor product of two projective resolutions is isomorphic to the Tor of the modules the two resolutions are resolving (see [21] theorem 9.3). In our setting this becomes easy, then, because $\mathbb{Z}$ is flat as a $\mathbb{Z}$-module and so $\mathrm{Tor}_n^{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) = 0$ for all $n > 0$. And so tensoring over the integers, everything works out. And this concludes the proof. □

I wanted prove this lemma to point out that items listed as straightforward or easy or obvious in research papers can sometimes be anything but. This is sometimes due to the steps to a proof being much longer than the author realizes. Sometimes it because it is often the case in research papers that it is never made clear which ring a tensor product is being constructed over (for example, probably in several places in this thesis) and this can lead to difficulties. For example, in the previous lemma, if we assume that the tensor product is being made over $\mathbb{Z}[L_1 \times L_2]$, treating the modules in each resolution as $\mathbb{Z}[L_1 \times L_2]$-modules, then the exactness of this new complex at dimension n is equivalent to $\mathrm{Tor}_n^{\mathbb{Z}[L_1 \times L_2]}(\mathbb{Z}, \mathbb{Z})$. By one of the definitions of group homology we can see that this is isomorphic to $H_n(L_1 \times L_2, \mathbb{Z})$, which we know is non-zero if n is less than or equal to the $\mathbb{Z}$-rank of $L_1 \times L_2$. And you're left wondering whether

1. You're mistaken about something (the correct choice, though perhaps not obviously so)
2. How this could possibly be true, much less "straightforward"
3. Could it be that some world-class mathematicians called something true and straightforward that is actually wrong?

It might sound like I am just complaining about having to do a little work. However, I believe that in all likelihood, the reader(s) of this document will be graduate students for whom the above questions will sound hauntingly familiar. And a little solidarity goes a long way...

Next, another lemma from [2]:

Lemma 2.3. If L is a finitely generated $\mathbb{Z}G_1$-lattice, $p : G_2 \to G_1$ a group homomorphism, and $\epsilon : F \to \mathbb{Z}$ is a $\mathbb{Z}[L]$-resolution of $\mathbb{Z}$ such that G_1 acts compatible on it by τ', then G_2 also acts compatibly on it by $\tau(g)f = \tau'(p(g))f$ for any $g \in G_1$.

The proof of this is simply a matter of tracing through definitions.[4]

These two lemmas significantly reduce the problem. The first lemma allows us to just focus on indecomposable representations and the second lemma allows us to just focus on

[4]↑A careful reader might notice that I just passed the buck on proving something after a lengthy diatribe about the negative aspects of passing the buck on proving things in technical research papers. I leave it as an exercise to the reader to determine how this is not actually hypocritical.

faithful representations.

As such, these lemmas leave very little to be determined about how putting together lattices fits within the framework of compatible actions. However, I came up with one interesting idea: replacing the direct sum in the first lemma with a tensor product. However, as we will see, some restrictions apply.

Lemma 2.4. Suppose that L_1 and L_2 are G-lattices and suppose that L_1 has an associated Koszul complex that admits a compatible G-action and that L_2 is a permutation representation[5] of G. Then $L_1 \otimes L_2$ has a Koszul complex that admits a compatible G-action.

This lemma, while most related to the information covered in this section, is proved using methods discussed in section 4 and so the proof will be postponed to that section.

[5] ↑Which therefore has a Koszul complex that admits a compatible G-action, see section 2.1

3. COHOMOLOGY AND GENERA

Genera is the plural of genus, which is a technical word used in the study of Integer Representation theory:

Definition 3.1. *Let R be a Dedekind domain with quotient field K with $R \neq K$, and let Λ be an R-order*[1] *in a separable finite dimensional K-algebra A. Two Λ lattices M and N are said to be in the same genus if $M_P \cong N_P$ for all prime ideals P of R, where subscript signifies localization. Stating that two lattices lie within the same genus is often denoted as $M \vee N$.*

This definition is equivalent if localization is replaced with P-adic completion.

Much of the theory of the integer representation theory is devoted to the extremely difficult task of finding invariants that distinguish lattices within the same genus, which demonstrates the complicated nature of these intra-genus lattices. And so reducing consideration to one $\mathbb{Z}G$-lattice for each genus of $\mathbb{Z}G$-lattices has been a crucial step in the work for computing the group cohomology and testing examples to determine the veracity[2] of conjectures. Examples of this can be found in [3] and [7]. The theoretical underpinnings for this reduction in that cyclic of prime order comes from Theorem 2.1 in [13]. This theorem is restricted only to group extensions of the form

$$0 \to L \to \Gamma \to \mathbb{Z}/p\mathbb{Z} \to 0 \tag{3.1}$$

where p is a rational prime. This was acceptable to Adem et al, as they were interested in the split extensions of this form. Petrosyan, in his thesis [7], appears to be using the fact that the theorem of Charlap and Vasquez easily extends to cyclic p-groups. With this extension of Charlap and Vasquez's result, one might ask how far the result can be extended. First we consider the split extension setting, then we broaden to the general extension.

[1] ↑See [12] Chapter 31 or Appendix A.
[2] ↑Or voracity, for that matter.

Consider two split short exact sequences of groups

$$0 \to \mathrm{L_i} \to \mathrm{L_i} \rtimes_{\alpha_i} \mathrm{G} \to \mathrm{G} \to 0,\ \mathrm{i} = 1,\ 2,$$

where G is any finite group and $\mathrm{L_i}$ is a $\mathbb{Z}$G-lattice on which G acts via α_i, respectively. Suppose further that $\mathrm{L_1} \vee \mathrm{L_2}$. There is a lemma of Roiter (found on page 660 of [12]) that will be useful:

Lemma 3.1. Let R be a Dedekind domain whose quotient field is global, Λ an R-order, and let M and N be Λ-lattices. If M∨N then for each nonzero ideal I of R, there exists a Λ-exact sequence

$$0 \to M \xrightarrow{f} N \to T \to 0 \tag{3.2}$$

for some R-torsion Λ-module T such that I + ann_RT = R.

Note that since $\mathbb{Z}$ is a Principal Ideal Domain, we can write its ideals as (n) for some element n$\in \mathbb{Z}$. The maximal ideals can be written as (p) for some rational prime p, and we denote the localization to this maximal ideal (p) as $\mathbb{Z}_{(p)}$. Now we can apply this lemma to our setting:

Corollary 3.1. Let $\mathrm{L_1}$ and $\mathrm{L_2}$ be $\mathbb{Z}$G-lattices and $\mathrm{L_1} \vee \mathrm{L_2}$. Then for each nonzero ideal (p) of $\mathbb{Z}$, there exists a $\mathbb{Z}$G-module homomorphism $\mathrm{f}_{(p)} : \mathrm{L_1} \to \mathrm{L_2}$ that becomes an isomorphism when tensored with $\mathbb{Z}_{(p)}$.

Proof. Consider equation (3.2) taking R to be $\mathbb{Z}$, Λ to be $\mathbb{Z}$G, I to be (p), M to be $\mathrm{L_1}$, and N to be $\mathrm{L_2}$. Then we tensor each term with $\mathbb{Z}_{(p)}$. Since $\mathbb{Z}_{(p)}$ is flat, we get a new short exact sequence:

$$0 \to \mathrm{L_1} \otimes \mathbb{Z}_{(p)} \xrightarrow{f_{(p)} \otimes \mathbb{Z}_{(p)}} \mathrm{L_2} \otimes \mathbb{Z}_{(p)} \to \mathrm{T} \otimes \mathbb{Z}_{(p)} \to 0.$$

But since $(p) + \mathrm{ann}_{\mathbb{Z}}\mathrm{T} = \mathbb{Z}$, and we have that T is a $\mathbb{Z}$-torsion module, we can see that $\mathrm{ann}_{\mathbb{Z}}\mathrm{T} = (p_T) \in \mathbb{Z}$ and that p_T and p are coprime. So if $\mathrm{t}\otimes\frac{x}{y}$ is a simple tensor in $\mathrm{T}\otimes\mathbb{Z}_{(p)}$, then we have the equality:

$$\begin{aligned} t \otimes \frac{x}{y} &= t \otimes p_T \frac{x}{p_T y} \\ &= t p_T \otimes \frac{x}{p_T y} \\ &= 0 \end{aligned}$$

But this was an arbitrary simple tensor, so $\mathrm{T}\otimes\mathbb{Z}_{(p)} = 0$, so $\mathrm{f}_{(p)} \otimes \mathbb{Z}_{(p)}$ is an isomorphism. $\square$

We also note that this map $\mathrm{f}_{(p)}$ from Corollary 3.1 is G-equivariant. We will use the following commutative diagram in which the horizontal maps are induced by $\mathrm{f}_{(p)}$:

$$\begin{tikzcd}
H^q(L_2, \mathbb{Z}_{(p)}) \arrow[r, "f^*_{(p)}"] \arrow[d, "UCT"', "\cong"] & H^q(L_1, \mathbb{Z}_{(p)}) \arrow[d, "UCT"', "\cong"] \\
Hom_{\mathbb{Z}}(H_q(L_2, \mathbb{Z}), \mathbb{Z}_{(p)}) \arrow[r] \arrow[d, "\mu^*"', "\cong"] & Hom_{\mathbb{Z}}(H_q(L_1, \mathbb{Z}), \mathbb{Z}_{(p)}) \arrow[d, "\mu^*"', "\cong"] \\
Hom_{\mathbb{Z}}(\Lambda^q H_1(L_2, \mathbb{Z}), \mathbb{Z}_{(p)}) \arrow[r] \arrow[d, "h^*"', "\cong"] & Hom_{\mathbb{Z}}(\Lambda^q H_1(L_1, \mathbb{Z}), \mathbb{Z}_{(p)}) \arrow[d, "h^*"', "\cong"] \\
Hom_{\mathbb{Z}}(\Lambda^q L_2, \mathbb{Z}_{(p)}) \arrow[r] \arrow[d, "\cong"] & Hom_{\mathbb{Z}}(\Lambda^q L_1, \mathbb{Z}_{(p)}) \arrow[d, "\cong"] \\
Hom_{\mathbb{Z}_{(p)}}((\Lambda^q L_2) \otimes \mathbb{Z}_{(p)}, \mathbb{Z}_{(p)}) \arrow[r] \arrow[d, "\cong"] & Hom_{\mathbb{Z}_{(p)}}((\Lambda^q L_1) \otimes \mathbb{Z}_{(p)}, \mathbb{Z}_{(p)}) \arrow[d, "\cong"] \\
Hom_{\mathbb{Z}_{(p)}}(\Lambda^q (L_2 \otimes \mathbb{Z}_{(p)}), \mathbb{Z}_{(p)}) \arrow[r] & Hom_{\mathbb{Z}_{(p)}}(\Lambda^q (L_1 \otimes \mathbb{Z}_{(p)}), \mathbb{Z}_{(p)})
\end{tikzcd}$$

Here UCT stands for the natural map appearing in the universal coefficient theorem which is an isomorphism because $\mathrm{H}_*(\mathrm{L_i}, \mathbb{Z})$ is free over $\mathbb{Z}$[3] since $\mathrm{L_i}$ is a free abelian group. Next, μ is the Pontryagin product from $\Lambda^q\mathrm{H}_1(\mathrm{L_i})$ to $\mathrm{H}_q(\mathrm{L_i})$ for each i, and h is the Hurewicz map. The vertical map following that is the Hom functor commuting with a flat base change. The vertical map following that is the commutation of the exterior algebra and the tensor

[3]↑This is important because this makes $\mathrm{Ext}_{\mathbb{Z}^{\mathrm{i}}}(\mathrm{H}_*(\mathrm{L_i}, \mathbb{Z}, \mathbb{Z}) = 0$ for $\mathrm{i} > 0$.

product. So everything is natural. Moreover, the bottom horizontal map is induced from $f_{(p)} \otimes \mathbb{Z}_{(p)}$ and is therefore an isomorphism, which informs us $f^*_{(p)} : H^q(L_2, \mathbb{Z}_{(p)}) \to H^q(L_1, \mathbb{Z}_{(p)})$ is an isomorphism that is natural and G-equivariant.

