STRONG ASYMPTOTICS OF HERMITE-PADÉ APPROXIMANTS FOR ANGELESCO SYSTEMS

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Abstract. In this work type II Hermite-Padé approximants for a vector of Cauchy transforms of smooth Jacobi-type densities are considered. It is assumed that densities are supported on mutually disjoint intervals (an Angelesco system with complex weights). The formulae of strong asymptotics are derived for any ray sequence of multi-indices.

Contents

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1 INTRODUCTION

Let $\vec{f} = (f_1, \ldots, f_p)$, $p \in \mathbb{N}$, be a vector of germs of holomorphic functions at infinity. Given a multi-index $\vec{n} \in \mathbb{N}^p$, *Hermite-Padé* approximant to $\vec{f}$ associated with $\vec{n}$, is a vector of rational functions

$$(1) \qquad [\,\vec{n}\,]_{\vec{f}} := \left(P_{\vec{n}}^{(1)}/Q_{\vec{n}}, \ldots, P_{\vec{n}}^{(p)}/Q_{\vec{n}}\right)$$

such that

$$(2) \qquad \begin{cases} \deg(Q_{\vec{n}}) = |\,\vec{n}\,| := n_1 + \cdots + n_p, \\ R_{\vec{n}}^{(i)}(z) := \left(Q_{\vec{n}} f_i - P_{\vec{n}}^{(i)}\right)(z) = \mathcal{O}\left(z^{-(n_i+1)}\right) \quad \text{as} \quad z \to \infty, \quad i \in \{1, \ldots, p\}. \end{cases}$$

It is quite simple to see that $[\,\vec{n}\,]_{\vec{f}}$ always exists since (2) can be rewritten as a linear system that has more unknowns than equations with coefficients coming from the Laurent expansions of f_i's at infinity. Hence, $Q_{\vec{n}}$ is never identically zero and, in what follows, we normalize $Q_{\vec{n}}$ to be monic.

The vector $\vec{f}$ is called an *Angelesco system* if

$$(3) \qquad f_i(z) = \int \frac{d\sigma_i(t)}{t - z}, \qquad i \in \{1, \ldots, p\},$$

where σ_i's are positive measures on the real line with mutually disjoint convex hulls of their supports, i.e., $[a_j, b_j] \cap [a_k, b_k] = \varnothing$ for $j \neq k$ where $[a_i, b_i]$ is the smallest interval containing $\operatorname{supp}(\sigma_i)$. Hermite-Padé approximants to such systems were initially considered by Angelesco [1] and later by Nikishin [23, 24]. The beauty of system (3) is that $Q_{\vec{n}}$, the denominator of $[\,\vec{n}\,]_{\vec{f}}$, turns out to be a multiple orthogonal polynomial satisfying

$$(4) \qquad \int Q_{\vec{n}}(x) x^k d\sigma_i(x) = 0, \qquad k \in \{0, \ldots, n_i - 1\}, \quad i \in \{1, \ldots, p\}.$$

When $p = 1$, Hermite-Padé approximant $[\,\vec{n}\,]_{\vec{f}}$ specializes to the diagonal Padé approximant, quite often denoted by $[n/n]_f$. It was shown by Markov [20] that if f is of the form (3) (now called a *Markov function*), then $[n/n]_f$ converge to f locally uniformly outside of $[a, b]$. Moreover, see [29, Thm. 6.1.6], it holds that

$$(5) \qquad \begin{cases} \lim\limits_{n\to\infty} n^{-1} \log|f - [n/n]_f| \leqslant -2(\ell - V^\omega) \\ \lim\limits_{n\to\infty} n^{-1} \log|Q_n| = -V^\omega \end{cases}$$

locally uniformly in $\overline{\mathbb{C}} \setminus [a, b]$, where $V^\omega(z) := -\int \log|z - t| d\omega(t)$ is the *logarithmic potential* of ω, while the measure ω and the constant ℓ are the unique solutions of the min/max problem:

$$(6) \qquad \ell := \min_{x \in [a,b]} V^\omega(x) = \max_{\nu \in M_1(a,b)} \min_{x \in [a,b]} V^\nu(x),$$

where $M_c(a, b)$ is the collection of all positive Borel measures of mass c supported on $[a, b]$. In fact, it also holds that ω is the *equilibrium distribution* and ℓ is the *Robin's constant* for the interval $[a, b]$. That is, ω is the unique measure on $[a, b]$ that solves the energy minimization problem:

$$(7) \qquad I[\omega] = \min_{\nu \in M_1(a,b)} I[\nu], \qquad \ell = I[\omega],$$

where $I[\nu] := -\int\int \log|z - t| d\nu(t) d\nu(z) = \int V^\nu d\nu$ is the *logarithmic energy* of ν (for the notions of logarithmic potential theory we use [27] and [28] as primary references).

It easily follows from (6)–(7) and properties of the superharmonic functions that

$$(8) \qquad \begin{cases} \ell - V^\omega \equiv 0 & \text{on} \quad [a, b], \\ \ell - V^\omega > 0 & \text{in} \quad \overline{\mathbb{C}} \setminus [a, b]. \end{cases}$$

Hence, the diagonal Padé approximants $[n/n]_f$ do indeed converge to f locally uniformly in $\overline{\mathbb{C}} \setminus [a, b]$. Moreover, if σ is a *regular* measure in the sense of Stahl and Totik [29, Sec. 3.1] (in particular, $\sigma' > 0$ almost everywhere on $[a, b]$ implies regularity), then the inequality in (5) can be replaced by equality.

The above results were extended by Gonchar and Rakhmanov [15] to Hermite-Padé approximants for Angelesco systems when multi-indices are such that

$$n_i = c_i|\vec{n}| + o(|\vec{n}|), \qquad \vec{c} = (c_1, \dots, c_p) \in (0,1)^p, \quad |\vec{c}| = 1, \tag{9}$$

as $|\vec{n}| \to \infty$, and the measures σ_i satisfy $\sigma_i' > 0$ almost everywhere on $[a_i, b_i]$, $i \in \{1, \dots, p\}$. The formulae for the errors of approximation are similar in appearance to (5) with measures coming not from a scalar but from a vector minimum energy problem. To describe it, define

$$M_{\vec{c}}\big(\{a_i, b_i\}_1^p\big) := \big\{\vec{\nu} = (\nu_1, \dots, \nu_p) : \ \nu_i \in M_{c_i}(a_i, b_i), \ i \in \{1, \dots, p\}\big\}.$$

Then it is known that there exists the unique vector of measures $\vec{\omega} \in M_{\vec{c}}\big(\{a_i, b_i\}_1^p\big)$ such that

$$I[\vec{\omega}] = \min_{\nu \in M_{\vec{c}}(\{a_i,b_i\}_1^p)} I[\vec{\nu}], \qquad I[\vec{\nu}] := \sum_{i=1}^p \Big(2I[\nu_i] + \sum_{k \neq i} I[\nu_i, \nu_k]\Big), \tag{10}$$

where $I[\nu_i, \nu_k] := -\int\int \log|z - t| d\nu_i(t) d\nu_k(z)$. The measures ω_i might no longer be supported on the whole intervals $[a_i, b_i]$ (the so-called *pushing effect*), but in general it holds that

$$\mathrm{supp}(\omega_i) = [a_{\vec{c},i}, b_{\vec{c},i}] \subseteq [a_i, b_i], \qquad i \in \{1, \dots, p\}. \tag{11}$$

Let $W^{\vec{\nu}}$ be a function on $\bigcup_{i=1}^p [a_i, b_i]$ such that its restriction to $[a_i, b_i]$ is equal to $V^{\nu_i + \nu}$ where $\nu = \sum_{i=1}^p \nu_i$ is a probability measure such that $\nu_{|[a_i,b_i]} = \nu_i$. Exactly as in (6), the equilibrium vector measure $\vec{\omega}$ can be characterized by the following property: if

$$\min_{x \in [a_i, b_i]} W^{\vec{\nu}}(x) \geqslant \min_{x \in [a_i, b_i]} W^{\vec{\omega}}(x) =: \ell_i \tag{12}$$

simultaneously for all $i \in \{1, \dots, p\}$ for some $\vec{\nu} \in M_{\vec{c}}\big(\{a_i, b_i\}_1^p\big)$, then $\vec{\nu} = \vec{\omega}$.

Having all the definitions at hand, we can formulate the main result of [15], which states that

$$\begin{cases} \lim\limits_{|\vec{n}| \to \infty} |\vec{n}|^{-1} \log\big|f_i - P_{\vec{n}}^{(i)}/Q_{\vec{n}}\big| = -\big(\ell_i - V^{\omega_i + \omega}\big), & i \in \{1, \dots, p\}, \\ \lim\limits_{|\vec{n}| \to \infty} |\vec{n}|^{-1} \log|Q_{\vec{n}}| = -V^{\omega} \end{cases} \tag{13}$$

locally uniformly in $\overline{\mathbb{C}} \setminus \bigcup_{i=1}^p [a_i, b_i]$[1]. Even though (13) looks exactly as (5), the convergence properties of the approximants are not as straightforward. Indeed, it is a direct consequence of the pushing effect ($[a_{\vec{c},i}, b_{\vec{c},i}] \subsetneq [a_i, b_i]$), when it occurs, of course, that the first relation in (8) is replaced now by

$$\begin{cases} \ell_i - V^{\omega_i + \omega} \equiv 0 & \text{on} \quad [a_{\vec{c},i}, b_{\vec{c},i}], \\ \ell_i - V^{\omega_i + \omega} < 0 & \text{on} \quad [a_i, b_i] \setminus [a_{\vec{c},i}, b_{\vec{c},i}]. \end{cases} \tag{14}$$

Further, set

$$\begin{cases} D_i^+ & := \quad \big\{z : \ \ell_i - V^{\omega_i + \omega}(z) > 0\big\}, \\ D_i^- & := \quad \big\{z : \ \ell_i - V^{\omega_i + \omega}(z) < 0\big\}. \end{cases} \tag{15}$$

Properties of the logarithmic potentials immediately imply that D_i^+ is an unbounded domain. This is exactly the domain in which the approximants $P_{\vec{n}}^{(i)}/Q_{\vec{n}}$ converge to f_i locally uniformly, while D_i^- is a bounded open set on which the approximants diverge to infinity. This set can be empty or not. The latter situation necessarily happens when $[a_{\vec{c},i}, b_{\vec{c},i}] \subsetneq [a_i, b_i]$ as can be clearly seen from the second line in (14); however, the pushing effect is not necessary for the divergence set to exist.

[1] (13) is consistent with (5) when $p = 1$, since in this case $I[\vec{\nu}] = 2I[\nu_1]$, $\ell_1 = 2\ell$, and $V^{\omega_1 + \omega} = 2V^{\omega}$.

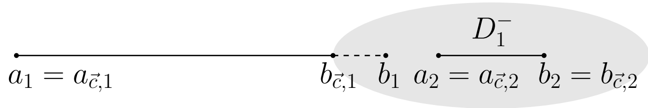

FIGURE 1. Schematic representation of the pushing effect in the case of 2 intervals (in Proposition 3 we shall show that this is the only possible situation for pushing effect in the case of 2 intervals; this is also explained in [15]). The shaded region is the divergence domain D_1^- while $D_2^- = \varnothing$.

The result of Gonchar and Rakhmanov (13) belongs to the realm of the so called *weak asymptotics* as to distinguish from *strong asymptotics* in which one establishes the existence of and identifies the limits

$$\begin{cases} \lim\limits_{|\vec{n}|\to\infty} \Big(\log\big|f_i - P_{\vec{n}}^{(i)}/Q_{\vec{n}}\big| + |\vec{n}|\big(\ell_i - V^{\omega_i+\omega}\big)\Big), \\ \lim\limits_{|\vec{n}|\to\infty} \big(\log|Q_{\vec{n}}| + |\vec{n}|V^{\omega}\big). \end{cases} \tag{16}$$

Not surprisingly, the first result completely answering the previous question was obtained for Padé approximants ($p = 1$) by Szegő. He proved that limit (16) takes place exactly when σ' satisfied $\int \log \sigma' d\omega > -\infty$, which is now known as *Szegő condition*[2]. The analog of the Szegő theorem for true Hermite-Padé approximants was proven by Aptekarev [2] when $p = 2$ and the multi-indices are diagonal ($\vec{n} = (n, n)$) with indications how one could carry the approach to any $p > 1$. A rigorous proof for any p and diagonal multi-indices was completed by Aptekarev and Lysov [4] for systems $\vec{f}$ of Markov functions generated by cyclic graphs (the so called *generalized Nikishin systems*), of which Angelesco systems are a particular example. The restriction on the measures σ_i is more stringent in [4], as it is required that

$$\sigma_i'(x) = h_i(x)(x - a_i)^{\alpha_i}(b_i - x)^{\beta_i}, \qquad \alpha_i, \beta_i > -1, \tag{17}$$

and h_i is holomorphic and non-vanishing in some neighborhood of $[a_i, b_i]$.

From the approximation theory point of view it is not natural to require the measures σ_i to be positive (as well as to be supported on the real line, but we shall not dwell on this point here). In the case of Padé approximants it was Nuttall [25] who proved the existence of and identified the limit in (16) for the set up (3) and (17) with $\alpha = \beta = -1/2$ and h being Hölder continuous, non-vanishing, and complex-valued on $[a, b]$. The proof of Szegő theorem for any parameters $\alpha, \beta > -1$ and h complex-valued, holomorphic, and non-vanishing around $[a, b]$ follows from Aptekarev [3] (this result was not the main focus of [3], there weighed approximation on one-arc S-contours was considered), and the condition of holomorphy of h was relaxed by Baratchart and the author in [5], where h is taken from a fractional Sobolev space that depends on the parameters α, β (again, the main focus of [5] was weighted (multipoint) Padé approximation on one-arc S-contours). The goal of this work is to extend the results of [4] to Angelesco systems with complex weights and Hermite-Padé approximants corresponding to multi-indices as in (9).

2 MAIN RESULTS

From now on, we fix a system of mutually disjoint intervals $\{[a_i, b_i]\}_{i=1}^p$ and a vector $\vec{c} \in (0, 1)^p$ such that $|\vec{c}| = 1$. We further denote by

$$\vec{\omega} = (\omega_1, \ldots, \omega_p), \quad \omega := \sum_{i=1}^p \omega_i, \quad \operatorname{supp}(\omega_i) = \big[a_{\vec{c},i}, b_{\vec{c},i}\big] \subseteq [a_i, b_i],$$

the equilibrium vector measure minimizing the energy functional (10).

[2] The word "completely" is slightly abused here as it was later realized by [31] that one can add any singular measure to $\sigma'(t)dt$, the absolutely continuous part, without changing (16).

To describe the forthcoming results we shall need a $(p+1)$-sheeted compact Riemann surface, say $\mathfrak{R}$, that we realize in the following way. Take $p+1$ copies of $\overline{\mathbb{C}}$. Cut one of them along the union $\bigcup_{i=1}^{p}[a_{\vec{c},i},b_{\vec{c},i}]$, which henceforth is denoted by $\mathfrak{R}^{(0)}$. Each of the remaining copies cut along only one interval $[a_{\vec{c},i},b_{\vec{c},i}]$ so that no two copies have the same cut and denote them by $\mathfrak{R}^{(i)}$. To form $\mathfrak{R}$, take $\mathfrak{R}^{(i)}$ and glue the banks of the cut $[a_{\vec{c},i},b_{\vec{c},i}]$ crosswise to the banks of the corresponding cut on $\mathfrak{R}^{(0)}$.