Since everything we used is natural and G-equivariant, we can then apply $H^*(G, -)$ to the above and obtain an isomorphism:

$$H^*(G, H^q(L_2, \mathbb{Z}_{(p)})) \xrightarrow{\cong} H^*(G, H^q(L_1, \mathbb{Z}_{(p)})).$$

But then Proposition B.1 tells us the abutments of the spectral sequences for which these groups represent the E_2 page of are isomorphic. That is,

$$H^*(L_1 \rtimes G, \mathbb{Z}_{(p)}) \cong H^*(L_2 \rtimes G, \mathbb{Z}_{(p)}). \tag{3.3}$$

To bring our coefficients back to the integers, we note:

$$H^*(L_i \rtimes G, \mathbb{Z}_{(p)}) = Hom(H_*(L_i \rtimes G, \mathbb{Z}), \mathbb{Z}_{(p)}) \oplus Ext(H_{*-1}(L_i \rtimes G, \mathbb{Z}), \mathbb{Z}_{(p)}) \tag{3.4}$$
$$= \mathbb{Z}_{(p)}^{\beta_*(L_i \rtimes G)} \oplus \bigoplus_{t=1}^{\infty} (\mathbb{Z}/p^t\mathbb{Z})^{rk_{p^t}(H_{*-1}(L_i \rtimes G))} \tag{3.5}$$

where $rk_{p^t}(X)$ means the $\mathbb{Z}/p^t\mathbb{Z}$-rank of X. The last summand in equation 3.5 is determined by the fact we know that $H_n(\Gamma_i, \mathbb{Z})$ is finitely generated for each n and each i by an inspection of the spectral sequence and that $Ext^1_{\mathbb{Z}}(\mathbb{Z}/k\mathbb{Z}, \mathbb{Z}_{(p)}) = 0$ if $k \neq p^t$, but $Ext^1_{\mathbb{Z}}(\mathbb{Z}_{p^t}, \mathbb{Z}_{(p)}) = \mathbb{Z}/p^t\mathbb{Z}$.

Meanwhile, we have

$$H^*(L_i \rtimes G, \mathbb{Z}) = \mathbb{Z}^{\beta^*(L_i \rtimes G)} \oplus \bigoplus_{t=1}^{\infty} \left[(\mathbb{Z}/p^t\mathbb{Z})^{rk_{p^t}(H^*(L_i \rtimes G))} \oplus \bigoplus_{j=1}^{\infty} (\mathbb{Z}/p_j^t\mathbb{Z})^{rk_{p_j^t}(H^*(L_i \rtimes G))} \right]$$

where p_j, j = 1, 2, ... are distinct primes coprime to p. Tensoring with $\mathbb{Z}_{(p)}$, we annihilate all $\mathbb{Z}/p_j^t\mathbb{Z}$ summands that are not $\mathbb{Z}/p^t\mathbb{Z}$, giving:

$$H^*(L_i \rtimes G, \mathbb{Z}) \otimes \mathbb{Z}_{(p)} = \mathbb{Z}_{(p)}^{\beta^*(L_i \rtimes G)} \oplus \bigoplus_{t=1}^{\infty} (\mathbb{Z}/p^t\mathbb{Z})^{rk_{p^t}(H^*(L_i \rtimes G))} \tag{3.6}$$

However, recalling that $\beta_*(\mathrm{L_i} \rtimes \mathrm{G}) = \beta^*(\mathrm{L_i} \rtimes \mathrm{G})$ and $\mathrm{rk}_{p^t}(\mathrm{H}^*(\mathrm{L_i} \rtimes \mathrm{G})) = \mathrm{rk}_{p^t}(\mathrm{H}_{*-1}(\mathrm{L_i} \rtimes \mathrm{G})$, we conclude

$$H^*(L_{\mathrm{i}} \rtimes G, \mathbb{Z}_{(p)}) \cong H^*(L_{\mathrm{i}} \rtimes G, \mathbb{Z}) \otimes \mathbb{Z}_{(p)}. \tag{3.7}$$

rearranging and with the (3.2) isomorphism gives:

$$H^*(L_1 \rtimes G, \mathbb{Z}) \otimes \mathbb{Z}_{(p)} \cong H^*(L_1 \rtimes G, \mathbb{Z}_{(p)}) \cong H^*(L_2 \rtimes G, \mathbb{Z}_{(p)}) \cong H^*(L_2 \rtimes G, \mathbb{Z}) \otimes \mathbb{Z}_{(p)}, \tag{3.8}$$

which shows that $H^*(\Gamma_1, \mathbb{Z})$ and $H^*(\Gamma_2, \mathbb{Z})$ can only differ by torsion prime to p.

Since (3.8) holds for each prime p, $H^*(L_1 \rtimes G, \mathbb{Z})$ and $H^*(L_2 \rtimes G, \mathbb{Z})$ agree on rank and for each torsion. Moreover, since $\mathrm{H}^p(\mathrm{G}, \mathrm{H}^q(\mathrm{L_i}, \mathbb{Z}))$ is finitely generated for each i and each p + q = n (since G is finite and $\mathrm{H}^q(\mathrm{L_i}, \mathbb{Z})$ is a G-module that is finitely generated as an abelian group), we know that $\mathrm{H}^n(\Gamma_\mathrm{i}, \mathbb{Z})$ is finitely generated for each n and each i by an inspection of the spectral sequence. But then the fundamental theorem of finitely generated abelian groups shows that

$$H^*(L_1 \rtimes G, \mathbb{Z}) \cong H^*(L_2 \rtimes G, \mathbb{Z}).$$

Stating what has been proved:

Proposition 3.1. *Suppose G is a finite group and L_1 and L_2 are $\mathbb{Z}G$-lattices in the same genus. Then*

$$H^*(L_1 \rtimes G, \mathbb{Z}) \cong H^*(L_2 \rtimes G, \mathbb{Z})$$

Which shows that the additive structure of group cohomology cannot distinguish between integral representations in the same genus.

This result can be extended with G infinite now, so long as G acts through a finite group and G satisfies some conditions.

Consider two split short exact sequences of groups

$$0 \to \mathrm{L_i} \to \mathrm{L_i} \rtimes_{\phi_i} \mathrm{H} \to \mathrm{H} \to 0, \ \mathrm{i} = 1, 2,$$

where H is a finite group and $\mathrm{L_i}$ is a $\mathbb{Z}$G-lattice on which H acts via ϕ_{i}, respectively. Suppose further that $\mathrm{L_1} \vee \mathrm{L_2}$. Now suppose G is an infinite group and there is a map $\psi : \mathrm{G} \to \mathrm{H}$ through which G acts on $\mathrm{L_i}$. From this action, we get the split short exact sequences:

$$0 \to \mathrm{L_i} \to \mathrm{L_i} \rtimes_{\phi_i \circ \psi} \mathrm{G} \to \mathrm{G} \to 0, \ \mathrm{i} = 1, 2.$$

Theorem 3.1. *Suppose G is a finite group and L_1 and L_2 are $\mathbb{Z}G$-lattices in the same genus. Then*

$$H^*(L_1 \rtimes_{\phi_1} G, \mathbb{Z}) \cong H^*(L_2 \rtimes_{phi_2} G, \mathbb{Z})$$

From here the proof is the same except we have to place a requirement that G have finitely generated cohomology groups when the coefficient modules are $\mathrm{H}^q(\mathrm{L_i}, \mathbb{Z}_{(p_j)})$ for any j and $\mathrm{H}^q(\mathrm{L_i}, \mathbb{Z})$. It is not known for exactly what groups with requirement holds, but the requirement has been shown to be met by large classes of groups in [22].

3.1 A Corollary for *p*-groups

If we assume that the finite group G from the previous section is a *p*-group for some prime *p*, then more can be said. Let $\mathrm{L_1} \vee \mathrm{L_2}$ be two $\mathbb{Z}$G-lattices.

In the previous section, the map $\mathrm{f}_{(p)}$ must be constructed for each nonzero prime ideal (p) of $\mathbb{Z}$. Each $\mathrm{f}_{(p)}$ then gives us a map that allows us to construct a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & L_1 \arrow[r] \arrow[d, "f_{(p)}"] & \Gamma_1 = L_1 \rtimes G \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L_2 \arrow[r] & \Gamma_2 = L_2 \rtimes G \arrow[r] & G \arrow[r] & 0
\end{tikzcd}$$

and we can run into problems. Let us consider the case where Γ_1 is a non-split extension of G by $\mathrm{L_1}$. The map $\mathrm{f}_{(p)}$ associated to the ideal (p) sits in a similar diagram to the above

$$\begin{tikzcd}
0 \arrow[r] & L_1 \arrow[r] \arrow[d, "f_{(p)}"] & \Gamma_1 \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L_2 \arrow[r] & \Gamma_2 \arrow[r] & G \arrow[r] & 0
\end{tikzcd}$$

where $\alpha \in H^2(G, L_1)$, the extension class for the top row is mapped to $\beta \in H^2(G, L_2)$, the extension class for the bottom row (see Exercise 1b in Chapter IV.3 of [23]).

Meanwhile, the map induced by the $f_{(q)}$ associated to a different ideal (q) might map α to a different extension class in $H^2(G, L_2)$. This would prevent us from "gluing" the maps together in the way we did in the previous section. We handled this by using the semidirect product for both Γ_1 and Γ_2, which corresponds to the zero extension class, and the homomorphism induced by $f_{(p)}$ sends zero to zero for whichever ideal (p) was used to produce $f_{(p)}$.

However, if G is a p-group, then there is only one prime ideal that ultimately produces anything, so gluing maps together ceases to be an issue and we can remove the requirement to work over semi-direct products and can instead consider general group extensions.

Corollary 3.2. Suppose G is a finite p-group, L_1 and L_2 are $\mathbb{Z}G$-lattices in the same genus, and we have the following group extensions:

$$0 \to L_1 \to \Gamma_1 \to G \to 0, \tag{3.9}$$

and

$$0 \to L_2 \to \Gamma_2 \to G \to 0. \tag{3.10}$$

Assume that extension class of (3.9) is the image of the extension class of (3.10) under the map induced from the map given by Roiter's Theorem. Then we have an isomorphism:

$$H^*(\Gamma_1, \mathbb{Z}) \cong H^*(\Gamma_2, \mathbb{Z}).$$

Moreover, since this map is induced from a map $\Gamma_1 \to \Gamma_2$, this map is even a ring homomorphism.

4. INDUCED MODULES

4.1 Ordinary Induction

First, a lemma:

Lemma 4.1. Suppose G is a finite group, H⊂G, M and N are $\mathbb{Z}$H-lattices, and M∨N. Then Ind_H^GM∨Ind_H^GN.

Proof. We have that $\mathrm{Ind}_H^G\mathrm{M} = \mathbb{Z}G \otimes_{\mathbb{Z}H}\mathrm{M}$ and $\mathrm{Ind}_H^G\mathrm{N} = \mathbb{Z}G \otimes_{\mathbb{Z}H}\mathrm{N}$. M∨N means that $\mathrm{M}\otimes\mathbb{Z}_P \cong \mathrm{N}\otimes\mathbb{Z}_P$ as $\mathbb{Z}$H-modules for each prime ideal P, where the subscript refers to localization. But then we have

$$\begin{aligned}
(\mathrm{Ind}_H^G M)_P &\cong (\mathbb{Z}G \otimes_{\mathbb{Z}H} M)_P \\
&\cong (\mathbb{Z}G \otimes_{\mathbb{Z}H} M) \otimes_{\mathbb{Z}} \mathbb{Z}_P \\
\text{(Associativity of Tensor Product)} \ &\cong \mathbb{Z}G \otimes_{\mathbb{Z}H} (M \otimes_{\mathbb{Z}} \mathbb{Z}_P) \\
\text{(Using M}\vee\text{N)} \ &\cong \mathbb{Z}G \otimes_{\mathbb{Z}H} (N \otimes_{\mathbb{Z}} \mathbb{Z}_P) \\
\text{(Associativity of Tensor Product)} \ &\cong (\mathbb{Z}G \otimes_{\mathbb{Z}H} N) \otimes_{\mathbb{Z}} \mathbb{Z}_P \\
&\cong (\mathbb{Z}G \otimes_{\mathbb{Z}H} N)_P \\
&\cong (\mathrm{Ind}_H^G N)_P.
\end{aligned}$$

So Ind_H^GM∨Ind_H^GN. □

Since $\mathbb{Z}$ and $\mathbb{Z}$H are permutation representations, we have that $\mathrm{Ind}_H^G\mathbb{Z}$ and $\mathrm{Ind}_H^G\mathbb{Z}\mathrm{H}$ are also permutation representations. Then, by 3.2 of [3], the Koszul complex, K_*, associated to $\mathrm{Ind}_H^G\mathbb{Z}$ and $\mathrm{Ind}_H^G\mathbb{Z}\mathrm{H}$ admit compatible actions a la Brady. Then, by Theorem 2.3 of [2], we have that the spectral sequence collapses without extension problems. In other words,

$$\mathrm{H}^n(\mathrm{Ind}_H^G\mathbb{Z} \rtimes G, \mathbb{Z}) \cong \bigoplus_{p+q=n} \mathrm{H}^{\mathrm{i}}(G, \mathrm{H}^{\mathrm{j}}(\mathrm{Ind}_H^G\mathbb{Z}, \mathbb{Z})),$$

and

$$\mathrm{H}^n(\mathrm{Ind}_H^G\mathbb{Z}\mathrm{H} \rtimes G, \mathbb{Z}) \cong \bigoplus_{p+q=n} \mathrm{H}^{\mathrm{i}}(G, \mathrm{H}^{\mathrm{j}}(\mathrm{Ind}_H^G\mathbb{Z}\mathrm{H}, \mathbb{Z})).$$

By Proposition 4.1, when we are considering the cohomology of a semi-direct product of a finite group H and a $\mathbb{Z}$H-lattice $\hat{L}$, we can instead work with any other $\mathbb{Z}$H-lattice, L, that is in the same genus as $\hat{L}$. If H is cyclic of prime order, this implies that we can always pick L to be of the form $(\mathbb{Z})^r \oplus (\mathbb{Z}\mathrm{H})^s \oplus (\mathrm{IH})^t$, for some r, s, and t, by [24].

From this, other than the augmentation ideal summands, we see that the an induced representation of an integer representation of a cyclic group of prime order will always cause the LHS spectral sequence to collapse.

Unfortunately, $\mathrm{Ind}_H^G \mathrm{IH}$ isn't quite as easy to describe. Fortunately, Langer and Lück developed an approach via free groups in [5] that will be helpful. In particular, they state and prove the theorem:

Theorem 4.1. *Let K be an arbitrary group acting on the lattice L. There is a compatible K-action on the Koszul resolution if and only if the K-action on L can be lifted to a K-action on* $F_n/\Gamma_2\Gamma_2 F_n$.

Here F_n is the free group on n letters (n being the rank of the lattice L), $\Gamma_2\mathrm{F}_n$ is the commutator subgroup of F_n, and $\Gamma_2\Gamma_2\mathrm{F}_n$ is the commutator subgroup of the commutator subgroup of F_n.

Naturally, if a group actions lifts all the way to F_n, then it lifts to $\mathrm{F}_n/\Gamma_2\Gamma_2\mathrm{F}_n$, and Langer and Lück with Examples 4.21 and 4.22 in [5] give examples of such lifts for L being a permutation G-module and G arbitrary finite, and L being a syzygy of a permutation module for G being a finite cyclic group, respectively. A lift to F_n interacts nicely with induced modules, as well:

Lemma 4.2. Suppose H$\subset$G and L is a $\mathbb{Z}$H-lattice of $\mathbb{Z}$-rank n. If the action of H on L lifts to F_n, then the induced action of G on $\mathrm{Ind}_H^G \mathrm{L}$ lifts to $\mathrm{F}_{n[G:H]}$.

Proof. For any choice of coset representatives $\{\gamma_{\mathrm{i}}\}_{\mathrm{i}=1}^{[G:H]}$ for G/H, we have: [1]

$$Ind_H^G L = \bigoplus_{\mathrm{i}=1}^{[G:H]} \gamma_{\mathrm{i}} L. \tag{4.1}$$

Since this is a G-module, we know that there is a G-action on it, which we can carefully write down. There are[2] set functions ρ : G×{1, ..., [G : H]} → {1, ..., [G : H]} and h : G×{1, ..., [G : H]} → H such that

$$g\gamma_{\mathrm{i}} = \gamma_{\rho(g,\mathrm{i})} h(g,\mathrm{i}). \tag{4.2}$$

And so an element γ_{i}l of Ind_H^GL is acted on in the following way:

$$\begin{aligned} g(g\gamma_{\mathrm{i}} l) &= g(\gamma_{\rho(g,\mathrm{i})} h(g,\mathrm{i}))l & (4.3)\\ &= \gamma_{\rho(g,\rho(g,\mathrm{i}))} h(g,\rho(g,\mathrm{i})) h(g,\mathrm{i}) l. & (4.4) \end{aligned}$$

Similarly,

$$(gg)(\gamma_{\mathrm{i}} l) = \gamma_{\rho(gg,\mathrm{i})} h(gg,\mathrm{i}) l. \tag{4.5}$$

The pertinent properties of the set functions ρ and h can be deduced by knowledge that this must be a group action. That is, that the equality must hold:

$$\gamma_{\rho(g,\rho(g,\mathrm{i}))} h(g,\rho(g,\mathrm{i})) h(g,\mathrm{i}) l = \gamma_{\rho(gg,\mathrm{i})} h(gg,\mathrm{i}) l \tag{4.6}$$

which implies (by setting l = 1) that

$$\rho(g,\rho(g,\mathrm{i})) = \rho(gg,\mathrm{i}) \tag{4.7}$$

[1]↑For more information on this construction in the context of group cohomology, see page 67 of [23]

[2]↑These functions are well defined because for each g ∈G we can write g uniquely as $\gamma_{\mathrm{i}_g} h_g$ for some i_g ∈{1, ..., [G : H]} and h_g ∈H.

and that

$$h(g,\rho(g,\mathrm{i}))h(g,\mathrm{i}) = h(gg,\mathrm{i}) \tag{4.8}$$

Next, by assumption we have that the action of H on L lifts to an action of H on $\mathrm{F}_n = \langle \mathrm{x}_1, \ldots, \mathrm{x}_n \rangle$. Now, for lack of better notation, we write

$$F_{n[G:H]} = (F_n)_{\gamma_1} \star \cdots \star (F_n)_{\gamma_{[G:H]}} \tag{4.9}$$

where words in the free group will be written as $(w)_{\gamma_{\mathrm{j}}}$ where γ_{j} indicates which copy F_n we are working with and w is a word in F_n. We can assume that the action of H lifts through the map $\mathrm{x}_{\mathrm{i}} \mapsto \mathrm{l}_{\mathrm{i}}$, where $\{\mathrm{l}_{\mathrm{i}}\}$ is a basis for L. With that out of the way, we can construct a G-action on this large free group that is a lift of the action of G on $\mathrm{Ind}_H^G\mathrm{L}$. Given an element $(\mathrm{x}_{\mathrm{i}})_{\gamma_{\mathrm{j}}}$, we can act by an element of G, g using the formula:

$$g(\mathrm{x}_{\mathrm{i}})_{\gamma_{\mathrm{j}}} = (h(g,\mathrm{j}){\cdot}\mathrm{x}_{\mathrm{i}})_{\gamma_{\rho(g,\mathrm{j})}}$$

This is an action because if g'∈G, we have:

$$\begin{aligned}
g(g(x_{\mathrm{i}})_{\gamma_{\mathrm{j}}}) &= g((h(g,\mathrm{j})\cdot x_{\mathrm{i}})_{\gamma_{\rho(g,\mathrm{j})}}) && (4.10)\\
&= (h(g,\rho(g,\mathrm{j}))h(g,\mathrm{j})\cdot x_{\mathrm{i}})_{\gamma_{\rho(g,\rho(g,\mathrm{j}))}} && (4.11)\\
\text{Using equations 4.7 and 4.8 } &= (h(gg,\mathrm{j})\cdot x_{\mathrm{i}})_{\gamma_{\rho(gg,\mathrm{j})}} && (4.12)\\
&= (gg)(x_{\mathrm{i}})_{\gamma_{\mathrm{j}}}. && (4.13)
\end{aligned}$$

And using the map $(\mathrm{x}_{\mathrm{i}})_{\gamma_{\mathrm{j}}} \mapsto \gamma_{\mathrm{j}}\otimes\mathrm{l}_{\mathrm{i}}$, we see that this G-action is a lift of the G-action on $\mathrm{Ind}_H^G\mathrm{L}$. □

From here we have the necessary results to prove a theorem:

Theorem 4.2. *If $H \subset G$, L is a $\mathbb{Z}H$-lattice of $\mathbb{Z}$-rank n, and the H action on L lifts to F_n, then the LHS spectral sequence associated to the SES:*

$$0 \to Ind_H^G L \to Ind_H^G L \rtimes G \to G \to 0$$

collapses.

Proof. From Lemma 5.2 we see that the action of G on $\mathrm{Ind}_H^G\mathrm{L}$ lifts to the free group $\mathrm{F}_{n[G:H]}$. The discussion following Theorem 5.1 informs us that this implies the action of G on $\mathrm{Ind}_H^G\mathrm{L}$ lifts to $\mathrm{F}_{n[G:H]}/\Gamma_2\Gamma_2\mathrm{F}_{n[G:H]}$. Theorem 4.1 then tells us that there is a compatible G-action on the Koszul resolution P_*. Then by Theorem 2.4 of [3], the spectral sequence collapses. □

Examples 4.21 and 4.22 of [5] tell us that if we have that H is a finite cyclic group and L is a permutation module or the augmentation ideal, then the H action on L lifts to the free group on a number of letters equal to the $\mathbb{Z}$-rank of L. Moreover, we know that $\mathbb{Z}$ and $\mathbb{Z}[\mathbb{Z}/(p)\mathbb{Z}]$ are permutation $\mathbb{Z}[\mathbb{Z}/(p)\mathbb{Z}]$-modules. These together give us a corollary to the previous theorem:

Corollary 4.1. If $\mathrm{H} = \mathbb{Z}/(p)\mathbb{Z}$ for some prime p, L is an H-module, and $\mathrm{H}\subset\mathrm{G}$ for some arbitrary group G, then the LHS spectral sequence associated to the short exact sequence

$$0 \to \mathrm{Ind}_H^G\mathrm{L} \to \mathrm{Ind}_H^G\mathrm{L}\rtimes\mathrm{G} \to \mathrm{G} \to 0$$

collapses.

More can be said about the interaction between induction and compatible actions, since Lemma 4.2 above can be improved:

Lemma 4.3. Suppose $\mathrm{H}\subset\mathrm{G}$ and L is a $\mathbb{Z}$H-lattice of $\mathbb{Z}$-rank n. If the action of H on L lifts to $\mathrm{F}_n/\Gamma_2\Gamma_2\mathrm{F}_n$, then the induced action of G on $\mathrm{Ind}_H^G\mathrm{L}$ lifts to $\mathrm{F}_{n[G:H]}/\Gamma_2\Gamma_2\mathrm{F}_{n[G:H]}$.