It can be easily verified that thus constructed Riemann surface has genus 0. Denote by π the natural projection from $\mathfrak{R}$ to $\overline{\mathbb{C}}$. We shall denote by $\boldsymbol{z},\boldsymbol{w},\boldsymbol{x},\boldsymbol{e}$ generic points on $\mathfrak{R}$ with natural projections z,w,x,e. We also shall employ the notation $z^{(i)}$ for a point on $\mathfrak{R}^{(i)}$ with $\pi(z^{(i)})=z$. This notation is well defined everywhere outside of the cycles $\boldsymbol{\Delta}_i:=\mathfrak{R}^{(0)}\cap\mathfrak{R}^{(i)}$. Clearly, $\pi(\boldsymbol{\Delta}_i)=[a_{\vec{c},i},b_{\vec{c},i}]$. It also will be convenient to denote by $\boldsymbol{a}_{\vec{c},i}$ and $\boldsymbol{b}_{\vec{c},i}$ the branch points of $\mathfrak{R}$ with respective projections $a_{\vec{c},i}$ and $b_{\vec{c},i}$, $i\in\{1,\dots,p\}$.

Unfortunately, to be able to handle general multi-indices of form (9), one Riemann surface is not sufficient. Let $\vec{n}\in\mathbb{N}^p$. Denote by

$$\vec{\omega}_{\vec{n}}=(\omega_{\vec{n},1},\dots,\omega_{\vec{n},p}),\quad \omega_{\vec{n}}:=\sum_{i=1}^{p}\omega_{\vec{n},i},\quad \mathsf{supp}(\omega_{\vec{n},i})=[a_{\vec{n},i},b_{\vec{n},i}]\subseteq[a_i,b_i],$$

the equilibrium vector measure minimizing the energy functional (10) where $\vec{c}$ is replaced by the vector $(n_1/|\vec{n}|,\dots,n_p/|\vec{n}|)$. The surface $\mathfrak{R}_{\vec{n}}$ is defined absolutely analogously to $\mathfrak{R}$. Notation $\boldsymbol{\Delta}_{\vec{n},i}$, $\boldsymbol{a}_{\vec{n},i}$, and $\boldsymbol{b}_{\vec{n},i}$, $i\in\{1,\dots,p\}$, is self-evident now.

Since each $\mathfrak{R}_{\vec{n}}$ has genus zero, one can arbitrarily prescribe zero/pole multisets of rational functions on $\mathfrak{R}_{\vec{n}}$ as long as the multisets have the same cardinality. Thus, given a multi-index $\vec{n}$, we shall denote $\Phi_{\vec{n}}$ a rational function on $\mathfrak{R}_{\vec{n}}$ which is non-zero and finite everywhere on $\mathfrak{R}_{\vec{n}}\setminus\bigcup_{k=0}^{p}\{\infty^{(k)}\}$, has a pole of order $|\vec{n}|$ at $\infty^{(0)}$, a zero of multiplicity n_i at each $\infty^{(i)}$, and satisfies

$$\prod_{k=0}^{p}\Phi_{\vec{n}}\big(z^{(k)}\big)\equiv 1. \tag{18}$$

Normalization (18) is possible since the function $\log\prod_{k=0}^{p}\big|\Phi_{\vec{n}}\big(z^{(k)}\big)\big|$ extends to a harmonic function on $\mathbb{C}$ which has a well defined limit at infinity. Hence, it is constant. Therefore, if (18) holds at one point, it holds throughout $\overline{\mathbb{C}}$. The importance of the function $\Phi_{\vec{n}}$ to our analysis lies in the following proposition.

Proposition 1. *With the above notation, it holds that*

$$\frac{1}{|\vec{n}|}\log\big|\Phi_{\vec{n}}(\boldsymbol{z})\big|=\begin{cases}-V^{\omega_{\vec{n}}}(z)+\frac{1}{p+1}\sum_{k=1}^{p}\ell_{\vec{n},k}, & \boldsymbol{z}\in\mathfrak{R}_{\vec{n}}^{(0)},\\ V^{\omega_{\vec{n},i}}(z)-\ell_{\vec{n},i}+\frac{1}{p+1}\sum_{k=1}^{p}\ell_{\vec{n},k}, & \boldsymbol{z}\in\mathfrak{R}_{\vec{n}}^{(i)},\quad i\in\{1,\dots,p\}.\end{cases}$$

If a sequence $\{\vec{n}\}$ satisfies (9), then the measures $\omega_{\vec{n}}$ converge to ω in the weak topology of measures as $|\vec{n}|\to\infty$ (in particular, this implies that $\ell_{\vec{n},i}\to\ell_i$, $a_{\vec{n},i}\to a_{\vec{c},i}$, and $b_{\vec{n},i}\to b_{\vec{c},i}$). Moreover, it holds that $V^{\omega_{\vec{n},i}}\to V^{\omega_i}$ uniformly on compact subsets of $\mathbb{C}$ for each $i\in\{1,\dots,p\}$.*

It immediately follows from Proposition 1 that

$$\frac{1}{|\vec{n}|}\log\left|\frac{\Phi_{\vec{n}}\big(z^{(i)}\big)}{\Phi_{\vec{n}}\big(z^{(0)}\big)}\right|=V^{\omega_{\vec{n},i}+\omega_{\vec{n}}}(z)-\ell_{\vec{n},i}=V^{\omega_i+\omega}(z)-\ell_i+o(1) \tag{19}$$

uniformly on compact subsets of $\mathbb{C}$ as $|\vec{n}|\to\infty$ for each $i\in\{1,\dots,p\}$.

The following corollary is an elementary consequence of Proposition 1. It describes the assumption with which (9) often replaced when strong asymptotics is discussed (most often $\vec{k}=(1,\dots,1)$).

Corollary 2. *Let $\vec{k}\in\mathbb{N}^p$. If $\vec{c}=\big(k_1/|\vec{k}|,\dots,k_p/|\vec{k}|\big)$ and $\vec{n}=n\vec{k}$, $n\in\mathbb{N}$, then $\vec{\omega}_{\vec{n}}=\vec{\omega}$ and $\Phi_{\vec{n}}=\Phi_{\vec{k}}^{n}$.*

Proposition 1 allows to recover $|\Phi_{\vec n}|$ via the vector equilibrium measure $\vec\omega_{\vec n}$. In order to do it for the function $\Phi_{\vec n}$ itself, let us define $h_{\vec n}$ on $\mathfrak{R}_{\vec n}$ by the rule

$$\text{(20)}\qquad \begin{cases} h_{\vec n}\big(z^{(0)}\big) := \displaystyle\int \frac{d\omega_{\vec n}(x)}{z-x}, & z\in\mathbb{C}\setminus\bigcup_{i=1}^p\big[a_{\vec n,i},b_{\vec n,i}\big], \\ h_{\vec n}\big(z^{(i)}\big) := \displaystyle\int \frac{d\omega_{\vec n,i}(x)}{x-z}, & z\in\mathbb{C}\setminus\big[a_{\vec n,i},b_{\vec n,i}\big], \quad i\in\{1,\dots,p\}. \end{cases}$$

We further define the function h on $\mathfrak{R}$ exactly as in (20) with $\vec\omega_{\vec n}$ replaced by $\vec\omega$. For brevity, we also denote by $\boldsymbol\gamma_{\vec n,i}$ (resp. $\boldsymbol\gamma_i$) the Jordan arc belonging to $\mathfrak{R}_{\vec n}^{(0)}$ (resp. $\mathfrak{R}^{(0)}$) such that $\pi(\boldsymbol\gamma_{\vec n,i}) = \big[b_{\vec n,i},a_{\vec n,i+1}\big]$ (resp. $\pi(\boldsymbol\gamma_i)=\big[b_{\vec c,i},a_{\vec c,i+1}\big]$), $i\in\{1,\dots,p-1\}$.

Proposition 3. *The function $h_{\vec n}$ is a rational function on $\mathfrak{R}_{\vec n}$ that has a simple zero at each $\infty^{(k)}$, $k\in\{0,\dots,p\}$, a single simple zero, say $\boldsymbol z_{\vec n,i}$, on each $\boldsymbol\gamma_{\vec n,i}$, $i\in\{1,\dots,p-1\}$, a simple pole[3] at each $\big\{\boldsymbol a_{\vec n,i},\boldsymbol b_{\vec n,i}\big\}_{i=1}^p$, and is otherwise non-vanishing and finite. Moreover,*

$$\boldsymbol z_{\vec n,i} = \boldsymbol b_{\vec n,i} \quad\Leftrightarrow\quad b_{\vec n,i}\in\partial D^-_{\vec n,i} \quad \textit{and} \quad \boldsymbol z_{\vec n,i}=\boldsymbol a_{\vec n,i+1} \quad\Leftrightarrow\quad a_{\vec n,i+1}\in\partial D^-_{\vec n,i+1},$$

where the sets $D^-_{\vec n,i}$ are defined as in (15). Absolutely analogous claims hold for h, $\mathfrak{R}$, and $\boldsymbol\gamma_i$. Furthermore, it holds that

$$\text{(21)}\qquad \Phi_{\vec n}(\boldsymbol z) = \exp\left\{|\vec n|\int^{\boldsymbol z} h_{\vec n}(\boldsymbol x)dx\right\},$$

where the initial bound for integration should be chosen so that (18) is satisfied. Finally, if we set $\mathfrak{R}_\delta$ to be $\mathfrak{R}$ with circular neighborhood of radius δ excised around each of its branch points, then $h_{\vec n}\to h$ uniformly on $\mathfrak{R}_\delta$ for each $\delta>0$, where $h_{\vec n}$ is carried over to $\mathfrak{R}_\delta$ with the help of natural projections.

Thus, knowing the logarithmic derivative of $\Phi_{\vec n}$, we can recover the vector equilibrium measure $\vec\omega_{\vec n}$ by

$$d\omega_{\vec n}(x) = \left(h^{(0)}_{\vec n-}(x) - h^{(0)}_{\vec n+}(x)\right)\frac{dx}{2\pi i},$$

as follows from Privalov's Lemma [26, Sec. III.2] (the above formula does not allow to recover $\vec\omega_{\vec n}$ via a purely geometric construction of $\Phi_{\vec n}$ as one needs to know the intervals $\big[a_{\vec n,i},b_{\vec n,i}\big]$ to construct $\mathfrak{R}_{\vec n}$). We prove Propositions 1 and 3 in Section 5.

The purpose of the following proposition is to identify the limits in (16), which are nothing but appropriate generalizations of the classical Szegő function. In order to do that we need to specify the conditions we placed on the considered densities. In what follows, it is assumed that

$$\text{(22)}\qquad \rho_i(x) = \rho_{r,i}(x)\rho_{s,i}(x),$$

where $\rho_{r,i}$ is the regular part, that is, it is holomorphic and non-vanishing in some neighborhood of $[a_i,b_i]$, and $\rho_{s,i}$ is the singular part consisting of finitely many Fisher-Hartwig singularities [8], i.e.,

$$\text{(23)}\qquad \rho_{s,i}(x) = \prod_{j=0}^{J_i}|x-x_{ij}|^{\alpha_{ij}}\prod_{j=1}^{J_i}\begin{Bmatrix} 1, & x<x_{ij} \\ \beta_{ij}, & x>x_{ij}\end{Bmatrix}$$

where $a_i = x_{i0} < x_{i1} < \cdots < x_{iJ_i-1} < x_{iJ_i} = b_i$, $\alpha_{ij}>-1$, $\beta_{ij}\in\mathbb{C}\setminus(-\infty,0]$. In what follows, we adopt the following convention: given a function F on $\mathfrak{R}$, we denote by $F^{(k)}$ its pull-back from $\mathfrak{R}^{(k)}\setminus\boldsymbol\Delta_k$, $k\in\{0,\dots,p\}$. That is, $F^{(k)}(z) := F\big(z^{(k)}\big)$, $z\in\overline{\mathbb{C}}\setminus\big[a_{\vec c,i},b_{\vec c,i}\big]$.

Proposition 4. *For each $i\in\{1,\dots,p\}$, let ρ_i be of the form (22)–(23). Further, let*

$$\text{(24)}\qquad w_i(z) := \sqrt{(z-a_{\vec c,i})(z-b_{\vec c,i})}$$

[3] Of course, if $\boldsymbol z_{\vec n,i}$ coincides with either $\boldsymbol b_{\vec n,i}$ or $\boldsymbol a_{\vec n,i+1}$, then it cancels the corresponding pole.

be the branch holomorphic outside of $[a_{\vec c,i}, b_{\vec c,i}]$ *normalized so that* $w_i(z)/z \to 1$ *as* $z \to \infty$. *Then there exists the unique function* S *non-vanishing and holomorphic in* $\mathfrak{R} \setminus \bigcup_{i=1}^p \boldsymbol{\Delta}_i$ *such that*

$$S_\pm^{(i)} = S_\mp^{(0)}(\rho_i w_{i+}) \quad \text{on} \quad (a_{\vec c,i}, b_{\vec c,i}) \setminus \{x_{ij}\}_{j=0}^{J_i}, \tag{25}$$

$i \in \{1,\dots,p\}$, *and that satisfies*

$$\left|S^{(0)}(z)\right| \sim \left|S^{(i)}(z)\right|^{-1} \sim |z-e|^{-(2\alpha+1)/4} \quad \text{as} \quad z \to e \in \{a_{\vec c,i}, b_{\vec c,i}\}, \tag{26}$$

$i \in \{1,\dots,p\}$, *where* $\alpha = \alpha_{ij}$ *if* $e = x_{ij}$ *and* $\alpha = 0$ *otherwise;*

$$\left|S^{(0)}(z)\right| \sim \left|S^{(i)}(z)\right|^{-1} \sim |z-x_{ij}|^{-(\alpha_{ij} \pm \arg(\beta_{ij})/\pi)/2} \quad \text{as} \quad z \to x_{ij} \in (a_{\vec c,i}, b_{\vec c,i}), \quad \pm \mathrm{Im}(z) > 0, \tag{27}$$

$i \in \{1,\dots,p\}$; *and* $\prod_{k=0}^p S^{(k)}(z) \equiv 1$.

We prove Proposition 4 in Section 6. Finally, we are ready to formulate our main result.

Theorem 5. *Let* $\vec f = (f_1,\dots,f_p)$ *be a vector of functions given by*

$$f_i(z) = \frac{1}{2\pi i}\int_{[a_i,b_i]} \frac{\rho_i(x)}{x-z}\,dx, \qquad z \in \overline{\mathbb{C}} \setminus [a_i,b_i], \tag{28}$$

for a system of mutually disjoint intervals $\{[a_i,b_i]\}_{i=1}^p$, *where the functions* ρ_i *are of the form* (22)–(23), $i \in \{1,\dots,p\}$. *Given* $\vec c \in (0,1)^p$ *such that* $|\vec c| = 1$ *and a sequence of multi-indices* $\{\vec n\}$ *satisfying* (9), *let* $[\vec n]_{\vec f}$ *be the corresponding Hermite-Padé approximant* (1)–(2). *Then*

$$\begin{cases} Q_{\vec n} &= C_{\vec n}[1+o(1)](S\Phi_{\vec n})^{(0)} \\ R_{\vec n}^{(i)} &= C_{\vec n}[1+o(1)](S\Phi_{\vec n})^{(i)}/w_i, \quad i \in \{1,\dots,p\}, \end{cases}$$

locally uniformly in $\overline{\mathbb{C}} \setminus \bigcup_{i=1}^p [a_i,b_i]$, *where the functions* $\Phi_{\vec n}$ *are as in Proposition 1, the functions* S *and* w_i *are as in Proposition 4, and* $\lim_{z\to\infty} C_{\vec n}(S\Phi_{\vec n})^{(0)}(z) z^{-|\vec n|} = 1$. *In particular,* $\deg(Q_{\vec n}) = |\vec n|$ *for all* $|\vec n|$ *large enough.*

Theorem 5 is proved in Section 8. It follows immediately from (2), (15), and (19) that

$$f_i - \frac{P_{\vec n}^{(i)}}{Q_{\vec n}} = \frac{1+o(1)}{w_i}\frac{(S\Phi_{\vec n})^{(i)}}{(S\Phi_{\vec n})^{(0)}}$$

is geometrically small locally uniformly in D_i^+ and is geometrically big locally uniformly in D_i^- whenever the latter is non-empty.

3 RIEMANN-HILBERT APPROACH

To prove Theorem 5 we use the extension to multiple orthogonal polynomials [14] of by now classical approach of Fokas, Its, and Kitaev [11, 12] connecting orthogonal polynomials to matrix Riemann-Hilbert problems. The RH problem is then analyzed via the non-linear steepest descent method of Deift and Zhou [10].

The Riemann-Hilbert approach of Fokas, Its, and Kitaev lies in the following. Assume that the multi-index $\vec n = (n_1,\dots,n_p)$ is such that

$$\deg(Q_{\vec n}) = |\vec n| \quad \text{and} \quad R^{(i)}_{\vec n - \vec e_i}(z) \sim z^{-n_i} \quad \text{as} \quad z \to \infty, \quad i \in \{1,\dots,p\}, \tag{29}$$

where all the entries of the vector $\vec e_i$ are zero except for the i-th one, which is 1. Set

$$\mathbf{Y} := \begin{pmatrix} Q_{\vec n} & R^{(1)}_{\vec n} & \cdots & R^{(p)}_{\vec n} \\ m_{\vec n,1} Q_{\vec n - \vec e_1} & m_{\vec n,1} R^{(1)}_{\vec n - \vec e_1} & \cdots & m_{\vec n,1} R^{(p)}_{\vec n - \vec e_1} \\ \vdots & \vdots & \ddots & \vdots \\ m_{\vec n,p} Q_{\vec n - \vec e_p} & m_{\vec n,p} R^{(1)}_{\vec n - \vec e_p} & \cdots & m_{\vec n,p} R^{(p)}_{\vec n - \vec e_p} \end{pmatrix}, \tag{30}$$

where $m_{\vec{n},i}$, $i \in \{1, \dots, p\}$, is a constant such that

$$\lim_{z\to\infty} m_{\vec{n},i} R^{(i)}_{\vec{n}-\vec{e}_i}(z) z^{n_i} = 1.$$

To capture the block structure of many matrices appearing below, let us introduce transformations T_i, $i \in \{1, \dots, p\}$, that act on 2×2 matrices:

$$T_i \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} := e_{11}\mathbf{E}_{1,1} + e_{12}\mathbf{E}_{1,i+1} + e_{21}\mathbf{E}_{i+1,1} + e_{22}\mathbf{E}_{i+1,i+1} + \sum_{j\neq 1,i+1} \mathbf{E}_{jj},$$

where $\mathbf{E}_{jk}$ is the matrix with all zero entries except for the (j,k)-th one which is 1. It can be easily checked that $T_i(\mathbf{AB}) = T_i(\mathbf{A})T_i(\mathbf{B})$ for any 2×2 matrices $\mathbf{A}, \mathbf{B}$.

The matrix-valued function $\mathbf{Y}$ solves the following Riemann-Hilbert problem (RHP-$\mathbf{Y}$) :

(a) $\mathbf{Y}$ is analytic in $\mathbb{C} \setminus \bigcup_{i=1}^p [a_i, b_i]$ and $\lim_{z\to\infty} \mathbf{Y}(z) z^{-\sigma(\vec{n})} = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix and $\sigma(\vec{n}) := \mathrm{diag}\,(|\vec{n}|, -n_1, \dots, -n_p)$;

(b) $\mathbf{Y}$ has continuous traces on each (a_i, b_i) that satisfy $\mathbf{Y}_+ = \mathbf{Y}_- T_i \begin{pmatrix} 1 & \rho_i \\ 0 & 1 \end{pmatrix}$;

(c) the entries of the $(i+1)$-st column of $\mathbf{Y}$ behave like $\mathcal{O}\left(\psi_{\alpha_{ij}}(z - x_{ij})\right)$ as $z \to x_{ij}$, $j \in \{0, \dots, J_i\}$, while the remaining entries stay bounded, where

$$\psi_\alpha(z) = \begin{cases} |z|^\alpha, & \text{if} \quad \alpha < 0, \\ \log|z|, & \text{if} \quad \alpha = 0, \\ 1, & \text{if} \quad \alpha > 0. \end{cases}$$

The property RHP-$\mathbf{Y}$(a) follows immediately from (2) and (29). The property RHP-$\mathbf{Y}$(b) is due to the equality

$$R^{(i)}_{\vec{n}+} - R^{(i)}_{\vec{n}-} = Q_{\vec{n}}(f_{i+} - f_{i-}) = Q_{\vec{n}}\rho_i \quad \text{on} \quad (a_i, b_i),$$

which in itself is a consequence of (2), (28), and the Sokhotski-Plemelj formulae [13, Section 4.2]. Finally, RHP-$\mathbf{Y}$(c) follows from the local analysis of Cauchy integrals in [13, Section 8.1].

Conversely, if $\mathbf{Y}$ is a solution of RHP-$\mathbf{Y}$, then it follows from RHP-$\mathbf{Y}$(b) and the normalization at infinity in RHP-$\mathbf{Y}$(a) that $[\mathbf{Y}]_{1,1}$ is a polynomial of degree exactly $|\vec{n}|$. It further follows from RHP-$\mathbf{Y}$(b) that $[\mathbf{Y}]_{1,i+1}$, $i \in \{1, \dots, p\}$, is holomorphic outside of $[a_i, b_i]$, vanishes at infinity with order $n_i + 1$, and satisfies

$$[\mathbf{Y}]_{1,i+1+} - [\mathbf{Y}]_{1,i+1-} = [\mathbf{Y}]_{1,1}\rho_i \quad \text{on} \quad (a_i, b_i).$$

Combining this with RHP-$\mathbf{Y}$(c), we see that $[\mathbf{Y}]_{1,i+1}$ is the Cauchy integral of $[\mathbf{Y}]_{1,1}\rho_i$ on $[a_i, b_i]$. Furthermore, from the order of vanishing at infinity one can easily infer that $[\mathbf{Y}]_{1,1}(x)$ is orthogonal to x^j, $j \in \{0, \dots, n_i - 1\}$, with respect to $\rho_i(x)dx$. Hence, $[\mathbf{Y}]_{1,1} = Q_{\vec{n}}$, $[\mathbf{Y}]_{1,i+1} = R^{(i)}_{\vec{n}}$, and (29) holds. Other rows of $\mathbf{Y}$ can be analyzed analogously. Altogether, the following proposition takes place.

Proposition 6. *If a solution of* RHP-$\mathbf{Y}$ *exists then it is unique. Moreover, in this case it is given by* (30) *where* $Q_{\vec{n}}$ *and* $R^{(i)}_{\vec{n}-\vec{e}_i}$ *satisfy* (29)*. Conversely, if* (29) *is fulfilled, then* (30) *solves* RHP-$\mathbf{Y}$.

4 MODEL RIEMANN-HILBERT PROBLEMS

As known, to analyze RHP-$\mathbf{Y}$ via steepest descent method of Deift and Zhou, one needs to construct local solutions around each singular point of the functions ρ_i and the endpoints of the support of each component of the vector equilibrium measure, see Section 9. In this section, we present all these model RH problems. In what follows we use the notation

$$\sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

4.1 *Singular Points of the Weights*

In what follows, we always assume that the real line as well as its subintervals is oriented from left to right. Further, we set

$$I_\pm := \{z: \arg(z) = \pm 2\pi/3\}, \quad J_\pm := \{z: \arg(z) = \pm\pi/3\}, \tag{31}$$

where the rays $I_\pm$ are oriented towards the origin and the rays $J_\pm$ are oriented away from the origin. Put

$$\Sigma(\mathbf{\Phi}_{\alpha,\beta}) := I_+ \cup I_- \cup J_+ \cup J_- \cup (-\infty, \infty)$$

and consider the following Riemann-Hilbert problem: given $\alpha > -1$ and $\beta \in \mathbb{C} \setminus (-\infty, 0]$, find a matrix-valued function $\mathbf{\Phi}_{\alpha,\beta}$ such that

(a) $\mathbf{\Phi}_{\alpha,\beta}$ is holomorphic in $\mathbb{C} \setminus \Sigma(\mathbf{\Phi}_{\alpha,\beta})$;
(b) $\mathbf{\Phi}_{\alpha,\beta}$ has continuous traces on $\Sigma(\mathbf{\Phi}_{\alpha,\beta}) \setminus \{0\}$ that satisfy

$$\mathbf{\Phi}_{\alpha,\beta+} = \mathbf{\Phi}_{\alpha,\beta-} \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{on} \quad (-\infty, 0), \\ \begin{pmatrix} 0 & \beta \\ -\beta^{-1} & 0 \end{pmatrix} & \text{on} \quad (0, \infty), \end{cases}$$

and

$$\mathbf{\Phi}_{\alpha,\beta+} = \mathbf{\Phi}_{\alpha,\beta-} \begin{cases} \begin{pmatrix} 1 & 0 \\ e^{\pm\alpha\pi i} & 1 \end{pmatrix} & \text{on} \quad I_\pm, \\ \begin{pmatrix} 1 & 0 \\ 1/\beta & 1 \end{pmatrix} & \text{on} \quad J_\pm; \end{cases}$$

(c) as $\zeta \to 0$ it holds that

$$\mathbf{\Phi}_{\alpha,\beta}(\zeta) = \mathcal{O}\begin{pmatrix} |\zeta|^{\alpha/2} & |\zeta|^{\alpha/2} + |\zeta|^{-\alpha/2} \\ |\zeta|^{\alpha/2} & |\zeta|^{\alpha/2} + |\zeta|^{-\alpha/2} \end{pmatrix} \quad \text{and} \quad \mathbf{\Phi}_{\alpha,\beta}(\zeta) = \mathcal{O}\begin{pmatrix} 1 & \log|\zeta| \\ 1 & \log|\zeta| \end{pmatrix}$$

when $\alpha \neq 0$ and $\alpha = 0$, respectively;
(d) $\mathbf{\Phi}_{\alpha,\beta}$ has the following behavior near ∞:

$$\mathbf{\Phi}_{\alpha,\beta}(\zeta) = \left(\mathbf{I} + \mathcal{O}\left(\zeta^{-1}\right)\right) (i\zeta)^{\log\beta\,\sigma_3/2\pi i} \mathbf{B}_\pm \exp\{\mp i\zeta\sigma_3/2\}, \quad \pm \mathrm{Im}(\zeta) > 0,$$

uniformly in $\mathbb{C} \setminus \Sigma(\mathbf{\Phi}_{\alpha,\beta})$, where $(i\zeta)^{\log\beta/2\pi i}$ has a branch cut along $(0,\infty)$ (observe also that $(i\zeta)_-^{\log\beta/2\pi i} = \beta (i\zeta)_+^{\log\beta/2\pi i}$ on $(0,\infty)$) and

$$\mathbf{B}_+ := \begin{pmatrix} \beta^{-1/2} & 0 \\ 0 & e^{-\alpha\pi i/2} \end{pmatrix} \beta^{\sigma_3} e^{\alpha\pi i\sigma_3}, \quad \mathbf{B}_- := \mathbf{B}_+ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The solution of RHP-$\mathbf{\Phi}_{\alpha,\beta}$ can be written explicitly with the help of confluent hypergeometric functions. It was done first in [30] for the case $\beta = 1$, then in [21, 22] for $\beta \in (0,\infty)$, and, in [8] for $\alpha \pm \log\beta/\pi i \notin \{-2, -4, \dots\}$ (of course, in all the cases $\alpha > -1$; parameters α_j and β_j in [8] correspond to $\alpha/2$ and $i\log\beta/2\pi$ above). To be more precise, one needs to take $\mathbf{\Phi}_{\alpha,\beta}\beta^{\sigma_3/4}$ multiply it by $e^{-\alpha\pi i\sigma_3/2}$ in the first quadrant, by $e^{\alpha\pi i\sigma_3/2}$ in the fourth quadrant, and then rotate the whole picture by $\pi/2$ to get the corresponding problem in [8].

4.2 *Hard Edge*

Given $\alpha > -1$, find a matrix-valued function $\mathbf{\Psi}_\alpha$ such that

(a) $\mathbf{\Psi}_\alpha$ is holomorphic in $\mathbb{C} \setminus (I_+ \cup I_- \cup (-\infty, 0])$;

(b) $\boldsymbol{\Psi}_\alpha$ has continuous traces on $I_+ \cup I_- \cup (-\infty, 0)$ that satisfy

$$\boldsymbol{\Psi}_{\alpha+} = \boldsymbol{\Psi}_{\alpha-} \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{on} \quad (-\infty, 0), \\ \begin{pmatrix} 1 & 0 \\ e^{\pm \pi i \alpha} & 1 \end{pmatrix} & \text{on} \quad I_\pm; \end{cases}$$

(c) as $\zeta \to 0$ it holds that

$$\boldsymbol{\Psi}_\alpha(\zeta) = \mathcal{O}\begin{pmatrix} |\zeta|^{\alpha/2} & |\zeta|^{\alpha/2} \\ |\zeta|^{\alpha/2} & |\zeta|^{\alpha/2} \end{pmatrix} \quad \text{and} \quad \boldsymbol{\Psi}_\alpha(\zeta) = \mathcal{O}\begin{pmatrix} \log|\zeta| & \log|\zeta| \\ \log|\zeta| & \log|\zeta| \end{pmatrix}$$

when $\alpha < 0$ and $\alpha = 0$, respectively, and

$$\boldsymbol{\Psi}_\alpha(\zeta) = \mathcal{O}\begin{pmatrix} |\zeta|^{\alpha/2} & |\zeta|^{-\alpha/2} \\ |\zeta|^{\alpha/2} & |\zeta|^{-\alpha/2} \end{pmatrix} \quad \text{and} \quad \boldsymbol{\Psi}_\alpha(\zeta) = \mathcal{O}\begin{pmatrix} |\zeta|^{-\alpha/2} & |\zeta|^{-\alpha/2} \\ |\zeta|^{-\alpha/2} & |\zeta|^{-\alpha/2} \end{pmatrix}$$

when $\alpha > 0$, for $|\arg(\zeta)| < 2\pi/3$ and $2\pi/3 < |\arg(\zeta)| < \pi$, respectively;

(d) $\boldsymbol{\Psi}_\alpha$ has the following behavior near ∞:

$$\boldsymbol{\Psi}_\alpha(\zeta) = \frac{\zeta^{-\sigma_3/4}}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \left(\mathbf{I} + \mathcal{O}\left(\zeta^{-1/2}\right) \right) \exp\left\{ 2\zeta^{1/2} \sigma_3 \right\}$$

uniformly in $\mathbb{C} \setminus \big(I_+ \cup I_- \cup (-\infty, 0]\big)$.