Proof. It can be seen, in the same way as in the proof of the previous lemma, that that G action on $\mathrm{Ind}_H^G\mathrm{L}$ can be lifted to $(\mathrm{F}_n/\Gamma_2\Gamma_2\mathrm{F}_n)\star\cdots\star(\mathrm{F}_n/\Gamma_2\Gamma_2\mathrm{F}_n)$ ([G : H] factors). But this group surjects down to $(\mathrm{F}_n\star\cdots\star\mathrm{F}_n)/\Gamma_2\Gamma_2(\mathrm{F}_n\star\cdots\star\mathrm{F}_n)$, with kernel the characteristic[3] subgroup $\Gamma_2\Gamma_2((\mathrm{F}_n/\Gamma_2\Gamma_2\mathrm{F}_n)\star\cdots\star(\mathrm{F}_n/\Gamma_2\Gamma_2\mathrm{F}_n))$.

Checking that this is indeed the kernel is made easier with the following diagram (using double primes to be $\Gamma_2\Gamma_2$, the double commutator subgroup, to conserve space):

[3]↑The subgroup being characteristic is important because of the lemma from any standard first year course in abstract algebra: If G acts on K and maps $\mathrm{N}\triangleleft\mathrm{K}$ to itself, then G acts on K/N.

$$\begin{tikzcd}
(F_n \star \cdots \star F_n) \arrow[r] \arrow[d, dashed] & F_n \star \cdots \star F_n \arrow[r] \arrow[d, "\psi"] & (F_n \star \cdots \star F_n)/(F_n \star \cdots \star F_n) \\
N = Ker(\phi) \arrow[r] & (F_n/F_n) \star \cdots \star (F_n/F_n) \arrow[ur, "\phi"'] &
\end{tikzcd}$$

The containment $\Gamma_2\Gamma_2((\mathrm{F}_n/\Gamma_2\Gamma_2\mathrm{F}_n)\star\cdots\star(\mathrm{F}_n/\Gamma_2\Gamma_2\mathrm{F}_n)) \subset \mathrm{N}$ is clear from the diagram because of the codomain of ϕ.

The reverse containment $\mathrm{N} \subset \Gamma_2\Gamma_2((\mathrm{F}_n/\Gamma_2\Gamma_2\mathrm{F}_n)\star\cdots\star(\mathrm{F}_n/\Gamma_2\Gamma_2\mathrm{F}_n))$ is provided by the fact that ψ is onto which makes the dashed arrow in the above diagram onto by the 5-lemma. □

From here it follows:

Theorem 4.3. *If $H \subset G$, L is a $\mathbb{Z}H$-lattice of $\mathbb{Z}$-rank n, and there is a compatible H-action on the Koszul resolution over $\mathbb{Z}L$, then there is a compatible G-action on the Koszul resolution over $\mathbb{Z}[Ind_H^G L]$. Consequently, the LHS spectral sequence associated to the SES:*

$$0 \to Ind_H^G L \to Ind_H^G L \rtimes G \to G \to 0$$

collapses without extension problems.

Proof. The proof is simply a combination of the above lemma 3, theorem 5.1, and theorem 2.4 of [3]. □

As a final note before moving on to more complicated things, I will point out that compatible actions are not as simple as they might appear. For example, the theorems proved in this section might lead one to, incorrectly, believe that in the setting of these specific types of group extensions

$$0 \to \mathrm{L} \to \mathrm{L} \rtimes \mathrm{H} \to \mathrm{H} \to 0,$$

where L is a $\mathbb{Z}$H-lattice of rank n and H is a finite group, there might be some connection between the second page of the associated LHS spectral sequence and the associated induced module up to some larger finite group G with H$\subset$G. That is to say, that we might be able to construct a map ϕ

$$\phi : \mathrm{H}^p(\mathrm{H}, \mathrm{H}^q(\mathrm{L}, \mathbb{Z})) \to \mathrm{H}^p(\mathrm{G}, \mathrm{H}^q(\mathrm{Ind}_H^G\mathrm{L}, \mathbb{Z})).$$

This seems reasonable because if the H^q(- , $\mathbb{Z}$) functor and the Ind$_H^G$- functor commute, we get an isomorphism through Shapiro's Lemma:

$$\begin{tikzcd}
H^p(H, H^q(L,\mathbb{Z})) \arrow[r, equal, "\text{Shapiro's Lemma}"] \arrow[d, equal] & H^p(G, Ind_H^G H^q(L,\mathbb{Z})) \arrow[d, equal, "Commute?"] \\
H^p(H, H^q(L,\mathbb{Z})) \arrow[r, dashed, "\phi"] & H^p(G, H^q(Ind_H^G L,\mathbb{Z}))
\end{tikzcd}$$

A lot of effort can be thrown into demonstrating the commutativity of the Ind functor and the cohomology functor in this setting of $\mathbb{Z}$H-lattices. For example, universal properties of exterior algebras and tensor products can be used to construct G-module maps out of direct products and these maps can then be made explicit be developing bases for each object. Then one simply has to show that the constructions are inverses of one another... A lot of thought and energy can be placed into this because there are a lot of choices involved and your initial inability to make the maps work together might be blamed on a bad choice of basis or a bad choice of map. I know that a lot of time and effort can be put into this because I put about a week of effort into it and wound up with nothing. I include this passage because I figure the most likely reader of this thesis is some poor graduate student desperately hoping that I found and documented an answer to a question you have, and I hope that knowing that I, too, spent weeks and months chasing proverbial wild geese might give you some measure of hope in your own chasing.

The reason that the maps don't ever seem to work together is that Ind$_H^G$H^q(L, $\mathbb{Z}$) and H^q(Ind$_H^G$L, $\mathbb{Z}$) are not even isomorphic as abelian groups. Taken as an abelian group, L $\cong$ $\mathbb{Z}^n$. So Ind$_H^G$L $\cong \mathbb{Z}^{[G:H]n}$, and H^q(Ind$_H^G$L, $\mathbb{Z}$) $\cong \mathbb{Z}^{\binom{[G:H]n}{q}}$.

Meanwhile, H^q(L, $\mathbb{Z}$) $\cong \mathbb{Z}^{\binom{n}{q}}$, so Ind$_H^G$H^q(L, $\mathbb{Z}$) $\cong \mathbb{Z}^{[G:H]\binom{n}{q}}$.

Unfortunately $\binom{[G:H]n}{q} = [G:H]\binom{n}{q}$ only at values of [G : H], n, or q that make the problem uninteresting. Drat. This, however, shows that the above results concerning the preservation of a property of the spectral sequence (its collapse) through induction is remarkable. Let's expand it.

4.2 Higher Induction

Letting H⊂G, L an H-module, and $\{\gamma_1, \ldots, \gamma_m\}$ a complete set of representative elements of the cosets of G/H, the ordinary induction used in the previous section can be defined:

$$\mathrm{Ind}_H^G \mathrm{L} = \bigoplus_{\mathrm{i}=1}^{m} \gamma_{\mathrm{i}} \mathrm{L}.$$

This construction creates a G-module because, as stated above, G will permute the coset representatives γ_{i} while acting on L via H using the unique description for each element of G as γ_{i}h for some i and some h∈H.

However, it turns out that this strategy can be used to construct different types of G-module structures by noticing that the important aspect of $\bigoplus_{\mathrm{i}=1}^{m} \gamma_{\mathrm{i}} \mathrm{L}$ is that the permutations created by the action of G will preserve the structure. And so any such structure, preserved by permutations, will similarly create a G-module. Thus, an induced G-module can be constructed from an H-module using any elementary symmetric polynomial, i.e.:

$${}^{(k)}\mathrm{Ind}_H^G \mathrm{L} = \bigoplus_{1 \leq \mathrm{i}_1 < \cdots < \mathrm{i}_k \leq m} \gamma_{\mathrm{i}_1} \mathrm{L} \otimes \cdots \otimes \gamma_{\mathrm{i}_k} \mathrm{L}$$

is the k-th higher induced G-module over the H-module L. The G-module structure is defined via[4]

$$\mathrm{g}(\gamma_{\mathrm{i}_1} \mathrm{l}_1 \otimes \cdots \otimes \gamma_{\mathrm{i}_k} \mathrm{l}_k) = [\mathrm{g}\gamma_{\mathrm{i}_1} \mathrm{l}_1 \otimes \cdots \otimes \mathrm{g}\gamma_{\mathrm{i}_k} \mathrm{l}_k].$$

Since g can be uniquely represented as γ_gh for some element γ_g in the list of coset representatives and h∈H, g acts on each γ_{i} via γ_g while the h passes through and acts on the element of L.

It is easy to see that ${}^{(1)}\mathrm{Ind}_H^G \mathrm{L} = \mathrm{Ind}_H^G \mathrm{L}$, the ordinary induction. It can also be seen that ${}^{(m)}\mathrm{Ind}_H^G \mathrm{L}$ is equal to what is known in the literature as "Tensor Induction".[5] We can prove some of the same things for these higher forms of induction as we proved for ordinary induction.

[4]↑The brackets on the right side of the equation indicate the element within the brackets reordered to be in the correct ordering.

[5]↑While I did not discover these other forms of induction, they did not appear to have a name in the literature.

Lemma 4.4. Suppose G is a finite group, H⊂G with [G : H] = n, $\{\gamma_1, \ldots, \gamma_m\}$ is a set of coset representatives of G/H, M and N are $\mathbb{Z}$H-lattices, and M∨N. Then ${}^{(k)}\mathrm{Ind}_H^G \mathrm{M} \vee {}^{(k)}\mathrm{Ind}_H^G \mathrm{N}$.

Proof. We have that ${}^{(k)}\mathrm{Ind}_H^G \mathrm{M} = \bigoplus_{1\leq \mathrm{i}_1<\cdots<\mathrm{i}_k\leq m} (\gamma_{\mathrm{i}_1}\mathrm{M})\otimes\cdots\otimes(\gamma_{\mathrm{i}_k}\mathrm{M})$. M∨N means that $\mathrm{M}\otimes\mathbb{Z}_P \cong \mathrm{N}\otimes\mathbb{Z}_P$ for each prime ideal P, where the subscript refers to localization. But then we have