The solution of this Riemann-Hilbert problem was constructed explicitly in [19] with the help of modified Bessel and Hankel functions.

4.3 *Soft-Type Edge*

To describe the model Riemann-Hilbert problem we need, it will be convenient to denote by Ω_1, Ω_2, Ω_3, and Ω_4 consecutive sectors of $\mathbb{C} \setminus \big((-\infty, \infty) \cup I_- \cup I_+\big)$ starting with the one containing the first quadrant and continuing counter clockwise. Given $\alpha \in \mathbb{R}$ and $\mathrm{Re}(\beta) \geqslant 0$, we are looking for a matrix-valued function $\boldsymbol{\Psi}_{\alpha,\beta}$ such that

(a) $\boldsymbol{\Psi}_{\alpha,\beta}$ is holomorphic in $\mathbb{C} \setminus \big(I_+ \cup I_- \cup (-\infty, \infty)\big)$;

(b) $\boldsymbol{\Psi}_{\alpha,\beta}$ has continuous traces on $I_+ \cup I_- \cup (-\infty, 0) \cup (0, \infty)$ that satisfy

$$\boldsymbol{\Psi}_{\alpha,\beta+} = \boldsymbol{\Psi}_{\alpha,\beta-} \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{on} \quad (-\infty, 0), \\ \begin{pmatrix} 1 & 0 \\ e^{\pm i\pi\alpha} & 1 \end{pmatrix} & \text{on} \quad I_\pm, \\ \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} & \text{on} \quad (0, \infty); \end{cases}$$

(c) as $\zeta \to 0$ it holds that

$$\boldsymbol{\Psi}_{\alpha,\beta}(\zeta) = \mathbf{E}(\zeta)\mathbf{S}_{\alpha,\beta}(\zeta)\mathbf{A}_j, \quad \zeta \in \Omega_j,$$

where $\mathbf{E}$ is a holomorphic matrix function,

$$\mathbf{A}_3 = \mathbf{A}_4 \begin{pmatrix} 1 & 0 \\ e^{-\alpha\pi i} & 1 \end{pmatrix}, \quad \mathbf{A}_4 = \mathbf{A}_1 \begin{pmatrix} 1 & -\beta \\ 0 & 1 \end{pmatrix}, \quad \mathbf{A}_1 = \mathbf{A}_2 \begin{pmatrix} 1 & 0 \\ e^{\alpha\pi i} & 1 \end{pmatrix},$$

and

$$\mathbf{A}_2 = \begin{pmatrix} \frac{1}{2\cos(\alpha\pi/2)} \frac{1-\beta e^{\alpha\pi i}}{1-e^{\alpha\pi i}} & \frac{1}{2\cos(\alpha\pi/2)} \frac{\beta - e^{\alpha\pi i}}{1-e^{\alpha\pi i}} \\ -e^{\alpha\pi i/2} & e^{-\alpha\pi i/2} \end{pmatrix} \quad \text{while} \quad \mathbf{S}_{\alpha,\beta}(\zeta) = \zeta^{\alpha\sigma_3/2}$$

when α is not an integer,

$$\mathbf{A}_2 = \begin{pmatrix} \frac{1}{2} e^{\alpha\pi i/2} & \frac{1}{2} e^{-\alpha\pi i/2} \\ -e^{\alpha\pi i/2} & e^{-\alpha\pi i/2} \end{pmatrix} \quad \text{while} \quad \mathbf{S}_{\alpha,\beta}(\zeta) = \begin{pmatrix} \zeta^{\alpha/2} & \frac{1-\beta}{2\pi i} \zeta^{\alpha/2} \log\zeta \\ 0 & \zeta^{-\alpha/2} \end{pmatrix}$$

when α is an even integer,

$$\mathbf{A}_2 = \begin{pmatrix} 0 & e^{-\alpha\pi i/2} \\ -e^{\alpha\pi i/2} & e^{-\alpha\pi i/2} \end{pmatrix} \quad \text{while} \quad \mathbf{S}_{\alpha,\beta}(\zeta) = \begin{pmatrix} \zeta^{\alpha/2} & \frac{1+\beta}{2\pi i}\zeta^{\alpha/2}\log\zeta \\ 0 & \zeta^{-\alpha/2} \end{pmatrix}$$

when α is an odd integer;

(d) $\mathbf{\Psi}_{\alpha,\beta}$ has the following behavior near ∞:

$$\mathbf{\Psi}_{\alpha,\beta}(\zeta;s) = \left(\mathbf{I} + \mathcal{O}\left(\zeta^{-1}\right)\right)\frac{\zeta^{-\sigma_3/4}}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\exp\left\{-\frac{2}{3}(\zeta+s)^{3/2}\sigma_3\right\}$$

uniformly in $\mathbb{C}\setminus\left(I_+\cup I_-\cup(-\infty,\infty)\right)$.

Besides RHP-$\mathbf{\Psi}_{\alpha,\beta}$, we shall also need RHP-$\widetilde{\mathbf{\Psi}}_{\alpha,\beta}$ obtained from RHP-$\mathbf{\Psi}_{\alpha,\beta}$ by replacing RHP-$\mathbf{\Psi}_{\alpha,\beta}$(d) with

($\tilde{\text{d}}$) $\widetilde{\mathbf{\Psi}}_{\alpha,\beta}$ has the following behavior near ∞:

$$\widetilde{\mathbf{\Psi}}_{\alpha,\beta}(\zeta;s) = \left(\mathbf{I} + \mathcal{O}\left(\zeta^{-1}\right)\right)\frac{\zeta^{-\sigma_3/4}}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\exp\left\{-\left(\frac{2}{3}\zeta^{3/2}+s\zeta^{1/2}\right)\sigma_3\right\}.$$

The problems RHP-$\mathbf{\Psi}_{\alpha,\beta}$ and RHP-$\widetilde{\mathbf{\Psi}}_{\alpha,\beta}$ are simultaneously uniquely solvable and the solutions are connected by

$$\widetilde{\mathbf{\Psi}}_{\alpha,\beta}(\zeta;s) = \begin{pmatrix} 1 & 0 \\ is^2/4 & 1 \end{pmatrix}\mathbf{\Psi}_{\alpha,\beta}(\zeta;s) \tag{32}$$

as follows from the estimate

$$\frac{2}{3}(\zeta+s)^{3/2} - \left(\frac{2}{3}\zeta^{3/2}+s\zeta^{1/2}\right) = \left(1+\mathcal{O}(s/\zeta)\right)\frac{s^2}{4\zeta^{1/2}} \quad \text{as} \quad \zeta\to\infty.$$

When $\alpha = 0$, $\beta = 1$, and $s = 0$, the above Riemann-Hilbert problem is well known [9] and is solved using Airy functions. When $\beta = 1$, the solvability of this problem for all $s \in \mathbb{R}$ was shown in [16] with further properties investigated in [17] (RHP-$\widetilde{\mathbf{\Psi}}_{\alpha,\beta}$ is associated with a solution of Painlevé XXXIV equation). The solvability of the case $\alpha = 0$, $\beta \in \mathbb{C}\setminus(-\infty,0)$, and $s \in \mathbb{R}$ was obtained in [32]. The latter case appeared in [6] as well. More generally, the following theorem holds.

Theorem 7. *Given $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{C}\setminus(-\infty,0)$, the RH-problems RHP-$\mathbf{\Psi}_{\alpha,\beta}$, and therefore RHP-$\widetilde{\mathbf{\Psi}}_{\alpha,\beta}$, is uniquely solvable for all $s \in \mathbb{R}$. Moreover, assuming $\beta \neq 0$, it holds that*

$$\mathbf{\Psi}_{\alpha,\beta}(\zeta;s) = \frac{\zeta^{-\sigma_3/4}}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\left(\mathbf{I}+\mathcal{O}\left(\sqrt{\frac{|s|+1}{|\zeta|+1}}\right)\right)\exp\left\{-\frac{2}{3}(\zeta+s)^{3/2}\sigma_3\right\} \tag{33}$$

uniformly for $\zeta \in \mathbb{C}\setminus\left(I_+\cup I_-\cup(-\infty,\infty)\right)$ and $s \in (-\infty,\infty)$, and it also holds uniformly for $s \in [0,\infty)$ when $\beta = 0$; furthermore, we have that

$$\widetilde{\mathbf{\Psi}}_{\alpha,0}(\zeta;s) = \frac{\zeta^{-\sigma_3/4}}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\left(\mathbf{I}+\mathcal{O}\left(\sqrt{\frac{|s|+1}{|\zeta|+1}}\right)\right)\exp\left\{-\left(\frac{2}{3}\zeta^{3/2}+s\zeta^{1/2}\right)\sigma_3\right\} \tag{34}$$

uniformly for $\zeta \in \mathbb{C}\setminus\left(I_+\cup I_-\cup(-\infty,0]\right)$ and $s \in (-\infty,0]$.

Theorem 7 is proved in Section 10.

5 GEOMETRY

In this section we prove Propositions 1 and 3.

5.1 *Proof of Proposition 1*

Set

$$O_i^{\pm} := \left\{z:\ \mathrm{Re}(z)\in\left(a_{\vec{n},i}, b_{\vec{n},i}\right) \text{ and } \pm\mathrm{Im}(z)>0\right\}.$$

Since the measures $\omega_{\vec{n},i}$ are supported on the real line, (14) and Schwarz reflection principle yield that the function

$$\begin{cases} \ell_{\vec{n},i} - V^{\omega_{\vec{n}}+\omega_{\vec{n},i}}(z), & z \in O_i^+, \\ V^{\omega_{\vec{n}}+\omega_{\vec{n},i}}(z) - \ell_{\vec{n},i}, & z \in O_i^-, \end{cases}$$

is harmonic across $(a_{\vec{n},i}, b_{\vec{n},i})$. As the support of $\omega_{\vec{n}} - \omega_{\vec{n},i}$ is disjoint from $[a_{\vec{n},i}, b_{\vec{n},i}]$, the function $\ell_{\vec{n},i} + V^{\omega_{\vec{n}}-\omega_{\vec{n},i}}$ is harmonic across $(a_{\vec{n},i}, b_{\vec{n},i})$ as well. By taking the difference of these two functions, we see that

$$\begin{cases} -2V^{\omega_{\vec{n}}}(z), & z \in O_i^+, \\ 2V^{\omega_{\vec{n},i}}(z) - 2\ell_{\vec{n},i}, & z \in O_i^-, \end{cases}$$

is harmonic in the same vertical strip. Thus, the function

$$H_{\vec{n}}(\boldsymbol{z}) := \begin{cases} -V^{\omega_{\vec{n}}}(z) + \frac{1}{p+1}\sum_{k=1}^{p} \ell_{\vec{n},k}, & \boldsymbol{z} \in \mathfrak{R}_{\vec{n}}^{(0)}, \\ V^{\omega_{\vec{n},i}}(z) - \ell_{\vec{n},i} + \frac{1}{p+1}\sum_{k=1}^{p} \ell_{\vec{n},k}, & \boldsymbol{z} \in \mathfrak{R}_{\vec{n}}^{(i)}, \quad i \in \{1,\dots,p\}, \end{cases} \tag{35}$$

is harmonic on $\mathfrak{R}_{\vec{n}} \setminus \bigcup_{k=0}^{p} \{\infty^{(k)}\}$. Since $V^{\nu}(z) = -|\nu|\log|z| + \mathcal{O}(1)$ as $z \to \infty$, we get that the difference

$$|\,\vec{n}\,|^{-1}\log|\Phi_{\vec{n}}(\boldsymbol{z})| - H_{\vec{n}}(\boldsymbol{z})$$

is harmonic on the whole surface $\mathfrak{R}_{\vec{n}}$ and therefore is a constant. Since $\sum_{k=0}^{p} H_{\vec{n}}(z^{(k)}) \equiv 0$ and $\Phi_{\vec{n}}$ is normalized so that (18) holds, the first claim of the proposition follows.

Let $\vec{\nu}$ be a weak* limit point of $\{\vec{\omega}_{\vec{n}}\}$. Since $\{\,\vec{n}\,\}$ satisfies (9), it holds that $\vec{\nu} \in M_{\vec{c}}(\{a_i, b_i\}_{i=1}^{p})$. Thus, if we show that $I[\,\vec{\omega}\,] \geqslant I[\,\vec{\nu}\,]$, then $\vec{\nu} = \vec{\omega}$ by (10). To this end, let $\alpha_{\vec{n},i}$ be positive constants such that $|\alpha_{\vec{n},i}\omega_i| = n_i/|\,\vec{n}\,|$, $i \in \{1,\dots,p\}$. By (9), $\alpha_{\vec{n},i} \to 1$ as $|\,\vec{n}\,| \to \infty$. Set $\vec{\nu}_{\vec{n}} := (\alpha_{\vec{n},1}\omega_1, \dots, \alpha_{\vec{n},p}\omega_p)$. Then it follows from (10) applied for the vector $(n_1/|\,\vec{n}\,|, \dots, n_p/|\,\vec{n}\,|)$ that

$$I[\,\vec{\omega}\,] = \lim_{|\,\vec{n}\,|\to\infty} I[\,\vec{\nu}_{\vec{n}}\,] \geqslant \liminf_{|\,\vec{n}\,|\to\infty} I[\,\vec{\omega}_{\vec{n}}\,].$$

Furthermore, the very definition of the weak* convergence implies that

$$\lim_{|\,\vec{n}\,|\to\infty} I[\omega_{\vec{n},j}, \omega_{\vec{n},k}] = I[\nu_j, \nu_k]$$

for $j \neq k$ as $\mathsf{supp}(\omega_{\vec{n},j}) \cap \mathsf{supp}(\omega_{\vec{n},k}) = \varnothing$ in this case. It also follows from the Principle of Descent [28, Thm. I.6.8] that

$$\liminf_{|\,\vec{n}\,|\to\infty} I[\omega_{\vec{n},i}] \geqslant I[\nu_i].$$

Altogether,

$$I[\,\vec{\omega}\,] \geqslant \liminf_{|\,\vec{n}\,|\to\infty} I[\,\vec{\omega}_{\vec{n}}\,] \geqslant I[\,\vec{\nu}\,],$$

which proves the claim about weak* convergence of measures.

Weak* convergence of measures implies convergence of minima of the corresponding potentials [15]. Hence, (12) yields that $\ell_{\vec{n},i} \to \ell_i$ for all $i \in \{1,\dots,p\}$. Moreover, weak* convergence also implies locally uniform convergence of $V^{\omega_{\vec{n},i}}$ to V^{ω_i} in $\mathbb{C} \setminus [a_{\vec{c},i}, b_{\vec{c},i}]$ (there is no convergence at infinity as, in general, $|\omega_{\vec{n},i}| \neq |\omega_i|$ for given $\vec{n}$). Thus, it remains to show that the convergence of the potentials is uniform on compact subsets of $\mathbb{C}$.

First let K be a continuum such that $a_{\vec{c},i}, b_{\vec{c},i} \notin K$ and either $\mathsf{Im}(z) \geqslant 0$ for all $z \in K$ or $\mathsf{Im}(z) \leqslant 0$ for all $z \in K$ (it can intersect $(a_{\vec{c},i}, b_{\vec{c},i})$). Then there exists a unique continuum $K^{(i)}$ such that $\pi(K^{(i)}) = K$ and $K^{(i)} \cap \mathfrak{R}^{(i)} \neq \varnothing$. Further, let U be a neighborhood of K such that $a_{\vec{c},i}, b_{\vec{c},i} \notin U$. Denote $U^{(i)}$ the neighborhood of $K^{(i)}$ such that $\pi(U^{(i)}) = U$.

Since $a_{\vec{n},i} \to a_{\vec{c},i}$ and $b_{\vec{n},i} \to b_{\vec{c},i}$ as $|\vec{n}| \to \infty$, we can analogously define $K_{\vec{n}}^{(i)}$ and $U_{\vec{n}}^{(i)}$ on $\mathfrak{R}_{\vec{n}}$. By definition,

$$\begin{cases} V_{|K}^{\omega_{\vec{n},i}} = H_{\vec{n}|K_{\vec{n}}^{(i)}} + \ell_{\vec{n},i} - \frac{1}{p+1}\sum_{j=1}^{p}\ell_{\vec{n},j} \\ V_{|K}^{\omega_i} = H_{|K^{(i)}} + \ell_i - \frac{1}{p+1}\sum_{j=1}^{p}\ell_j, \end{cases}$$

where H is defined on $\mathfrak{R}$ exactly as $H_{\vec{n}}$ was defined on $\mathfrak{R}_{\vec{n}}$. Hence, to show that $V^{\omega_{\vec{n},i}}$ converge to V^{ω_i} uniformly on K it is enough to show that the pull backs of $H_{\vec{n}}$ from $U_{\vec{n}}^{(i)}$ to U converge locally uniformly to the pull back of H. We do know that such a convergence takes place locally uniformly on $U \cap \{\mathrm{Im}(z) > 0\}$ and $U \cap \{\mathrm{Im}(z) < 0\}$. The full claim will follow from Harnack's theorem if we show that the pull backs of $H_{\vec{n}}$, which are harmonic in U, form a uniformly bounded family there. The latter is true since each $H_{\vec{n}}^{(k)}$ converges to $H^{(k)}$ on any Jordan curve J that encloses $\bigcup_{i=1}^{p}[a_i, b_i]$. Hence, the moduli $|H_{\vec{n}}|$ are bounded on the lift of J to $\mathfrak{R}_{\vec{n}}$ and the bound is independent of $\vec{n}$. The maximum principle propagates this estimate through the region of $\mathfrak{R}_{\vec{n}}$ containing $U_{\vec{n}}^{(i)}$ and bounded by the lift of J.

Assume now that K is a continuum that contains one of the points $\{a_{\vec{c},i}, b_{\vec{c},i}\}$, say $b_{\vec{c},i}$ for definiteness. It is sufficient to assume that K is contained in a disk, say U, centered at the $b_{\vec{c},i}$ of radius small enough so that no other point from $\bigcup_{j=1}^{p}\{a_{\vec{c},j}, b_{\vec{c},j}\}$ belongs to U. We can define $K^{(i)}$ and $K_{\vec{n}}^{(i)}$ analogously to the previous case. Let $U^{(i)}$ and $U_{\vec{n}}^{(i)}$ be the circular neighborhoods of $\boldsymbol{b}_{\vec{c},i}$ and $\boldsymbol{b}_{\vec{n},i}$, respectively, with the natural projection U (clearly, they cover U twice). Let V be a disk centered at the origin of radius smaller than the one of U, but large enough so that the translation of V to $b_{\vec{c},i}$ still contains K. Then the functions $\phi_{\vec{n}}(z) = (z + b_{\vec{n},i})^2$ and $\phi(z) = (z + b_{\vec{c},i})^2$ provide one-to-one correspondents between V and some subdomains of $U_{\vec{n}}^{(i)}$ and $U^{(i)}$, respectively. These subdomains still contain $K_{\vec{n}}^{(i)}$ and $K^{(i)}$. Since $b_{\vec{n},i} \to b_{\vec{c},i}$ as $|\vec{n}| \to \infty$, we can establish exactly as above that $H_{\vec{n}} \circ \phi_{\vec{n}}$ converges to $H \circ \phi$ locally uniformly in V, which again yields that $V^{\omega_{\vec{n},i}}$ converges to V^{ω_i} uniformly on K. Clearly, the considered cases are sufficient to establish the uniform convergence on compact subsets of $\mathbb{C}$.

5.2 *Proof of Proposition 3*

Observe that

$$\begin{aligned} h_{\vec{n}}^{(0)}(z) &= \int \frac{d\omega_{\vec{n}}(x)}{z-x} = -2\partial_z V^{\omega_{\vec{n}}}(z) = 2|\vec{n}|^{-1}\partial_z \log\left|\Phi_{\vec{n}}^{(0)}(z)\right| \\ &= |\vec{n}|^{-1}\big(\Phi_{\vec{n}}^{(0)}(z)\big)'/\Phi_{\vec{n}}^{(0)}(z) \end{aligned}$$

by Proposition 1 and direct computation, where $2\partial_z := \partial_x - i\partial_y$. Clearly, analogous formulae hold for $h_{\vec{n}}^{(i)}$. That is, $h_{\vec{n}}$ is the logarithmic derivative of $\Phi_{\vec{n}}$, in particular, (21) holds. Therefore, $h_{\vec{n}}$ is holomorphic around each point of $\mathfrak{R}_{\vec{n}} \setminus \{\boldsymbol{a}_{\vec{n},i}, \boldsymbol{b}_{\vec{n},i}\}_{i=1}^{p}$ and clearly has a simple zero at each $\infty^{(k)}$, $k \in \{0, \dots, p\}$. Since $\mathfrak{R}_{\vec{n}}$ has square root branching at each ramification point, $\Phi_{\vec{n}}^{(0)}$ has Puiseux expansion in non-negative powers of $1/2$ at each of them. Hence, $h_{\vec{n}}^{(0)}$ has such an expansion as well and the smallest exponent is $-1/2$. Thus, $h_{\vec{n}}$ has at most a simple pole at each $\{\boldsymbol{a}_{\vec{n},i}, \boldsymbol{b}_{\vec{n},i}\}_{i=1}^{p}$ and, in particular, is a rational function on $\mathfrak{R}_{\vec{n}}$.

The number of zeros and poles, including multiplicities, of a rational function should be the same. Therefore, $h_{\vec{n}}$ has at most $2p$ and at least $p+1$ poles (the lower bound comes from the number of zeros at "infinities") and at most $p-1$ "finite" zeros. Let us now show that each of $p-1$ arcs $\gamma_{\vec{n},i}$ contains exactly one of those "finite" zeros (we slightly abuse the notion of a zero here since a simple zero at the endpoint means cancelation of the corresponding pole). Clearly, this is equivalent to showing that $h_{\vec{n}}^{(0)}$ has a single simple zero in each gap $[b_{\vec{n},i}, a_{\vec{n},i+1}]$ (again, a "zero" at the endpoint means that $h^{(0)}_{\vec{n}}$ is locally bounded there).

Assume to the contrary that there is at least one gap, say $[b_{\vec{n},j}, a_{\vec{n},j+1}]$, without a zero. Then $h^{(0)}_{\vec{n}}$ would be infinite at both endpoints $b_{\vec{n},j}, a_{\vec{n},j+1}$. However, since $\omega_{\vec{n}}$ is a positive measure, the very definition (20) yields that $h^{(0)}_{\vec{n}}$ is decreasing on $(b_{\vec{n},j}, a_{\vec{n},j+1})$. The latter is possible only if

$$\lim_{x\to b_{\vec{n},j}} h^{(0)}_{\vec{n}}(x) = -\lim_{x\to a_{\vec{n},j+1}} h^{(0)}_{\vec{n}}(x) = \infty. \tag{36}$$

As $h^{(0)}_{\vec{n}}$ is continuous on $(b_{\vec{n},j}, a_{\vec{n},j+1})$ it must vanish there. Since there are exactly $p-1$ gaps and $p-1$ "free" zeros, this contradiction proves the claim.

Let us now show the correspondence between occurrence of the zeros at the endpoints of the gaps and the fact that divergence domains are touching the support. To this end, notice that (21) combined with (19) yields that

$$\ell_{\vec{n},i} - V^{\omega_{\vec{n},i}+\omega_{\vec{n}}}(x) = \int_{b_{\vec{n},i}}^{x} \left(h^{(0)}_{\vec{n}} - h^{(i)}_{\vec{n}}\right)(y)dy. \tag{37}$$

If the zero of $h^{(0)}_{\vec{n}}$ on $[b_{\vec{n},i}, a_{\vec{n},i+1}]$ does not coincide with $b_{\vec{n},i}$, then

$$\begin{cases} h^{(0)}_{\vec{n}}(y) &=& c_{\vec{n}}\left(y-b_{\vec{n},i}\right)^{-1/2} + \mathcal{O}(1) \\ h^{(i)}_{\vec{n}}(y) &=& -c_{\vec{n}}\left(y-b_{\vec{n},i}\right)^{-1/2} + \mathcal{O}(1) \end{cases}$$

for $y - b_{\vec{n},i} > 0$ and small enough, where $c_{\vec{n}} > 0$, see (36). Hence,

$$\ell_{\vec{n},i} - V^{\omega_{\vec{n},i}+\omega_{\vec{n}}}(x) = 4c_{\vec{n}}\left(x-b_{\vec{n},i}\right)^{1/2} + \mathcal{O}\left(|x-b_{\vec{n},i}|^{3/2}\right) > 0 \tag{38}$$

for $x - b_{\vec{n},i} > 0$ and small enough. On the other hand, if the zero coincides with $b_{\vec{n},i}$, then

$$\begin{cases} h^{(0)}_{\vec{n}}(y) &=& \tilde{c}_{\vec{n}} - c'_{\vec{n}}\left(y-b_{\vec{n},i}\right)^{1/2} + \mathcal{O}\left(|y-b_{\vec{n},i}|\right) \\ h^{(i)}_{\vec{n}}(y) &=& \tilde{c}_{\vec{n}} + c'_{\vec{n}}\left(y-b_{\vec{n},i}\right)^{1/2} + \mathcal{O}\left(|y-b_{\vec{n},i}|\right) \end{cases}$$

for $y - b_{\vec{n},i} > 0$ and small enough, where $c'_{\vec{n}} > 0$ (recall that $h^{(0)}_{\vec{n}}$ is a decreasing function in each gap). Therefore,

$$\ell_{\vec{n},i} - V^{\omega_{\vec{n},i}+\omega_{\vec{n}}}(x) = -(4c'_{\vec{n}}/3)\left(x-b_{\vec{n},i}\right)^{3/2} + \mathcal{O}\left(|x-b_{\vec{n},i}|^{5/2}\right) < 0 \tag{39}$$

for $x - b_{\vec{n},i} > 0$ and small enough. Thus, if the zero from $[b_{\vec{n},i}, a_{\vec{n},i+1}]$ coincides with $b_{\vec{n},i}$, then $b_{\vec{n},i} \in \partial D^-_{\vec{n},i}$ and if it does not, then $b_{\vec{n},i} \notin \partial D^-_{\vec{n},i}$, see (15). As the analysis near $a_{\vec{n},i}$ can be completed similarly, this finishes the proof of the claim.

Let now $H_{\vec{n}}$ be defined by (35) and H be defined analogously on $\mathfrak{R}$. We have shown during the course of the proof of Proposition 1 that $H_{\vec{n}} \to H$ uniformly on $\mathfrak{R}_\delta$, where $H_{\vec{n}}$ is carried over to $\mathfrak{R}_\delta$ with the help of natural projections. Since $h_{\vec{n}} = 2\partial_z H_{\vec{n}}$ and $h = 2\partial_z H$, we get that $h_{\vec{n}} \to h$ uniformly on $\mathfrak{R}_\delta$. This implies that h is a rational function on $\mathfrak{R}$. The claim about zero/pole distribution of h follows from the analogous statement for $h_{\vec{n}}$ and analysis similar to (37)–(39).

6 Szegő function

In this section we prove Proposition 4. Let $\boldsymbol{z}, \boldsymbol{w} \in \mathfrak{R}$. Denote by $d\Omega_{\boldsymbol{z},\boldsymbol{w}}$ the unique abelian differential of the third kind which is holomorphic on $\mathfrak{R}\setminus\{\boldsymbol{z},\boldsymbol{w}\}$ and has simple poles at $\boldsymbol{z}$ and $\boldsymbol{w}$ of respective residues $+1$ and -1. Define

$$dC_{\boldsymbol{z}} := p\,d\Omega_{\boldsymbol{z},\boldsymbol{w}} - \sum_{i=1}^{p} d\Omega_{\boldsymbol{z}_i,\boldsymbol{w}}, \tag{40}$$

where $\pi^{-1}(z) = \{\boldsymbol{z}, \boldsymbol{z}_1, \dots, \boldsymbol{z}_p\}$ for each z which is not a projection of a branch point of $\mathfrak{R}$. The differential $dC_{\boldsymbol{z}}$ does not depend on the choice of $\boldsymbol{w}$ as it is simply the normalized third kind differential with $p+1$ simple poles at $z, z_1, \dots, z_p$ having respective residues $p, -1, \dots, -1$.