$$
\begin{aligned}
({}^{(k)}Ind_H^G M)_P &\cong \Big(\bigoplus_{1\leq \mathrm{i}_1<\cdots<\mathrm{i}_k\leq m} (\gamma_{\mathrm{i}_1}M)\otimes\cdots\otimes(\gamma_{\mathrm{i}_k}M)\Big)_P \\
&\cong \Big(\bigoplus_{1\leq \mathrm{i}_1<\cdots<\mathrm{i}_k\leq m} (\gamma_{\mathrm{i}_1}M)\otimes\cdots\otimes(\gamma_{\mathrm{i}_k}M)\Big)\otimes\mathbb{Z}_P \\
&\cong \bigoplus_{1\leq \mathrm{i}_1<\cdots<\mathrm{i}_k\leq m} \Big((\gamma_{\mathrm{i}_1}M)\otimes\cdots\otimes(\gamma_{\mathrm{i}_k}M)\Big)\otimes\mathbb{Z}_P \\
&\cong \bigoplus_{1\leq \mathrm{i}_1<\cdots<\mathrm{i}_k\leq m} \Big((\gamma_{\mathrm{i}_1}M)\otimes\mathbb{Z}_P\otimes\cdots\otimes(\gamma_{\mathrm{i}_k}M)\otimes\mathbb{Z}_P\Big) \\
&\cong \bigoplus_{1\leq \mathrm{i}_1<\cdots<\mathrm{i}_k\leq m} \Big((\gamma_{\mathrm{i}_1}M\otimes\mathbb{Z}_P)\otimes\cdots\otimes(\gamma_{\mathrm{i}_k}M\otimes\mathbb{Z}_P)\Big) \\
&\cong \bigoplus_{1\leq \mathrm{i}_1<\cdots<\mathrm{i}_k\leq m} \Big((\gamma_{\mathrm{i}_1}N\otimes\mathbb{Z}_P)\otimes\cdots\otimes(\gamma_{\mathrm{i}_k}N\otimes\mathbb{Z}_P)\Big) \\
&\cong \bigoplus_{1\leq \mathrm{i}_1<\cdots<\mathrm{i}_k\leq m} \Big((\gamma_{\mathrm{i}_1}N)\otimes\mathbb{Z}_P\otimes\cdots\otimes(\gamma_{\mathrm{i}_k}N)\otimes\mathbb{Z}_P\Big) \\
&\cong \bigoplus_{1\leq \mathrm{i}_1<\cdots<\mathrm{i}_k\leq m} \Big((\gamma_{\mathrm{i}_1}N)\otimes\cdots\otimes(\gamma_{\mathrm{i}_k}N)\Big)\otimes\mathbb{Z}_P \\
&\cong \Big(\bigoplus_{1\leq \mathrm{i}_1<\cdots<\mathrm{i}_k\leq m} (\gamma_{\mathrm{i}_1}N)\otimes\cdots\otimes(\gamma_{\mathrm{i}_k}N)\Big)\otimes\mathbb{Z}_P \\
&\cong \Big(\bigoplus_{1\leq \mathrm{i}_1<\cdots<\mathrm{i}_k\leq m} (\gamma_{\mathrm{i}_1}N)\otimes\cdots\otimes(\gamma_{\mathrm{i}_k}N)\Big)_P \\
&\cong^{(k)} Ind_H^G(N)_P
\end{aligned}
$$

□

Next, we may move immediately to the same questions of preservation of lifting to free groups as before in ordinary induction:

Lemma 4.5. Suppose H⊂G and L is a $\mathbb{Z}$H-lattice of $\mathbb{Z}$-rank n. If the action of H on L lifts to F_n, then the induced action of G on ${}^{(k)}\mathrm{Ind}_H^G \mathrm{L}$ lifts to $\mathrm{F}_{n^k\binom{m}{k}}$, which can be better written as:

$$(\mathrm{F}_{n^k})\star\cdots\star(\mathrm{F}_{n^k})$$

i.e., the free product of $\binom{m}{k}$ free groups on n^k each.

Proof. Take $\mathrm{l}_1, \ldots, \mathrm{l}_n$ to be a basis for L, $\mathrm{x}_1, \ldots, \mathrm{x}_n$ to be a free generating set for F_n, and take ordered list of length k with entries in $\mathrm{x}_1, \ldots, \mathrm{x}_n$ to be a free generating set for F_{n^k}. A free generating set for the full $\mathrm{F}_{n^k\binom{m}{k}}$ will be all the ordered sets in each F_{n^k} with a label to describe which F_{n^k} the list comes from. As an example: $(\mathrm{x}_{\mathrm{j}_1}, \ldots, \mathrm{x}_{\mathrm{j}_k})_{\mathrm{i}_1,\ldots,\mathrm{i}_k}$ is an element of the free generating set for $\mathrm{F}_{n^k\binom{m}{k}}$. Also, take $\{\gamma_1, \ldots, \gamma_m\}$ to be a fixed choice of representatives of G/H. Then we have

$$^{(k)}\mathrm{Ind}_H^G\mathrm{L} = \bigoplus_{1\leq \mathrm{i}_1<\cdots<\mathrm{i}_k\leq m}(\gamma_{\mathrm{i}_1}\mathrm{L})\otimes\cdots\otimes(\gamma_{\mathrm{i}_k}\mathrm{L}).$$

We can express the G-module structure in terms of the data provided by the functions $\rho : \mathrm{G}\times\{1, \ldots, \mathrm{m}\} \to \{1, \ldots, \mathrm{m}\}$ and $\mathrm{h} : \mathrm{G}\times\{1, \ldots, \mathrm{m}\} \to \mathrm{H}$ described in the previous section on Ordinary Induction:

$$g(\gamma_{\mathrm{i}_1}l_{\mathrm{i}_1})\otimes\cdots\otimes(\gamma_{\mathrm{i}_k}l_{\mathrm{i}_k}))) = (g(\gamma_{\mathrm{i}_1}l_{\mathrm{i}_1}))\otimes\cdots\otimes(g(\gamma_{\mathrm{i}_k}l_{\mathrm{i}_k})) \tag{4.14}$$

$$= (\gamma_{\rho(g,1)}h(g,\mathrm{i})\cdot l_1)\otimes\cdots\otimes(\gamma_{\rho(g,k)}h(g,k)\cdot l_k). \tag{4.15}$$

Next, by assumption we have that the action of H on L lifts to an action of H on F_n. We can write

$$\mathrm{F}_{n^k\binom{m}{k}} = \bigstar_{1\leq \mathrm{i}_1<\cdots<\mathrm{i}_k\leq m}\Big((\mathrm{F}_{n^k})_{\gamma_{\mathrm{i}_1},\ldots,\gamma_{\mathrm{i}_k}}\Big)$$

where the big star indicates free product and there exists a H-equivariant map

$$(\mathrm{F}_{n^k})_{\gamma_{\mathrm{i}_1},\ldots,\gamma_{\mathrm{i}_k}} \to (\gamma_{\mathrm{i}_1}\mathrm{L})\otimes\cdots\otimes(\gamma_{\mathrm{i}_k}\mathrm{L}).$$

that sends $(\mathrm{x}_{\mathrm{j}_1}, \ldots, \mathrm{x}_{\mathrm{j}_k})_{\gamma_{\mathrm{i}_1},\ldots,\gamma_{\mathrm{i}_k}}$ to $\gamma_{\mathrm{i}_1}\mathrm{l}_{\mathrm{j}_1}\otimes\cdots\otimes\gamma_{\mathrm{i}_k}\mathrm{l}_{\mathrm{j}_k}$.

With that out of the way, we can construct a G-action on this large free group that is a lift of the action of G on ${}^{(k)}\mathrm{Ind}_H^G$L. Given a basis element $(\mathrm{x}_{\mathrm{j}_1}, \ldots, \mathrm{x}_{\mathrm{j}_k})_{\mathrm{i}_1,\ldots,\mathrm{i}_k}$, we can act by an element g∈G using the formula:

$$g(x_{\mathrm{j}_1},\ldots,x_{\mathrm{j}_k})_{\mathrm{i}_1,\ldots,\mathrm{i}_k} = g(x_{\mathrm{j}_1},\ldots,x_{\mathrm{j}_k})_{\mathrm{i}_1,\ldots,\mathrm{i}_k} \tag{4.16}$$

$$= (h(g,\mathrm{i}_1)x_{\mathrm{j}_1},\ldots,h(g,\mathrm{i}_k)x_{\mathrm{j}_k})_{\rho(g,\mathrm{i}_1),\ldots,\rho(g,\mathrm{i}_k)} \tag{4.17}$$

This can be seen to give a group action through the following formal manipulation:

$$g(g((x_{\mathrm{j}_1},\ldots,x_{\mathrm{j}_k})_{\mathrm{i}_1,\ldots,\mathrm{i}_k})) = g((h(g,\mathrm{i}_1)x_{\mathrm{j}_1},\ldots,h(g,\mathrm{i}_k)x_{\mathrm{j}_k})_{\rho(g,\mathrm{i}_1),\ldots,\rho(g,\mathrm{i}_k)} \tag{4.18}$$

$$= g(h(g,\mathrm{i}_1)x_{\mathrm{j}_1},\ldots,h(g,\mathrm{i}_k)x_{\mathrm{j}_k})_{\rho(g,\mathrm{i}_1),\ldots,\rho(g,\mathrm{i}_k)} \tag{4.19}$$

$$= (h(g,\rho(g,\mathrm{i}_1))h(g,\mathrm{i}_1)x_{\mathrm{j}_1},\ldots,h(g,\rho(g,\mathrm{i}_k))h(g,\mathrm{i}_k)x_{\mathrm{j}_k})_{\rho(g,\rho(g,\mathrm{i}_1)),\ldots,\rho(g,\rho(g,\mathrm{i}_k))} \tag{4.20}$$

$$= (h(gg,\mathrm{i}_1)x_{\mathrm{j}_1},\ldots h(gg,\mathrm{i}_k)x_{\mathrm{j}_k})_{\rho(gg,\mathrm{i}_1),\ldots,\rho(gg,\mathrm{i}_k)} \tag{4.21}$$

$$= (gg)((x_{\mathrm{j}_1},\ldots,x_{\mathrm{j}_k})_{\mathrm{i}_1,\ldots,\mathrm{i}_k})) \tag{4.22}$$

□

Of course this then gives a higher induction form of Theorem 4.2:

Theorem 4.1. *If $H \subset G$, L is a $\mathbb{Z}H$-lattice of $\mathbb{Z}$-rank n, and the H action on L lifts to F_n, then the LHS spectral sequence associated to the SES:*

$$0 \to {}^{(k)}Ind_H^G L \to {}^{(k)}Ind_H^G L \rtimes G \to G \to 0$$

collapses.

Proof. The proof is similar to the proof of Theorem 4.2, using Lemma 4.4 instead of Lemma 4.2. □

More can be said about the interaction between higher induction and compatible actions, since Lemma 4 can be improved. Remembering the Γ_2X = [X, X], the commutator subgroup, we have:

Lemma 4.6. Suppose H⊂G and L is a $\mathbb{Z}$H-lattice of $\mathbb{Z}$-rank n. If the action of H on L lifts to $\mathrm{F}_n/\Gamma_2\Gamma_2\mathrm{F}_n$, then the induced action of G on ${}^{(k)}\mathrm{Ind}_H^G$L lifts to $\mathrm{F}_{n^k\binom{m}{k}}/\Gamma_2\Gamma_2\mathrm{F}_{n^k\binom{m}{k}}$

Proof. This is similar to the proof of Lemma 4.3. It can be seen, in the same way as in the proof of Lemma 4.3, that the G action on ${}^{(k)}\mathrm{Ind}_H^G \mathrm{L}$ can be lifted to

$$\mathop{\bigstar}_{1 \leq \mathrm{i}_1 < \cdots < \mathrm{i}_k \leq m} \Big((\mathrm{F}_{n^k})/\Gamma_2\Gamma_2\mathrm{F}_{n^k} \Big)_{\mathrm{i}_1,\ldots,\mathrm{i}_k}.$$

But this group surjects down to $(\mathrm{F}_{n^k} \star \cdots \star \mathrm{F}_{n^k})/\Gamma_2\Gamma_2(\mathrm{F}_{n^k} \star \cdots \star \mathrm{F}_{n^k}) = \mathrm{F}_{n^k\binom{m}{k}}/\Gamma_2\Gamma_2\mathrm{F}_{n^k\binom{m}{k}}$, with kernel the characteristic subgroup $\Gamma_2\Gamma_2(\mathop{\bigstar}_{1 \leq \mathrm{i}_1 < \cdots < \mathrm{i}_k \leq m} \Big((\mathrm{F}_{n^k})/\Gamma_2\Gamma_2\mathrm{F}_{n^k} \Big)_{\mathrm{i}_1,\ldots,\mathrm{i}_k})$. □

From here it follows:

Theorem 4.2. *If $H \subset G$, L is a $\mathbb{Z}H$-lattice of $\mathbb{Z}$-rank n, and there is a compatible H-action on the Koszul resolution over $\mathbb{Z}L$, then there is a compatible G-action on the Koszul resolution over $\mathbb{Z}[{}^{(k)}\mathit{Ind}_H^G L]$. Consequently, the LHS spectral sequence associated to the SES:*

$$0 \to {}^{(k)}\mathit{Ind}_H^G L \to {}^{(k)}\mathit{Ind}_H^G L \rtimes G \to G \to 0$$

collapses without extension problems.