For each $\mathbf{x} \in \boldsymbol{\Delta}_i$, which is not a branch point of $\mathfrak{R}$, we shall denote by $\mathbf{x}^*$ a point on $\boldsymbol{\Delta}_i$ having the same canonical projection, i.e., $\pi(\mathbf{x}) = \pi(\mathbf{x}^*)$. When $\mathbf{x} \in \boldsymbol{\Delta}_i$ is a branch point of the surface, we simply set $\mathbf{x}^* = \mathbf{x}$. Let λ be a Hölder continuous function on $\boldsymbol{\Delta} := \bigcup_{i=1}^p \boldsymbol{\Delta}_i$. Define

$$\Lambda(z) := \frac{1}{2(p+1)\pi i} \oint_{\boldsymbol{\Delta}} \lambda dC_z, \qquad z \in \mathfrak{R} \setminus \pi^{-1}\big(\pi(\boldsymbol{\Delta})\big). \tag{41}$$

The function Λ is holomorphic in the domain of its definition. Further, if $z \to \mathbf{x} \in \boldsymbol{\Delta}^{\pm}$, then $z_j \to \mathbf{x}^* \in \boldsymbol{\Delta}^{\mp}$ for some $j \in \{1, \dots, p\}$ and

$$\Lambda_+(\mathbf{x}) - \Lambda_-(\mathbf{x}) = \frac{p\lambda(\mathbf{x}) + \lambda(\mathbf{x}^*)}{p+1},$$

according to [33, Eq. (2.8)]. On the other hand, if $z \to \tilde{\mathbf{x}} \notin \boldsymbol{\Delta}$, while $z_j \to \mathbf{x} \in \boldsymbol{\Delta}^{\pm}$ and $z_k \to \mathbf{x}^* \in \boldsymbol{\Delta}^{\mp}$ for some $j, k \in \{1, \dots, p\}$, then

$$\Lambda_+(\tilde{\mathbf{x}}) - \Lambda_-(\tilde{\mathbf{x}}) = \frac{\lambda(\mathbf{x}^*) - \lambda(\mathbf{x})}{p+1}.$$

Thus, if we additionally require that $\lambda(\mathbf{x}) = \lambda(\mathbf{x}^*)$, then Λ is a holomorphic function in $\mathfrak{R} \setminus \boldsymbol{\Delta}$ such that

$$\Lambda_+(\mathbf{x}) - \Lambda_-(\mathbf{x}) = \lambda(\mathbf{x}), \quad \mathbf{x} \in \boldsymbol{\Delta}. \tag{42}$$

It also can be readily verified using (40) and (41) that

$$\Lambda(z) + \sum_{i=1}^p \Lambda(z_i) \equiv 0 \quad \text{on} \quad \mathfrak{R}. \tag{43}$$

The above construction works for discontinuous function as well. Moreover, it is known that the continuity of $\Lambda_\pm$, in fact, Hölder continuity, depends on Hölder continuity of λ only locally. That is, if λ is Hölder continuous on some open subarc of $\boldsymbol{\Delta}$, so are the traces $\Lambda_\pm$ on this subarc irrespectively of the smoothness of λ on the remaining part of $\boldsymbol{\Delta}$. To capture the behavior of Λ around the points where λ is not continuous, we define a local approximation to the Cauchy differential dC_z. To this end, fix $i \in \{1, \dots, p\}$ and denote by $\mathbf{U}$ a connected annular neighborhood of $\boldsymbol{\Delta}_i$ disjoint from other $\boldsymbol{\Delta}_j$ such that every point in $\pi(\mathbf{U})$ has exactly two preimages (except for the branch points, of course). Write $\mathbf{U}^+ \cup \mathbf{U}^- = \mathbf{U} \setminus \boldsymbol{\Delta}$, where $\mathbf{U}^+ \cap \mathbf{U}^- = \varnothing$, $\mathbf{U}^{\pm}$ are connected and partially bounded by $\boldsymbol{\Delta}_i^{\pm}$. Set $\widetilde{w}_i(z) := \pm w_i(z)$, $z \in \mathbf{U}^{\pm}$, where w_i is given by (24). Then $\widetilde{w}_i$ is holomorphic in $\mathbf{U}$. Further, put

$$d\widetilde{\Omega}_z(\mathbf{x}) := \frac{1}{2}\frac{\widetilde{w}_i(\mathbf{x}) + \widetilde{w}_i(z)}{x - z}\frac{dx}{\widetilde{w}_i(\mathbf{x})},$$

which is a holomorphic differential on $\mathbf{U} \setminus \{z\}$ that has a simple pole at z with residue 1. Then the difference $dC_z - pd\widetilde{\Omega}_z + d\widetilde{\Omega}_{z_i}$ is a holomorphic differential in $\mathbf{U}$ and therefore the function $\Lambda - \widetilde{\Lambda}$ is holomorphic $\mathbf{U}$, where

$$\widetilde{\Lambda}(z) := \frac{1}{2(p+1)\pi i} \oint_{\boldsymbol{\Delta}_i} \lambda d\big(p\widetilde{\Omega}_z - \widetilde{\Omega}_{z^*}\big)$$

and $z^* \neq z$ is a point in $\mathbf{U}$ such that $\pi(z) = \pi(z^*)$. Thus, to understand the local behavior of Λ is sufficient to study $\widetilde{\Lambda}$. Since $\widetilde{w}_i(z^*) = -\widetilde{w}_i(z)$ for $z \in \mathbf{U}$, and $w_{i-}(x) = -w_{i+}(x)$ for $x \in \big(a_{\vec{c},i}, b_{\vec{c},i}\big)$, it holds for $\lambda(\mathbf{x}) = \lambda(x)$ that

$$\widetilde{\Lambda}(z) = \frac{\widetilde{w}_i(z)}{2\pi i} \int_{\Delta_i} \frac{\lambda(x)}{w_{i+}(x)} \frac{dx}{x - z}, \quad z \in \mathbf{U} \setminus \boldsymbol{\Delta}. \tag{44}$$

The first type of singularities we are interested in is of the form

$$\lambda(\mathbf{x}) = \alpha \log|x - x_0|, \quad \mathbf{x} \in \boldsymbol{\Delta}_i, \tag{45}$$

where $x_0 \in [a_{\vec{c},i}, b_{\vec{c},i}]$. Carefully tracing the implications of [13, Sec. I.8.5–6] to the integrals of the form (44) and (45), we get that

$$\widetilde{\Lambda}(\boldsymbol{z}) = \pm\frac{\alpha}{2}\log(z - x_0) + \mathcal{O}(1), \quad \mathbf{U}^{\pm} \ni \boldsymbol{z} \to \boldsymbol{x}_0. \tag{46}$$

The second type of the singular behavior we want to consider is given by

$$\lambda(\boldsymbol{x}) = (\log\beta)\chi_{x_0}(x), \quad \boldsymbol{x} \in \boldsymbol{\Delta}_i, \tag{47}$$

where $x_0 \in (a_{\vec{c},i}, b_{\vec{c},i})$ and χ_{x_0} is the characteristic function of $[x_0, b_{\vec{c},i}]$. It follows from the analysis in [13, Sec. I.8.6] that

$$\left\{\begin{array}{lcl} \widetilde{\Lambda}(z^{(0)}) & = & \mp\frac{\log\beta}{2\pi i}\log(z - x_0) + \mathcal{O}(1), \\ \widetilde{\Lambda}(z^{(i)}) & = & \pm\frac{\log\beta}{2\pi i}\log(z - x_0) + \mathcal{O}(1), \end{array}\right. \quad z \to x_0, \quad \pm\mathrm{Im}(z) > 0. \tag{48}$$

Now, let the functions ρ_i be of the form (22)–(23). Set

$$\lambda_\rho(\boldsymbol{x}) := -\log\left(\rho_i(x) w_{i+}(x)\right), \quad \boldsymbol{x} \in \boldsymbol{\Delta}_i.$$

By using the identity $w_{i+}(x) = i|w_i(x)|$ and the explicit expressions (23), we can then write

$$\lambda_\rho(\boldsymbol{x}) = -\log\left(i\rho_{r,i}(x)\right) - \sum_{i=0}^{J_i}\left(\alpha_{ij}\log|x - x_{ij}| + \log\beta_{ij}\chi_{x_{ij}}(x)\right) \\ - (1/2)\log\left|x - a_{\vec{c},i}\right| - (1/2)\log\left|x - b_{\vec{c},i}\right|.$$

Clearly, the singular behavior of λ_ρ is precisely of the form (45) and (47). Define Λ_ρ as in (41) and set $S := \exp\{\Lambda_\rho\}$. Then (25) is a consequence of (42) since

$$\left(S_{\pm}^{(i)}/S_{\mp}^{(0)}\right)(x) = \exp\left\{(\Lambda_{\rho-} - \Lambda_{\rho+})(\boldsymbol{x})\right\}.$$

Moreover, (26) and (27) clearly follow from (46) and (48). Finally, the last claim of the proposition follows from (43).

7 AUXILIARY RESULTS

Below we prove auxiliary estimates (50) and (51) that will be needed in Section 8.4 to finish the proof of Theorem 5. They are presented here in a separate section as the arguments used to prove them are disconnected from the techniques of the steepest descent method employed in Section 8.

Let $\boldsymbol{x}, \boldsymbol{w} \in \mathfrak{R}$ be such that $\boldsymbol{x}$ is not a branch point of $\mathfrak{R}$. There exists a unique, up to multiplicative normalization, rational function on $\mathfrak{R}$, say Ψ, with a simple pole at $\boldsymbol{x}$, a simple zero at $\boldsymbol{w}$, and otherwise non-vanishing and finite. For uniqueness, we normalize $\Psi(z) = z + \{\text{holomorphic part}\}$ around $\boldsymbol{x}$ if $\boldsymbol{x}$ is a point above infinity, and $\Psi(\boldsymbol{z}) = (z - x)^{-1} + \{\text{holomorphic part}\}$ around $\boldsymbol{x}$ otherwise.

Let $\boldsymbol{x}_{\vec{n}}, \boldsymbol{w}_{\vec{n}} \in \mathfrak{R}_{\vec{n}}$ be such that they have the same canonical projections and belong to the sheets with the same labels as $\boldsymbol{x}, \boldsymbol{w}$, respectively, when the latter are not branch points of $\mathfrak{R}$ (points on $\bigcup_{i=1}^p \boldsymbol{\Delta}_i$ need to be identified with the sequences of points convergent to them to set up the correspondence). If $\boldsymbol{w}$ is a branch point, we set $\boldsymbol{w}_{\vec{n}}$ to be the branch point of $\mathfrak{R}_{\vec{n}}$ whose projection converges to or coincides with the one of $\boldsymbol{w}$. We define $\Psi_{\vec{n}}$ to be similarly normalized rational function on $\mathfrak{R}_{\vec{n}}$ with a pole at $\boldsymbol{x}_{\vec{n}}$ and a zero at $\boldsymbol{w}_{\vec{n}}$.

As the statement of Proposition 3, let $\mathfrak{R}_\delta$ be the subsets of $\mathfrak{R}$ obtained by removing circular neighborhoods of radius δ around each branch point. We assume that δ is small enough so that $\boldsymbol{x} \in \mathfrak{R}_\delta$ and $\boldsymbol{w} \in \mathfrak{R}_\delta$ when $\boldsymbol{w}$ is not a branch point. Using natural projections we can redefine $\Psi_{\vec{n}}$ as a function on $\mathfrak{R}_\delta$. Naturally, it will have a pole at $\boldsymbol{x}$ and a zero at $\boldsymbol{w}$ if the latter belong to $\mathfrak{R}_\delta$. Then, regarding $\Psi_{\vec{n}}$ as a function on $\mathfrak{R}_\delta$, we have that

$$\Psi_{\vec{n}} = [1 + o(1)]\Psi \tag{49}$$

uniformly on $\mathfrak{R}_\delta$ as $|\vec{n}| \to \infty$. Indeed, assume first that $\boldsymbol{w} \in \mathfrak{R}_\delta$. Let $\mathfrak{U}_{\boldsymbol{x}} \subset \mathfrak{R}_\delta$ be a circular neighborhood of $\boldsymbol{x}$ such that $\boldsymbol{w} \notin \mathfrak{U}_{\boldsymbol{x}}$. Observe that Ψ is a univalent function on $\mathfrak{R}$. Thus, by applying Koebe's $1/4$ theorem to $1/\Psi$, we see that $|\Psi| < C$ on $\partial\mathfrak{U}_{\boldsymbol{x}}$ for some constant $C > 0$ that depends only on the radius of $\mathfrak{U}_{\boldsymbol{x}}$. Moreover, the maximum modulus principle implies that $|\Psi| < C$ on $\mathfrak{R} \setminus \mathfrak{U}_{\boldsymbol{x}}$. Clearly, absolutely analogous considerations apply to $\Psi_{\vec{n}}$ on $\mathfrak{R}_{\vec{n}}$ and the constant C remains the same. Hence, the ratio $\Psi_{\vec{n}}/\Psi$ is a holomorphic function on $\mathfrak{R}_\delta$ such that $|\Psi_{\vec{n}}/\Psi| < C/\tilde{C}$ by the maximum modulus principle, where $0 < \tilde{C} \leqslant \min_{\mathfrak{R}\setminus\mathfrak{R}_\delta} |\Psi|$ and this constant can be chosen independently of δ. Picking a discrete sequence $\delta_n \to 0$ and using the diagonal argument as well as the normal family argument, we see that any subsequence of $\{\Psi_{\vec{n}}/\Psi\}$ contains a subsequence convergent to a function holomorphic on $\mathfrak{R} \setminus \bigcup_{i=1}^p \{\boldsymbol{a}_{\vec{c},i}, \boldsymbol{b}_{\vec{c},i}\}$. Moreover, this function is necessarily bounded around the branch points and therefore holomorphically extends to the entire Riemann surface $\mathfrak{R}$. Thus, this function must be a constant and the normalization at $\boldsymbol{x}$ yields that this constant is 1. This finishes the proof of (49) in the case $\boldsymbol{w} \in \mathfrak{R}_\delta$. When $\boldsymbol{w}$ is a branch point, the first half of the above considerations yields that $\{\Psi - \Psi_{\vec{n}}\}$ is a family of holomorphic function on $\mathfrak{R}_\delta$ with uniformly and independently of δ bounded moduli. Therefore, the same argument yields that $\Psi_{\vec{n}} = \Psi + o(1)$ uniformly on $\mathfrak{R}_\delta$. As Ψ is non-vanishing in $\mathfrak{R}_\delta$, this estimate implies (49).

Let $\Upsilon_{\vec{n},i}$ (resp. Υ_i), $i \in \{1,\dots,p\}$, be rational functions on $\mathfrak{R}_{\vec{n}}$ (resp. $\mathfrak{R}$) with a simple pole at $\infty^{(i)}$, a simple zero at $\infty^{(0)}$, otherwise non-vanishing and finite, and normalized so $\Upsilon_{\vec{n},i}^{(i)}(z)/z \to 1$ as $z \to \infty$. Then (49) immediately yields

$$\Upsilon_{\vec{n},i} = \big[1 + o(1)\big]\Upsilon_i \tag{50}$$

uniformly on each $\mathfrak{R}_\delta$ as $|\vec{n}| \to \infty$.