Proof. The proof is simply a combination of the above Lemma 4.6, Theorem 4.1, and Theorem 2.4 of [3]. □

This leaves us with a question: is there a larger generalization of techniques to construct bigger modules out of smaller modules that preserves compatible actions that both induction and direct sum/product are special cases of?

4.2.1 Proof of Lemma 2.4

In section 2.3, we stated Lemma 2.4 as:

Lemma 4.7. Suppose that L_1 and L_2 are G-lattices and suppose that L_1 has an associated Koszul complex that admits a compatible G-action and that L_2 is a permutation representation[6] of G. Then $\mathrm{L}_1 \otimes \mathrm{L}_2$ has a Koszul complex that admits a compatible G-action.

[6]↑Which therefore has a Koszul complex that admits a compatible G-action, see section 2.1

In the previous section we have seen several proofs of similar statements using induction and the proof of this lemma will be very similar, hinging on the theorem of Langer and Lück, proved in [5]:

Theorem 4.3. *Let K be an arbitrary group acting on the lattice L. There is a compatible K-action on the Koszul resolution if and only if the K-action on L can be lifted to a K-action on $F_n/\Gamma_2\Gamma_2 F_n$.*

Where Γ_2X is the double commutator subgroup of X. Let's prove the lemma:

Proof. First, suppose L_1 has $\mathbb{Z}$-rank n_1 with basis $\mathrm{l}_{1_1}, \ldots, \mathrm{l}_{1_{n_1}}$ and L_2 has $\mathbb{Z}$-rank n_2 with basis $\mathrm{l}_{2_1}, \ldots, \mathrm{l}_{2_{n_2}}$. Then $\mathrm{L}_1 \otimes \mathrm{L}_2$ has $\mathbb{Z}$-rank $\mathrm{n}_1\mathrm{n}_2$, and the G action on it must lift to $\mathrm{F}_{n_1 n_2}/\Gamma_2\Gamma_2\mathrm{F}_{n_1 n_2}$.

We assumed that L_1 has an associated Koszul complex that admits a compatible G-action, so the action of G on L_1 lifts to $\mathrm{F}_{n_1}/\Gamma_2\Gamma_2\mathrm{F}_{n_1}$. Similarly, since L_2 is a permutation module of G, we know that the action of G on L_2 lifts all the way to F_{n_2}. So we will make n_2 copies of $\mathrm{F}_{n_1}/\Gamma_2\Gamma_2\mathrm{F}_{n_1}$ and free product them together:

$$\mathop{\bigstar}_{\mathrm{i}=1}^{n_2}(\mathrm{F}_{n_1}/\Gamma_2\Gamma_2\mathrm{F}_{n_1})_{\mathrm{i}}$$

We can see immediately that this surjects down to $\mathrm{F}_{n_1 n_2}/\Gamma_2\Gamma_2\mathrm{F}_{n_1 n_2}$ with kernel the characteristic subgroup $\Gamma_2\Gamma_2(\mathop{\bigstar}_{\mathrm{i}=1}^{n_2}(\mathrm{F}_{n_1}/\Gamma_2\Gamma_2\mathrm{F}_{n_1})_{\mathrm{i}})$ by referring to Lemma 4.6.

We can also see that $\mathop{\bigstar}_{\mathrm{i}=1}^{n_2}(\mathrm{F}_{n_1}/\Gamma_2\Gamma_2\mathrm{F}_{n_1})_{\mathrm{i}}$ has a natural G-action associated to it that is a lift of the G-action on $\mathrm{L}_1 \otimes \mathrm{L}_2$. This G-action is defined as:

$$\mathrm{g}\cdot \mathop{\bigstar}_{\mathrm{i}=1}^{n_2}(\mathrm{f}_{\mathrm{i}})_{l_{2_{\mathrm{i}}}} = \mathop{\bigstar}_{\mathrm{i}=1}^{n_2}(\mathrm{g}\cdot\mathrm{f}_{\mathrm{i}})_{\sigma_g(\mathrm{i})}$$

for each $\mathrm{f}_{\mathrm{i}} \in \mathrm{F}_{n_1}/\Gamma_2\Gamma_2\mathrm{F}_{n_1}$. Here $\mathrm{g}\cdot\mathrm{f}_{\mathrm{i}}$ to be the lift of the action of G on L_1, and here σ_g are the elements of S_{n_2} that align with the action of G on L_2. And we see that this is a lift of the G action on $\mathrm{L}_1 \otimes \mathrm{L}_2$ to

$$\mathop{\bigstar}_{\mathrm{i}=1}^{n_2}(\mathrm{F}_{n_1}/\Gamma_2\Gamma_2\mathrm{F}_{n_1})_{l_{2_{\mathrm{i}}}}.$$

□

4.3 Cocycle Induction

Finally, there is a third type of induction that is significantly newer and less studied than the other types of induction inspected thus far. This is cocycle induction. For our definition and introduction to the topic, we will be making heavy use of [25].

Definition 4.1. *Suppose $G \curvearrowright X$ is an action of a group G on a set X, and H is another group. A cocycle for the action into H is a map $\alpha : G \times X \to H$ such that*

$$\alpha(g_1 g_2,\ x) = \alpha(g_1,\ g_2 x)\alpha(g_2,\ x),$$

for all $g_1,\ g_2 \in G$ and $x \in X$.

Two cocycles $\alpha, \beta :$ G×X → H are cohomologous if there is a map $\xi :$ X → K such that

$$\alpha(\mathrm{g},\ \mathrm{x}) = \xi(\mathrm{gx})\beta(\mathrm{g},\ \mathrm{x})\xi(\mathrm{x})^{-1},$$

for all g∈G, x∈X. Using this, we can define a cocycle induction of a ℤH-lattice L by alpha:

$${}_{\alpha}\mathrm{Ind}_H^G \mathrm{L} := \bigoplus_{x \in X} \mathrm{L}_x$$

Here each L_x is a copy of the original ℤH-module L with the subscript serving only to distinguish its position in the sum. We denote an element of L_x as an element l of L with a subscript x: $\mathrm{l}_x \in \mathrm{L}_x$. The ℤG-module structure of this induction is given via the cocycle in the formula:

$$g(l_x) = (\alpha(g, x)l)_{g \cdot x} \tag{4.23}$$

with the action extending to the full module by linearity.

There are two natural ways in which cocycles arise in this area of mathematics.

First, suppose H and G are groups with $\mathrm{G} \curvearrowright \mathrm{X} \curvearrowleft \mathrm{H}$ being a pair of commuting[7] actions. Peterson, in [25], explains how to obtain a cocycle (uniquely up to cohomology[8]) $\alpha :$ G×X/H → H.

[7]↑In this setting, commuting actions means that for each g∈G, h∈H, and x∈X, we have (gx)h = g(xh).
[8]↑I understand that this word is being abused here. I apologize.

Suppose H is a subgroup of G. Then we can let X = G with actions given by left and right multiplication and we obtain $\alpha :$ G$\times$(G/H) $\to$ H. Peterson's construction then gives a cocycle. This cocycle is identical to the function h : G$\times\{1, \ldots, [\mathrm{G : H}]\} \to$ H from section 4.1, and we can see that the cocycle induction corresponding to this example is identical to the ordinary induction handled in section 4.1:

$$_{\alpha}\mathrm{Ind}_H^G\mathrm{L} = \bigoplus_{\gamma_i \in G/H} \gamma_i\mathrm{L} = \mathrm{Ind}_H^G\mathrm{L}.$$

The second way these come about is if there are group actions G$\curvearrowright$X and H$\curvearrowright$X with the action of H is free (if h$\neq$id$_H$ then hx $\neq$ x for all x$\in$X), and if Gx $\subset$ Hx for all x$\in$X, then a cocycle $\alpha :$ G$\times$X $\to$ H can be defined by setting α(g, x) to be the unique element in H such that

$$\mathrm{gx} = \alpha(\mathrm{g, x})\mathrm{x}.$$

Theorem 4.1. *Suppose that L is a $\mathbb{Z}$H-lattice and H with an associated Koszul complex that admits a compatible action and X is an H-set. Suppose further that there is a cocycle*

$$\alpha : G\times X \to H.$$

Then the Koszul complex associated to $_{\alpha}Ind_H^G L$ admits a compatible action.

The proof of this theorem is almost identical to the proof of Theorem 4.3.

We can also jazz up the definition of cocycle induction to make it more aligned with higher induction by first choosing a total ordering on X, call it $<$. Then we can still take alpha to be our general cocycle:

$$\alpha : \mathrm{G}\times\mathrm{X} \to \mathrm{H},$$

but now we take as our induction:

$$^{(k)}_{\alpha}\mathrm{Ind}_H^G\mathrm{L} := \bigoplus_{x_1 < \cdots < x_k} \mathrm{L}_{x_1} \otimes \cdots \otimes \mathrm{L}_{x_k}.$$

This produces a $\mathbb{Z}$G-module with the action provided as:

$$g(l_{x_1} \otimes \cdots \otimes l_{x_k}) = (\alpha(g, x_1)l)_{g\cdot x_1} \otimes \cdots \otimes (\alpha(g, x_k)l)_{g\cdot x_k}. \tag{4.24}$$

Technically, these might be out of order after g acts, but there is a unique permutation of the tensor factors that puts them in the correct order.[9]

Theorem 4.2. *Suppose that L is a $\mathbb{Z}H$-lattice and H with an associated Koszul complex that admits a compatible action and X is an H-set. Suppose further that there is a cocycle*

$$\alpha \,:\, G{\times}X \to H.$$

Then the Koszul complex associated to ${}^{k}_{\alpha}\mathrm{Ind}_H^G L$ admits a compatible action.

Proof. The proof of this theorem is almost identical to the proof of Theorem 4.5 □

To illustrate how this works we show how the axioms for cocycle induction compare with the construction of the action developed for higher induction: Given g, g'∈G, and the set of coset representatives $\{\gamma_{\mathrm{i}_1}, \ldots,$ chosen in the above equation, we can act on an element, $(\gamma_{\mathrm{i}_1} l_{\mathrm{i}_1}) \otimes \cdots \otimes (\gamma_{\mathrm{i}_k} l_{\mathrm{i}_k})$, of ${}^{(k)}\mathrm{Ind}_H^G$L (again, ignoring the tensor products between the γ's and the l's) by using the formula