Further, let $d\Omega_{\boldsymbol{z},\boldsymbol{w}}^{\vec{n}}$ be the unique abelian differential of the third kind which is holomorphic on $\mathfrak{R}_{\vec{n}} \setminus \{\boldsymbol{z},\boldsymbol{w}\}$, has simple poles at $\boldsymbol{z}$ and $\boldsymbol{w}$ with respective residues $+1$ and -1. It is known that such a differential can be written as $d\Omega_{\boldsymbol{z},\boldsymbol{w}}^{\vec{n}}(\boldsymbol{x}) = \Psi_{\boldsymbol{z},\boldsymbol{w}}^{\vec{n}}(\boldsymbol{x})dx$, where $\Psi_{\boldsymbol{z},\boldsymbol{w}}^{\vec{n}}$ is the unique rational function on $\mathfrak{R}_{\vec{n}}$ with double zero at each $\infty^{(k)}$, $k \in \{0,\dots,p\}$, a simple pole at each $\bigcup_{i=1}^p \{\boldsymbol{a}_{\vec{n},i}, \boldsymbol{b}_{\vec{n},i}\}$, simple poles at $\boldsymbol{z}$ and $\boldsymbol{w}$, otherwise non-vanishing and finite, and normalized to have residue 1 at $\boldsymbol{z}$. Writing $1/\Psi_{\boldsymbol{z},\boldsymbol{w}}^{\vec{n}}$ as a product of terms with one zero and one pole and applying (49) to these factors, we see that

$$\Psi_{\boldsymbol{z},\boldsymbol{w}}^{\vec{n}} = \big[1 + o(1)\big]\Psi_{\boldsymbol{z},\boldsymbol{w}}$$

uniformly on each $\mathfrak{R}_\delta$ as $|\vec{n}| \to \infty$, where $d\Omega_{\boldsymbol{z},\boldsymbol{w}}(\boldsymbol{x}) = \Psi_{\boldsymbol{z},\boldsymbol{w}}(\boldsymbol{x})dx$ is the corresponding differential on $\mathfrak{R}$. Then, defining $\Lambda_{\vec{n}}$ via analogs of (40) and (41) for $\mathfrak{R}_{\vec{n}}$, we get that $\Lambda_{\vec{n}}(\boldsymbol{z}) = \Lambda(\boldsymbol{z}) + o(1)$ uniformly in $\mathfrak{R} \setminus \mathfrak{N}$ for each neighborhood $\mathfrak{N}$ of $\bigcup_{i=1}^p \boldsymbol{\Delta}$. Therefore, if we define $S_{\vec{n}}$ on $\mathfrak{R}_{\vec{n}}$ exactly as S was defined on $\mathfrak{R}$ and consider $S_{\vec{n}}$ as function on $\mathfrak{R} \setminus \mathfrak{N}$, then

$$S_{\vec{n}} = \big[1 + o(1)\big]S \tag{51}$$

uniformly there. Moreover, $S_{\vec{n}}$ obeys all the conclusions of Proposition 4 with respect to $\mathfrak{R}_{\vec{n}}$.

8 NON-LINEAR STEEPEST DESCENT ANALYSIS

In this section we prove Theorem 5 with some technical details relegated to Section 9.

8.1 *Opening of the Lenses*

Since we shall use these sets quite often, put

$$(52)\qquad \begin{cases} E_{\vec c} & := \bigcup_{i=1}^{p}\{a_{\vec c,i}, b_{\vec c,i}\}, \\ E_{\mathsf{in}} & := \bigcup_{i=1}^{p}(\{x_{ij}\}\cap(a_{\vec c,i}, b_{\vec c,i})) \\ E_{\mathsf{out}} & := \bigcup_{i=1}^{p}\{x_{ij}:\ x_{ij}\notin[a_{\vec c,i}, b_{\vec c,i}] \text{ and } \alpha_{ij}\leqslant 0\}. \end{cases}$$

That is, E_{in} consists of the singular points x_{ij} that belong to the support of $\vec\omega$, and E_{out} consists of those singular points outside of the support for which the densities ρ_i are unbounded.

To proceed with the factorization of the jump matrices in RHP-Y(b), we need to construct the so-called "lens" around $\bigcup_{i=1}^{p}[a_i, b_i]$. To this end, given $e \in E_{\mathsf{out}}\cup E_{\mathsf{in}}\cup E_{\vec c}$, let U_e be a disk centered at e. We assume that the radii of these disks are small enough so that $\overline{U}_{e_1}\cap\overline{U}_{e_2}=\varnothing$ for $e_1\neq e_2$. We also assume that $\overline{U}_e\subset D_i^-$ when $e\in E_{\mathsf{out}}$.

Now, let e_0, e_1 be the j-th pair of two consecutive points from $(E_{\mathsf{in}}\cup E_{\vec c})\cap[a_{\vec c,i}, b_{\vec c,i}]$. We choose arcs $\Gamma_{ij}^{\pm}$ incident with e_0 and e_1 and lying in the upper $(+)$ and lower $(-)$ half-planes in the following way: if $e_k\in E_{\vec c}$, then it should hold that

$$(53)\qquad \zeta_{e_k}(\Gamma_{ij}^{\pm}\cap U_{e_k})\subset I_{\pm},$$

where the rays $I_\pm$ are defined in (31) and ζ_{e_k} is a certain conformal function in U_{e_k} constructed further below in (69) or (76) (depending on the considered case); if $e_k\in E_{\mathsf{in}}$, it should hold that

$$(54)\qquad \zeta_{e_k}(\Gamma_{ij+k-1}^{\pm}\cap U_{e_k})\subset I_{\pm} \quad\text{and}\quad \zeta_{e_k}(\Gamma_{ij+k}^{\pm}\cap U_{e_k})\subset J_{\pm},$$

where ζ_{e_k} is a conformal function in U_{e_k} constructed further below in (63) and the rays $J_\pm$ are also defined in (31). Outside $U_{e_0}\cup U_{e_1}$ we choose $\Gamma_{ij}^{\pm}$ to be segments joining the corresponding points on ∂U_{e_0} and ∂U_{e_1}, see Figure 2. We further set $\Gamma_i^{\pm}:=\bigcup_j\Gamma_{ij}^{\pm}$.

Since the geometry of the problem might depend on each particular index $\vec n$ (and not only on $\vec c$), we construct in a similar fashion arcs $\Gamma_{\vec n,ij}^{\pm}$ and $\Gamma_{\vec n,i}^{\pm}$, where this time the maps ζ_{e_k} are replaced by $\zeta_{\vec n,e_k}$, see (64), (70), (77), or (85). As we show later in (65), the arcs $\Gamma_{\vec n,i}^{\pm}$ converge to $\Gamma_i^{\pm}$ in Hausdorff metric. Finally, we denote by $\Omega_{\vec n,ij}^{\pm}$ the domains delimited by $\Gamma_{\vec n,ij}^{\pm}$ and $[a_{\vec n,i}, b_{\vec n,i}]$, and set $\Omega_{\vec n,i}^{\pm}:=\bigcup_j\Omega_{\vec n,ij}^{\pm}$.

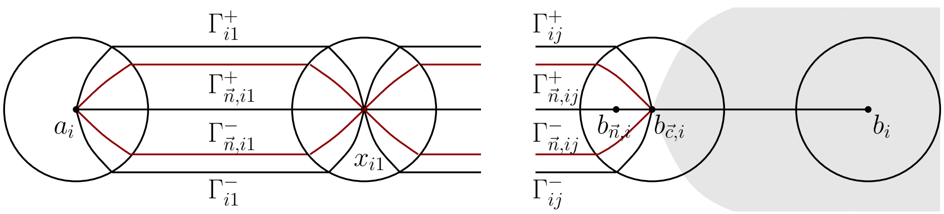

FIGURE 2. The arcs $\Gamma_{ij}^{\pm}$ and $\Gamma_{\vec n,ij}^{\pm}$ in the case where there is at least one point in E_{in}, $b_{\vec n,i}<b_{\vec c,i}<b_i$, and $b_i\in E_{\mathsf{out}}$.

Fix $\Gamma_{\vec n,il}^{\pm}$ with endpoints $e_1<e_2$. There exists an index k such that $x_{ij}\leqslant e_1$ for $j<k$ and $x_{ij}\geqslant e_2$ for $j\geqslant k$. Then it follows from (22) and (23) that the function ρ_i holomorphically extends to $\Omega_{\vec n,il}^{\pm}$ by

$$(55)\qquad \rho_i(z):=\rho_{r,i}(z)\prod_{j<k}\beta_{ij}\prod_{j<k}(z-x_{ij})^{\alpha_{ij}}\prod_{j\geqslant k}(x_{ij}-z)^{\alpha_{ij}},$$

where $(z-x_{ij})^{\alpha_{ij}}$ is holomorphic off $(-\infty, x_{ij}]$ and $(x_{ij}-z)^{\alpha_{ij}}$ is holomorphic off $[x_{ij}, \infty)$. Using these extensions, set

$$\mathbf{X} := \mathbf{Y}\begin{cases} \mathrm{T}_i\begin{pmatrix} 1 & 0 \\ \mp 1/\rho_i & 1 \end{pmatrix} & \text{in} \quad \Omega^{\pm}_{\vec{n},i}, \\ \mathbf{I} & \text{otherwise}, \end{cases} \tag{56}$$

where $\mathbf{Y}$ is a matrix-function that solves RHP-$\mathbf{Y}$ (if it exists). It can be readily verified that $\mathbf{X}$ solves the following Riemann-Hilbert problem (RHP-$\mathbf{X}$):

(a) $\mathbf{X}$ is analytic in $\mathbb{C}\setminus\bigcup_{i=1}^{p}\left([a_i,b_i]\cup\Gamma^{+}_{\vec{n},i}\cup\Gamma^{-}_{\vec{n},i}\right)$ and $\lim_{z\to\infty}\mathbf{X}(z)z^{-\sigma(\vec{n})} = \mathbf{I}$;

(b) $\mathbf{X}$ has continuous traces on $\bigcup_{i=1}^{p}\left((a_i,b_i)\cup\Gamma^{+}_{\vec{n},i}\cup\Gamma^{-}_{\vec{n},i}\right)$ that satisfy

$$\mathbf{X}_+ = \mathbf{X}_-\begin{cases} \mathrm{T}_i\begin{pmatrix} 0 & \rho_i \\ -1/\rho_i & 0 \end{pmatrix} & \text{on} \quad [a_{\vec{c},i}, b_{\vec{c},i}], \\ \mathrm{T}_i\begin{pmatrix} 1 & \rho_i \\ 0 & 1 \end{pmatrix} & \text{on} \quad (a_i,b_i)\setminus[a_{\vec{c},i}, b_{\vec{c},i}], \\ \mathrm{T}_i\begin{pmatrix} 1 & 0 \\ 1/\rho_i & 1 \end{pmatrix} & \text{on} \quad \Gamma^{+}_{\vec{n},i}\cup\Gamma^{-}_{\vec{n},i}; \end{cases}$$

(c) $\mathbf{X}$ has the following behavior near $e\in E_{\vec{c}}\cup E_{\text{in}}\cup E_{\text{out}}$:
- if $e\in E_{\text{out}}$, then $\mathbf{X}$ satisfies RHP-$\mathbf{Y}$(c) with $\mathbf{Y}$ replaced by $\mathbf{X}$;
- if $e\in E_{\vec{c}}\setminus\{x_{ij}\}$, then all the entries of $\mathbf{X}$ are bounded at e;
- if $e\in E_{\text{in}}$ or $e\in E_{\vec{c}}\cap\{x_{ij}\}$, then $\mathbf{X}$ satisfies RHP-$\mathbf{Y}$(c) with $\mathbf{Y}$ replaced by $\mathbf{X}$ outside of $\overline{\Omega^{+}_{\vec{n},i}\cup\Omega^{-}_{\vec{n},i}}$ while inside it behaves exactly as in RHP-$\mathbf{Y}$(c) when $\alpha_{ij}<0$, the entries of the first and $(i+1)$-st column behave like $\mathcal{O}(\psi_0(z-x_{ij}))$ and the rest of the entries are bounded when $\alpha_{ij}=0$, and the entries of the first column behave like $\mathcal{O}(\psi_{-\alpha_{ij}}(z-x_{ij}))$ and the rest of the entries are bounded when $\alpha_{ij}>0$.

Due to the block structure of the jumps in RHP-$\mathbf{Y}$(b), [5, Lemma 17] can be carried over word for word to the present case to prove:

Lemma 8. *RHP-$\mathbf{X}$ is solvable if and only if RHP-$\mathbf{Y}$ is solvable. When solutions of RHP-$\mathbf{X}$ and RHP-$\mathbf{Y}$ exist, they are unique and connected by (56).*

8.2 *Auxiliary Parametrices*

To solve RHP-$\mathbf{X}$, we construct parametrices that asymptotically describe the behavior of $\mathbf{X}$ away from and around each point in $E_{\text{in}}\cup E_{\text{out}}\cup E_{\vec{c}}$. To this end, we construct a matrix-valued function $\mathbf{N}$ that solves the following Riemann-Hilbert problem (RHP-$\mathbf{N}$):

(a) $\mathbf{N}$ is analytic in $\mathbb{C}\setminus\bigcup_{i=1}^{p}[a_{\vec{n},i}, b_{\vec{n},i}]$ and $\lim_{z\to\infty}\mathbf{N}(z)z^{-\sigma(\vec{n})} = \mathbf{I}$;

(b) $\mathbf{N}$ has continuous traces on $(a_{\vec{n},i}, b_{\vec{n},i})$ that satisfy $\mathbf{N}_+ = \mathbf{N}_-\mathrm{T}_i\begin{pmatrix} 0 & \rho_i \\ -1/\rho_i & 0 \end{pmatrix}$.

Let $\Phi_{\vec{n}}$ be the functions from Proposition 1 while $S_{\vec{n}}$ and $\Upsilon_{\vec{n},i}$, $i\in\{1,\dots,p\}$, be the functions introduced in Section 7. Set

$$\mathbf{N} := \mathbf{CMD}, \tag{57}$$

where $\mathbf{D} := \operatorname{diag}\left(\Phi^{(0)}_{\vec{n}},\dots,\Phi^{(p)}_{\vec{n}}\right)$, $\mathbf{C} := \operatorname{diag}\left(C_{\vec{n},0},\dots,C_{\vec{n},p}\right)$ with the constant $C_{\vec{n},k}$ defined by

$$\begin{cases} \lim_{z\to\infty} C_{\vec{n},0}\left(S_{\vec{n}}\Phi_{\vec{n}}\right)^{(0)}(z)z^{-|\vec{n}|} & = & 1 \\ \lim_{z\to\infty} C_{\vec{n},i}\left(S_{\vec{n}}\Phi_{\vec{n}}\right)^{(i)}(z)z^{n_i} & = & 1, \quad i\in\{1,\dots,p\}, \end{cases} \tag{58}$$

and the matrix $\mathbf{M}$ is given by

$$\mathbf{M} := \begin{pmatrix} S_{\vec{n}}^{(0)} & S_{\vec{n}}^{(1)}/w_{\vec{n},1} & \cdots & S_{\vec{n}}^{(p)}/w_{\vec{n},p} \\ (S_{\vec{n}}\Upsilon_{\vec{n},1})^{(0)} & (S_{\vec{n}}\Upsilon_{\vec{n},1})^{(1)}/w_{\vec{n},1} & \cdots & (S_{\vec{n}}\Upsilon_{\vec{n},1})^{(p)}/w_{\vec{n},p} \\ \vdots & \vdots & \ddots & \vdots \\ (S_{\vec{n}}\Upsilon_{\vec{n},p})^{(0)} & (S_{\vec{n}}\Upsilon_{\vec{n},p})^{(1)}/w_{\vec{n},1} & \cdots & (S_{\vec{n}}\Upsilon_{\vec{n},p})^{(p)}/w_{\vec{n},p} \end{pmatrix}. \tag{59}$$

Then (57) solves RHP-N. Indeed, RHP-N(a) follows immediately from the analyticity properties of $S_{\vec{n}}$, $\Upsilon_{\vec{n},i}$, and $\Phi_{\vec{n}}$ as well as from (58). Observe that the multiplication by

$$T_i \begin{pmatrix} 0 & \rho_i \\ -1/\rho_i & 0 \end{pmatrix}$$

on the right replaces the first column by the $(i+1)$-st one multiplied by ρ_i, while $(i+1)$-st column is replaced by the first one multiplied by $-1/\rho_i$. Hence, RHP-N(b) follows from the analog of (25) for $S_{\vec{n}}$ and the fact that any rational function Ψ on $\mathfrak{R}_{\vec{n}}$ satisfies $\Psi_{\pm}^{(0)} = \Psi_{\mp}^{(i)}$ on $(a_{\vec{n},i}, b_{\vec{n},i})$.