This is an action because if g ∈G as well, we have:

$$g(g((\gamma_{\mathrm{i}_1} \otimes l_{\mathrm{i}_1}) \otimes \cdots \otimes (\gamma_{\mathrm{i}_k} \otimes l_{\mathrm{i}_k}))) = g((g(\gamma_{\mathrm{i}_1} \otimes l_{\mathrm{i}_1})) \otimes \cdots \otimes (g(\gamma_{\mathrm{i}_k} \otimes l_{\mathrm{i}_k}))) \tag{4.25}$$
$$= g((\gamma_{\rho(g,\mathrm{i}_1)} \otimes h(g,\mathrm{i}_1) l_{\mathrm{i}_1}) \otimes \cdots \otimes (\gamma_{\rho(g,\mathrm{i}_k)} \otimes h(g,\mathrm{i}_k) l_{\mathrm{i}_k})) \tag{4.26}$$
$$= g(\gamma_{\rho(g,\mathrm{i}_1)} \otimes h(g,\mathrm{i}_1) l_{\mathrm{i}_1}) \otimes \cdots \otimes g(\gamma_{\rho(g,\mathrm{i}_k)} \otimes h(g,\mathrm{i}_k) l_{\mathrm{i}_k})) \tag{4.27}$$
$$= (\gamma_{\rho(g,\rho(g,\mathrm{i}_1))} \otimes h(g,\rho(g,\mathrm{i}_1)) h(g,\mathrm{i}_1) l_{\mathrm{i}_1}) \otimes \cdots \tag{4.28}$$
$$\cdots \otimes (\gamma_{\rho(g,\rho(g,\mathrm{i}_k))} \otimes h(g,\rho(g,\mathrm{i}_k)) h(g,\mathrm{i}_k) l_{\mathrm{i}_k}) \tag{4.29}$$

and similarly,

$$(gg)((\gamma_{\mathrm{i}_1} \otimes l_{\mathrm{i}_1}) \otimes \cdots \otimes (\gamma_{\mathrm{i}_k} \otimes l_{\mathrm{i}_k})) = ((gg)(\gamma_{\mathrm{i}_1} \otimes l_{\mathrm{i}_1})) \otimes \cdots \otimes ((gg)(\gamma_{\mathrm{i}_k} \otimes l_{\mathrm{i}_k})) \tag{4.30}$$
$$= (\gamma_{\rho(gg,\mathrm{i}_1)} \otimes h(gg,\mathrm{i}_1) l_{\mathrm{i}_1}) \otimes \cdots \otimes (\gamma_{\rho(gg,\mathrm{i}_k)} \otimes h(gg,\mathrm{i}_k) l_{\mathrm{i}_k}) \tag{4.31}$$

and if we have:

[9]↑The notation is ugly enough as it is, so we elected to leave this permutation out.

$$(\gamma_{\rho(g,\rho(g,\mathrm{i}_1))}h(g,\rho(g,\mathrm{i}_1))h(g,\mathrm{i}_1)l_{\mathrm{i}_1})\otimes\cdots\otimes(\gamma_{\rho(g,\rho(g,\mathrm{i}_k))}h(g,\rho(g,\mathrm{i}_k))h(g,\mathrm{i}_k)l_{\mathrm{i}_k}) \tag{4.32}$$

equals

$$(\gamma_{\rho(gg,\mathrm{i}_1)}h(gg,\mathrm{i}_1)l_{\mathrm{i}_1})\otimes\cdots\otimes(\gamma_{\rho(gg,\mathrm{i}_k)}h(gg,\mathrm{i}_k)l_{\mathrm{i}_k}) \tag{4.33}$$

then this defines a G-action and ${}^{(k)}\mathrm{Ind}_H^G\mathrm{L}$ is a G-module.

Notice that the axioms of cocycle induction could be weakened further while maintaining the G-module structure.

5. A SAMPLE COMPUTATION

In this section, I will be computing the cohomology of groups Γ that fit into the short exact sequence

$$0 \to \mathbb{Z}^n \to \Gamma \to \mathbb{Z}/\mathrm{p}^2\mathbb{Z},$$

for various values of n and primes p. With this in mind, the author recommends perusing the first appendix on integer representation theory before beginning.

5.1 Collapsing Spectral Sequences for $\mathbb{Z}/\mathrm{p}^2\mathbb{Z}$ Actions

For the benefit of the reader, I will list the indecomposable $\mathbb{Z}/\mathrm{p}^2\mathbb{Z}$ representations here:

$$\begin{aligned}
\Gamma_1 &: t \mapsto \tilde{\alpha}, \\
\Gamma_2 &: t \mapsto \tilde{\beta}, \\
\Gamma_3 &: t \mapsto 1, \\
\Gamma_4 &: t \mapsto \begin{pmatrix} \tilde{\alpha} & \langle\{\mathrm{i}\}\rangle \\ 0 & \tilde{\beta} \end{pmatrix} \\
\Gamma_5 &: t \mapsto \begin{pmatrix} \tilde{\alpha} & \langle\{1\}\rangle \\ 0 & 1 \end{pmatrix} \\
\Gamma_6 &: t \mapsto \begin{pmatrix} \tilde{\beta} & \langle 1\rangle \\ 0 & 1 \end{pmatrix} \\
\Gamma_7 &: t \mapsto \begin{pmatrix} \tilde{\alpha} & \langle\{\mathrm{i}\}\rangle & 0 \\ 0 & \tilde{\beta} & \langle 1\rangle \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

$$\Gamma_8 : t \mapsto \begin{pmatrix} \tilde{\alpha} & \langle\{\mathrm{i}\}\rangle & \langle 1\rangle \\ 0 & \tilde{\beta} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Gamma_9 : t \mapsto \begin{pmatrix} \tilde{\alpha} & 0 & \langle 1\rangle \\ 0 & \tilde{\beta} & \langle 1\rangle \\ 0 & 0 & 1 \end{pmatrix}$$

(i = 0, 1, $\cdots$, p - 2) If p $\neq$ 2 then we get a Γ_{10}:

$$\Gamma_{10} : t \mapsto \begin{pmatrix} \tilde{\alpha} & \langle\{\mathrm{i}\}\rangle & 0 & \langle 1\rangle \\ 0 & \tilde{\beta} & \langle 1\rangle & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(i = 1, $\cdots$, p -2)

We can see immediately that if $\phi = \Gamma_3$, then the spectral sequence will collapse. Since Γ_2 and Γ_6 are indecomposable representations of $\mathbb{Z}/\mathrm{p}\mathbb{Z}$, we have by a theorem of Adem et al. that they have associated compatible actions, which cause the spectral sequence to collapse. Since the eigenvalues of Γ_1 are always the primitive p^2-th roots of unity, we have that 1 is not an eigenvalue of Γ_1. This implies that there are no fixed points outside of the origin for the action given by Γ_1 which, by a theorem of Langer and Lück, tells us that if ϕ is Γ_1 then the spectral sequence will collapse.

Since the eigenvalues of a block upper triangular matrix is the union of the sets of eigenvalues of the blocks, we see that Γ_4 also provides no fixed points for any value of i and therefore we may be led down the treacherous path of unthinkingly believing that the spectral sequence associated to that the semi-direct product with that action collapses. However, upon inspection, we see that the matrix associated to Γ_4 has eigenvalues equal to 1 when cubed. And so the action Γ_4 is not free outside the origin and we cannot use Langer and Lück's result. Whether or not the spectral sequence collapses when the action is Γ_4 is unknown.

5.2 The Cohomology of $\mathbb{Z}^{p^2-p} \rtimes_{\Gamma_1} \mathbb{Z}/\mathbf{p}^2\mathbb{Z}$

We can find more, but we need to first compute the cohomology of Γ_1. Since the spectral sequence collapses, we know that

$$\begin{aligned} H^1(\mathbb{Z}^{p^2-p} \rtimes_{\Gamma_1} G) &\cong H^1(G, H^0(\mathbb{Z}^{p^2-p}) \oplus H^0(G, H^1(\mathbb{Z}^{p^2-p})) \\ &= 0 \oplus (\mathbb{Z}^{p^2-p})^G = 0 \end{aligned}$$

where we have the first zero because G is a cyclic group, and we have the second zero because the action of G is free. We also know that

$$\mathrm{H}^2(\mathrm{G}\rtimes_{\Gamma_1}\mathbb{Z}^{p^2-p}) = \mathrm{H}^2(\mathrm{G},\ \mathrm{H}^0(\mathbb{Z}^{p^2-p}))\oplus\mathrm{H}^1(\mathrm{G},\ \mathrm{H}^1(\mathbb{Z}^{p^2-p}))\oplus\mathrm{H}^0(\mathrm{G},\ \mathrm{H}^2(\mathbb{Z}^{p^2-p})).$$

The first summand is given by $\mathrm{H}^2(\mathrm{G},\ \mathrm{H}^0(\mathbb{Z}^{p^2-p})) \cong \mathrm{H}^2(\mathrm{G},\ \mathbb{Z}) = \mathbb{Z}/\mathrm{p}^2\mathbb{Z}$. The last summand is given by $\mathrm{H}^0(\mathrm{G},\ \mathrm{H}^2(\mathbb{Z}^{p^2-p})) = \mathrm{H}^2(\mathbb{Z}^{p^2-p}) = \mathbb{Z}^{\binom{p^2-p}{2}}$.

Taking $\overline{N}$ to be the norm map, as defined on page 58 of [23], the middle summand can be determined as follows:

$$\begin{aligned} H^1(G, H^1(\mathbb{Z}^{p^2-p})) &= ker\{\overline{N} : H^1(\mathbb{Z}^{p^2-p})_G \to H^1(\mathbb{Z}^{p^2-p})^G\} \\ &= ker\{\overline{N} : \bigwedge^1(\mathbb{Z}^{p^2-p})_G \to \bigwedge^1(\mathbb{Z}^{p^2-p})^G\} \\ &= ker\{\overline{N} : \bigwedge^1(\mathbb{Z}^{p^2-p})_G \to 0\} \\ &= \bigwedge^1(\mathbb{Z}^{p^2-p})_G \\ &= \mathbb{Z}^{p^2-p}/Im(\Gamma_1 - I) \end{aligned}$$

We can determine that $\mathbb{Z}^{p^2-p}/\mathrm{Im}(\Gamma_1$ - I) is $\mathbb{Z}_p$ by computing the Smith Normal Form of I - Γ_1. For easier visual interpretation, I will perform this computation for p = 3.

$$I - \Gamma_1 = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

Using row and then column operations with the first row/column, we obtain:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

Do same thing with the second row/column:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

Swap the fourth and third rows and multiply the new third row by -1:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

Perform row/column operations:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

Exchange rows and multiply by -1:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

Perform row/column operations:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

Exchange rows and multiply by -1:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 2 & 1 \end{pmatrix}$$

Perform row and column operations:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

And we see that $\mathbb{Z}^6/\mathrm{Im}(\Gamma_1$ - I) will be isomorphic to $\mathbb{Z}_3$. This generalizes to arbitrary prime p by noticing that the cause of the increase of the diagonal term to p in the example computation above was the p - 1 entries of -1 in row 1 of Γ_1 along with the one extra -1 from subtracting the identity matrix to get a total of p.

This generalization can be seen by noting that the at the start of the process the (p)×(p) submatrix in the top left corner is always of the form:

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix}.$$

Applying the Smith Normal Form algorithm like we did above gives:

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 2 \end{pmatrix}$$

At the x-th step (x > 1), we use a p×p submatrix that consists of the p rows starting at the $[\text{xp} + 1]^{th}$ row and the p columns starting at the $[\text{xp} + 1]^{th}$ column:

$$\begin{pmatrix} x & 0 & 0 & \cdots & 0 & 1 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix}.$$

Applying the Smith Normal Form algorithm like we did above gives:

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & x+1 \end{pmatrix}.$$

Since there are p - 1 such steps, we end with the (p^2 - p)$\times$(p^2 - p) matrix:

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & p \end{pmatrix}.$$

This provides us with knowledge that in general $\mathbb{Z}^{p^2-p}/\mathrm{Im}(\Gamma_1 - \mathrm{I}) \cong \mathbb{Z}_p$.

And so we see that

$$\mathrm{H}^2(\mathbb{Z}^{p^2-p} \rtimes_{\Gamma_1} \mathrm{G}, \mathbb{Z}) = (\mathbb{Z}/\mathrm{p}^2\mathbb{Z}) \oplus (\mathbb{Z}/\mathrm{p}\mathbb{Z}) \oplus (\mathbb{Z}^{\binom{p^2-p}{2}})$$

These techniques can be continued to be used to compute higher cohomology groups. A visual inspection of the collapsing spectral sequence and knowledge of the periodic nature of the cohomology of finite cyclic groups shows that for t > p^2 - p, $\mathrm{H}^t(\mathbb{Z}^{p^2-p} \rtimes_{\Gamma_1} \mathrm{G}, \mathbb{Z})$ is 2-periodic. So a full computation of the cohomology of this group requires only p^2 - p + 2 computations.

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A. A BRIEF LOOK INTO INTEGER REPRESENTATION THEORY

Throughout this section we will assume that R is a Dedekind domain with field of quotients K.

The integer representation theory we use in this paper is largely covered by [12], and we refer the interested reader there. However, the topic might be somewhat obscure to the group cohomologist audience of this paper so some basic information on the subject will be introduced here.

Integer representation theory can be defined in many ways, but perhaps the simplest is that it is the study of representations of finite groups as matrices with entries elements in rings of integers. It is in many ways a bridge between ordinary and modular representation theories in that the underlying ring is not a field, but rather just a Dedekind domain, R. This Dedekind domain can be taken to be a discrete valuation ring with quotient field K and residue class field k, which provide us with the usual ordinary and modular representation theories.

The investigation of R-representations of a finite group G can also, as in the more typical theories, be described as modules over the group-ring RG. However, in integer representation theory, we concern ourselves with Λ-lattices, where Λ is an R-order. We define these terms as:

Definition A.1. *An R-order is a ring Λ whose center contains R and such that Λ is finitely generated and projective over R.*

Definition A.2. *If Λ is an R-order, a Λ-lattice is a Λ-module which is finitely generated and projective as an R-module.*

As an example, the group ring RG happens to be an R-order if G is finite, and it is the R-order that we are most interested in. The concept of $\mathbb{Z}$G-lattices being isomorphic when localized to a prime is not a new one, and has been informing integer representation theorists for decades. It is seen as so important that it is given a name:

Definition A.3. *Let R be a Dedekind domain and let Λ be an R-order in a separable K-algebra A, where K is the quotient field of R. Letting P range over the maximal ideals of R, we let R_P denote the localization of R at P. Two Λ-lattices M and N are placed in the same genus, denoted $M \vee N$, if $M_P \cong N_P$ for each maximal ideal P of R, where $M_P \cong M \otimes R_P$ and $N_P \cong N \otimes R_P$.*

Understanding genera is essential to working on the central problems of classical integer representation theory, as expressed in the introduction of [26], but we will see that they will also be useful to the study of group cohomology.

Since it will be important later, we point out here a well known fact that if M$\vee$N, then it follows that for each P we can find a $\hat{\Lambda}_P$-isomorphism (where the hat signifies P-adic completion)

$$\mathrm{f}_P : \hat{M}_P \xrightarrow{\cong} \hat{N}_P$$

In Heller and Reiner's paper, [19], we learn that the R_P-indecomposable representations for a cyclic group of order p^2, $\mathrm{G} = \langle \mathrm{g} \rangle$, for some prime p can be constructed as follows:

First, let H be a subgroup of G of order p, generated by h.

Next, consider the following:

(i) $\mathrm{A} \coloneqq \mathrm{RH}/(\mathrm{h} - 1)\mathrm{RH} \cong \mathrm{R}$

(ii) $\mathrm{B} \coloneqq \mathrm{RH}/(\mathrm{h}^{p-1} + \cdots + \mathrm{h} + 1)\mathrm{RH} \cong \mathrm{IH}$

(iii) $\mathrm{E} \coloneqq \mathrm{RG}/(\mathrm{g}^p - 1)\mathrm{RG} \cong \mathrm{RH}$

(iv) $\mathrm{C} := \mathrm{RG}/(\mathrm{g}^{p(p-1)} + \mathrm{g}^{p(p-2)} + \cdots + \mathrm{g}^{p} + 1)\mathrm{RG} \cong \mathrm{IG}$

Suppose we are in the setting of $\mathrm{G} = \mathbb{Z}/\mathrm{p}^2\mathbb{Z}$ for some rational prime p. We would like to know the cohomology, specifically if the LHS spectral sequence collapses, of the group extension Γ given by

$$0 \to \mathrm{L} \to \Gamma = \mathrm{L} \rtimes_{\phi} \mathrm{G} \to \mathrm{G} \to 0,$$

where L is a $\mathbb{Z}$G-lattice whose G-action is induced from $\phi : \mathrm{G} \to \mathrm{GL}_n(\mathbb{Z})$.

Letting α be a primitive p^2-th root of unity and β be a primitive p-th root of unity, we can find matrices, $\tilde{\alpha}$ and $\tilde{\beta}$, corresponding to α and β in the following way: It is well known that the cyclotomic polynomial for a primitive p^k-th root of unity is given by the equation:

$$\Phi_{p^k}(x) = \sum_{\mathrm{i}=0}^{p-1} x^{\mathrm{i}p^{k-1}}. \tag{A.1}$$

Moreover, it is well known that the characteristic polynomial for a matrix that is an nth root of the unit matrix must have the nth cyclotomic polynomial as an irreducible factor of its characteristic polynomial. Letting k = 2 and k = 1 in the above equation, we get the characteristic polynomials for α and β as

$$\text{char.pol.}(\alpha) = \Phi_{p^2}(\mathrm{x}) = \sum_{\mathrm{i}=0}^{p-1} x^{\mathrm{i}p},$$

and

$$\text{char.pol.}(\beta) = \Phi_{p}(\mathrm{x}) = \sum_{\mathrm{i}=0}^{p-1} x^{\mathrm{i}}.$$

The earliest source I could find using this methodology to construct integer matrices that are roots of the unit matrix is [27]. In that paper, the author demonstrated the now well known fact that if

$$\text{char.pol.}(\lambda) = \mathrm{x}^n - \mathrm{a}_{n-1}\mathrm{x}^{n-1} - \cdots - \mathrm{a}_1\mathrm{x} - \mathrm{a}_0,$$

then the matrix corresponding to λ can be given as

$$\begin{pmatrix} a_n & a_{n-1} & \cdots & a_1 & a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

and is of size $(\phi(\lambda))\times(\phi(\lambda))$, where ϕ is the Euler totient function. Applying this to α and β, we find that the matrix corresponding to β, hereby denoted $\tilde{\beta}$, will be a (p - 1)×(p - 1) matrix of the form:

$$\begin{pmatrix} -1 & -1 & \cdots & -1 & -1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

and the matrix corresponding to α, hereby denoted $\tilde{\alpha}$, will be a (p^2 - p)×(p^2 - p) matrix, similarly defined except that the first row will only have -1 in the columns that are a multiple of p and will otherwise be zero. For example, if p = 3, then

$$\tilde{\alpha} = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Berman and Gudikov, in [28], gave a complete classification of the indecomposable pairwise nonequivalent p-adic representations of $\mathbb{Z}/\mathrm{p}^2\mathbb{Z} = \langle \mathrm{t} \mid \mathrm{t}^{p^2} = 1\rangle$. If we let $\langle\{\mathrm{i}\}\rangle$ be the matrix where all columns are zero except for the last column, which is given as the coordinates of i in the $\mathbb{Z}_{(p)}$-basis 1, α, $\ldots$, α^{p^2-p-1} of $\mathbb{Z}_{(p)}[\alpha]$, then the representations are given as:

$$\Gamma_1 : t \mapsto \tilde{\alpha},$$

$$\Gamma_2 : t \mapsto \tilde{\beta},$$

$$\Gamma_3 : t \mapsto 1,$$

$$\Gamma_4 : t \mapsto \begin{pmatrix} \tilde{\alpha} & \langle\{\mathrm{i}\}\rangle \\ 0 & \tilde{\beta} \end{pmatrix}$$

$$\Gamma_5 : t \mapsto \begin{pmatrix} \tilde{\alpha} & \langle\{1\}\rangle \\ 0 & 1 \end{pmatrix}$$

$$\Gamma_6 : t \mapsto \begin{pmatrix} \tilde{\beta} & \langle 1\rangle \\ 0 & 1 \end{pmatrix}$$

$$\Gamma_7 : t \mapsto \begin{pmatrix} \tilde{\alpha} & \langle\{\mathrm{i}\}\rangle & 0 \\ 0 & \tilde{\beta} & \langle 1\rangle \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Gamma_8 : t \mapsto \begin{pmatrix} \tilde{\alpha} & \langle\{\mathrm{i}\}\rangle & \langle 1\rangle \\ 0 & \tilde{\beta} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Gamma_9 : t \mapsto \begin{pmatrix} \tilde{\alpha} & 0 & \langle 1\rangle \\ 0 & \tilde{\beta} & \langle 1\rangle \\ 0 & 0 & 1 \end{pmatrix}$$

(i = 0, 1, $\cdots$, p - 2) If p $\neq$ 2 then we get a Γ_{10}:

$$\Gamma_{10} : t \mapsto \begin{pmatrix} \tilde{\alpha} & \langle\{\mathrm{i}\}\rangle & 0 & \langle 1\rangle \\ 0 & \tilde{\beta} & \langle 1\rangle & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(i = 1, $\cdots$, p -2)

B. A BRIEF LOOK INTO GROUP COHOMOLOGY

Group cohomology might similarly be obscure for the representation theorists audience of this paper, so some basic information on the subject will be introduced here. All of the information in this section is covered by [23] and [29], and we refer the interested reader there.

Group cohomology can similarly be defined in many ways, but in this paper we will define it as follows:

Let R be an arbitrary ring and M an R-module. We call the exact sequence

$$\cdots \xrightarrow{\partial_2} \mathrm{F}_1 \xrightarrow{\partial_1} \mathrm{F}_0 \xrightarrow{\epsilon} \mathrm{M} \to 0.$$

a free resolution of M over R if $\mathrm{F_i}$ is a free R-module. Applying the functor $\mathrm{Hom}_{RG}(\text{ - },\mathrm{N})$ to this free resolution, where N is a R-module, we get the cochain complex:

$$\mathrm{Hom}_{RG}(\mathrm{F}_0, \mathrm{N}) \to \mathrm{Hom}_{RG}(\mathrm{F}_1, \mathrm{N}) \to \cdots.$$

If we let R be the group ring $\mathbb{Z}\mathrm{G} = \{\sum_{h \in G} \mathrm{c}_h \mathrm{h} : \mathrm{c}_h \in \mathrm{R}\}$ (This object is called a group ring. If G is infinite, we require that all but finitely many c_h's be zero) and let M be the $\mathbb{Z}$G-module $\mathbb{Z}$ obtained by the trivial action, then by taking the cohomology of this complex at the i-th place, we get $\mathrm{H^i}(\mathrm{G}, \mathrm{N})$.

There are many methods and tools that can be used to assist in the computation of these cohomology groups. One of the more important tools is the Lyndon-Hochschild-Serre (LHS) Spectral Sequence mentioned in the introduction. This is a first quadrant spectral sequence that starts with a short exact sequence of groups:

$$0 \to \mathrm{L} \to \Gamma \to \mathrm{G} \to 0,$$

and provides a spectral sequence:

$$\mathrm{E}_2^{p,q}(\mathrm{M}) = \mathrm{H}^p(\mathrm{G}, \mathrm{H}^q(\mathrm{L}, \mathrm{M})) \implies \mathrm{H}^{p+q}(\Gamma, \mathrm{M}).$$

Methods for working with spectral sequences are similarly numerous. However, while there are many resources for studying spectral sequences and the tools useful to their study, my experience with the topic is that it still has a very feudal guild vibe of ”tricks of the trade”. I state this as warning to future graduate students considering a venture into their realm.

To begin, let us introduce a proposition that is central to the usefulness of spectral sequences:

Proposition B.1. *If $E_2^{p,q}$ and $\tilde{E}_2^{p,q}$ are two spectral sequences and there exists a map of spectral sequences $\mu : E_r^{p,q} \to \tilde{E}_r^{p,q}$, that becomes an isomorphism when restricted to some page r, then the abutments of the spectral sequences are isomorphic as well.*

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