Since the jump matrices in RHP-N(b) have determinant 1, $\det(\mathbf{N})$ is a holomorphic function in $\overline{\mathbb{C}} \setminus \bigcup_i \{a_{\vec{n},i}, b_{\vec{n},i}\}$ and $\det(\mathbf{N})(\infty) = 1$. Moreover, it follows from the analogs of (26) and (27) for $S_{\vec{n}}$ that each entry of the first column of $\mathbf{N}$ behaves like

$$\mathcal{O}\left(|z-e|^{-(2\alpha+1)/4}\right) \quad \text{and} \quad \mathcal{O}\left(|z-x_{ij}|^{-(\alpha_{ij}\mp\arg(\beta_{ij})/\pi)/2}\right)$$

for $e \in \{a_{\vec{n},i}, b_{\vec{n},i}\}$ ($\alpha = \alpha_{ij}$ if $e = x_{ij}$ and $\alpha = 0$ otherwise) and for $x_{ij} \in (a_{\vec{n},i}, b_{\vec{n},i})$ ($\pm\mathrm{Im}(z) > 0$), respectively, the entries of the $(i+1)$-st column behave like

$$\mathcal{O}\left(|z-e|^{(2\alpha-1)/4}\right) \quad \text{and} \quad \mathcal{O}\left(|z-x_{ij}|^{(\alpha_{ij}\mp\arg(\beta_{ij})/\pi)/2}\right)$$

there, and the rest of the entries are bounded. Thus, the determinant has at most square root singularities at these points and therefore is a bounded entire function. That is, $\det(\mathbf{N}) \equiv 1$ as follows from the normalization at infinity.

Further, for each $e \in E_{\mathrm{in}} \cup E_{\mathrm{out}} \cup E_{\vec{c}}$, we want to solve RHP-X locally in U_e. That is, we are seeking a solution of the following RHP-$\mathbf{P}_e$:

(a,b,c) $\mathbf{P}_e$ satisfies RHP-X(a,b,c) within U_e;

(d) $\mathbf{P}_e = \mathbf{M}\left(\mathbf{I} + \mathcal{O}(\varepsilon_{e,\vec{n}})\right)\mathbf{D}$ uniformly on $\partial U_e \setminus \left([a_i, b_i] \cup \bigcup_{i=1}^p \Gamma_{\vec{n},i}^+ \cup \Gamma_{\vec{n},i}^-\right)$, where $0 < \varepsilon_{e,\vec{n}} \to 0$ as $|\vec{n}| \to \infty$.

Since the construction of $\mathbf{P}_e$ solving RHP-$\mathbf{P}_e$ is rather lengthy, it is carried out separately in Section 9 further below.

8.3 *Final R-H Problem*

Denote by $\Omega_{\vec{n},ij}$ the domain delimited by $\Gamma_{\vec{n},ij}^+$ and $\Gamma_{\vec{n},ij}^-$ (in particular, $\Omega_{\vec{n},ij}^{\pm} \subset \Omega_{\vec{n},ij}$). Set $\Omega_{\vec{n}} := \bigcup_{ij} \Omega_{\vec{n},ij}$ and $U := \bigcup_{e \in E_{\mathrm{in}} \cup E_{\mathrm{out}} \cup E_{\vec{c}}} U_e$. Define

$$\Sigma_{\vec{n}} := \partial U \cup \left[\bigcup_{i=1}^p \left(\Gamma_{\vec{n},i}^+ \cup \Gamma_{\vec{n},i}^-\right) \setminus U\right] \cup \left[\bigcup_{i=1}^p [a_i, b_i] \setminus (U \cup \Omega_{\vec{n}})\right].$$

Moreover, we define Σ by replacing $\Gamma_{\vec{n},i}^{\pm}$ with $\Gamma_i^{\pm}$ in the definition of $\Sigma_{\vec{n}}$ see Figure 3. Given matrices $\mathbf{N}$ and $\mathbf{P}_e$, $e \in E_{\mathrm{in}} \cup E_{\mathrm{out}} \cup E_{\vec{c}}$, from the previous section, consider the following Riemann-Hilbert Problem (RHP-$\mathbf{Z}$):

(a) $\mathbf{Z}$ is a holomorphic matrix function in $\overline{\mathbb{C}} \setminus \Sigma_{\vec{n}}$ and $\mathbf{Z}(\infty) = \mathbf{I}$;

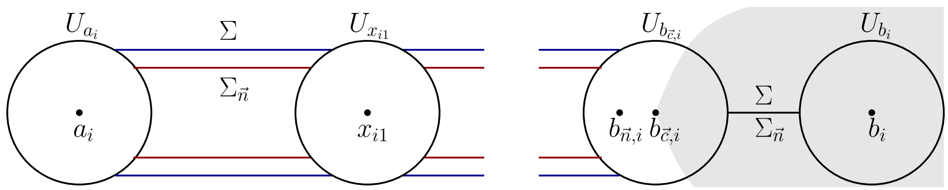

FIGURE 3. Contours Σ (black and blue lines) and $\Sigma_{\vec n}$ (black and red lines).

(b) $\mathbf{Z}$ has continuous traces on $\Sigma_{\vec n}$ that satisfy

$$\mathbf{Z}_+ = \mathbf{Z}_- \begin{cases} \mathbf{MD}\mathrm{T}_i \begin{pmatrix} 1 & 0 \\ 1/\rho_i & 1 \end{pmatrix} (\mathbf{MD})^{-1} & \text{on} \quad \left(\Gamma^+_{\vec n,i} \cup \Gamma^-_{\vec n,i}\right) \setminus \overline{U}, \\ \mathbf{MD}\mathrm{T}_i \begin{pmatrix} 1 & \rho_i \\ 0 & 1 \end{pmatrix} (\mathbf{MD})^{-1} & \text{on} \quad [a_i, b_i] \setminus \left(\overline{U \cup \Omega_{\vec n}}\right) \\ \mathbf{P}_e (\mathbf{MD})^{-1} & \text{on} \quad \partial U_e. \end{cases}$$

Then the following lemma takes place.

Lemma 9. *The solution of RHP-Z exists for all $|\vec n|$ large enough and satisfies*

$$\mathbf{Z} = \mathbf{I} + \mathcal{O}(\varepsilon_{\vec n}) \tag{60}$$

uniformly in $\overline{\mathbb{C}}$, where $\varepsilon_{\vec n} = \min_e \varepsilon_{e,\vec n}$.

Proof. Analyticity of ρ_i yields that $\mathbf{Z}$ can be analytically continued to be holomorphic outside of Σ. To do that one simply needs to multiply $\mathbf{Z}$ by the first jump matrix in RHP-Z(b) or its inverse (the jump matrices have determinate 1 and therefore are invertible). We shall show that the jump matrices are locally uniformly geometrically small in D_i^+. This would imply that the new problem is solvable if and only if the initial problem is solvable and the bound (60) remains valid regardless the contour. Hence, in what follows we shall consider RHP-$\mathbf{Z}$ on Σ rather than on $\Sigma_{\vec n}$.

It was shown in Section 8.2 that $\det(\mathbf{N}) \equiv 1$. Moreover, it follows from (18) that $\det(\mathbf{D}) \equiv 1$ while the equality $\prod_{k=0}^p S_{\vec n}^{(k)} \equiv 1$ and (58) imply that $\det(\mathbf{C}) \equiv 1$. Hence, $\det(\mathbf{M}) \equiv 1$ and it follows from RHP-$\mathbf{P}_e$(d), (51), and (50) that

$$\mathbf{P}_e(\mathbf{MD})^{-1} = \mathbf{I} + \mathbf{M}\mathcal{O}(\varepsilon_{e,\vec n})\mathbf{M}^{-1} = \mathbf{I} + \mathcal{O}(\varepsilon_{e,\vec n})$$

holds uniformly on each ∂U_e. On the other hand, it holds on $\Gamma_i^\pm \setminus \overline{U}$ that

$$\mathbf{MD}\mathrm{T}_i \begin{pmatrix} 1 & 0 \\ 1/\rho_i & 1 \end{pmatrix} (\mathbf{MD})^{-1} = \mathbf{I} + \frac{1}{\rho_i}\frac{\Phi_{\vec n}^{(i)}}{\Phi_{\vec n}^{(0)}}\mathbf{M}\mathbf{E}_{i+1,1}\mathbf{M}^{-1} = \mathbf{I} + \mathcal{O}\left(C_i^{-|\vec n|}\right)$$

for some constant $C_i > 1$ by (15), (19), and Proposition 1. Analogously, we get that

$$\mathbf{MD}\mathrm{T}_i \begin{pmatrix} 1 & \rho_i \\ 0 & 1 \end{pmatrix} (\mathbf{MD})^{-1} = \mathbf{I} + \rho_i \frac{\Phi_{\vec n}^{(0)}}{\Phi_{\vec n}^{(i)}}\mathbf{M}\mathbf{E}_{1,i+1}\mathbf{M}^{-1} = \mathbf{I} + \mathcal{O}\left(\widetilde C_i^{-|\vec n|}\right)$$

on $[a_i, b_i] \setminus \left(\overline{U \cup \Omega_{\vec n}}\right)$ for some $\widetilde C_i > 1$ by (19) and (14). That is, all the jump matrices for $\mathbf{Z}$ asymptotically behave like $\mathbf{I} + \mathcal{O}(\varepsilon_{\vec n})$ (as will be clear in Section 9, the decay of $\varepsilon_{\vec n}$ is of power type and not exponential). The conclusion of the lemma follows from the same argument as in [7, Corollary 7.108]. □

8.4 *Proof of Theorem 5*

Let $\mathbf{Z}$ be the solution of RHP-Z granted by Lemma 9, $\mathbf{P}_e$ be solutions of RHP-$\mathbf{P}_e$, and $\mathbf{N} = \mathbf{CMD}$ be the matrix constructed in (57). Then it can be easily checked that

$$\mathbf{X} = \mathbf{CZ} \begin{cases} \mathbf{MD} & \text{in} \quad \mathbb{C} \setminus \left(\overline{U} \cup \bigcup [a_{\vec c,i} b_{\vec c,i}]\right), \\ \mathbf{P}_e & \text{in} \quad U_e, \quad e \in E_{\mathrm{out}} \cup E_{\mathrm{in}} \cup E_{\vec c}, \end{cases} \tag{61}$$

solves RHP-X for all $|\vec n|$ large enough. Given a closed set K in $\overline{\mathbb{C}}\setminus\bigcup_{i=1}^p[a_i,b_i]$, we can always shrink the lens so that $K\subset\mathbb{C}\setminus\left(\overline{U\cup\Omega_{\vec n}}\right)$. In this case $\mathbf{Y}=\mathbf{X}$ on K by Lemma 8. Write the first row of $\mathbf{Z}$ as $\left(1+\upsilon_{\vec n,0},\upsilon_{\vec n,1},\dots,\upsilon_{\vec n,p}\right)$. Then $(1,j+1)$-st entry of $\mathbf{ZM}$ is equal to

$$\left(1+\upsilon_{\vec n,0}+\sum_{i=1}^p\upsilon_{\vec n,i}\Upsilon_{\vec n,i}^{(j)}\right)S_{\vec n}^{(j)}/w_{\vec n,j}=\left(1+\mathcal{O}(\varepsilon_{\vec n})\right)S_{\vec n}^{(j)}/w_{\vec n,j}$$

by Lemma 9 and (50), where $w_{\vec n,0}\equiv 1$. Therefore, it follows from Proposition 6 that

$$\begin{cases} Q_{\vec n} & = & C_{\vec n,0}\left[1+\mathcal{O}(\varepsilon_{\vec n})\right]\left(S_{\vec n}\Phi_{\vec n}\right)^{(0)} \\ R_{\vec n}^{(j)} & = & C_{\vec n,0}\left[1+\mathcal{O}(\varepsilon_{\vec n})\right]\left(S_{\vec n}\Phi_{\vec n}\right)^{(j)}/w_{\vec n,j}. \end{cases}$$

Theorem 5 now follows from (51), since $C_{\vec n,0}=(1+o(1))C_{\vec n}$ again by (51) and $w_{\vec n,j}\to w_j$ uniformly on K.

9 LOCAL RIEMANN-HILBERT ANALYSIS

The goal of this section is to construct solutions to RHP-$\mathbf{P}_e$.

9.1 *Local Parametrices around Points in* $E_{\rm out}$

Let $e\in E_{\rm out}$, see (52). A solution of RHP-$\mathbf{P}_e$ is given by

$$\mathbf{P}_e:=\mathbf{M}\mathsf{T}_i\begin{pmatrix}1 & \mathcal{C}_i\Phi_{\vec n}^{(0)}/\Phi_{\vec n}^{(i)}\\ 0 & 1\end{pmatrix}\mathbf{D}, \tag{62}$$

where $\mathcal{C}_i(z):=\frac{1}{2\pi i}\int_{[a_i,b_i]}\frac{\rho_i(x)}{x-z}dx$. Indeed, since the matrices $\mathbf{M}$ and $\mathbf{D}$ are holomorphic in U_e, and $\mathcal{C}_i$ has a jump only across $(a_i,b_i)\cap U_e$, the matrix above satisfies RHP-$\mathbf{P}_e$(a). As $\left(\mathcal{C}_i^+-\mathcal{C}_i^-\right)(x)=\rho_i(x)$ for $x\in(a_i,b_i)\setminus\{x_{ij}\}$, RHP-$\mathbf{P}_e$(b) follows. RHP-$\mathbf{P}_e$(c) is a consequence of the fact that $\left|\mathcal{C}_i(z)(z-x_{ij})^{-\alpha_{ij}}\right|$ is bounded in the vicinity of x_{ij} for $\alpha_{ij}<0$, [13, Sec. 8.3]. Finally, RHP-$\mathbf{P}_e$(d) is easily deduced from the inclusion $\overline{U}_e\subset D_i^-$, (19) and (14).

9.2 *Local Parametrices around Points in* $E_{\rm in}$

The construction below is known [30, 21, 22, 8].

9.2.1 *Conformal Maps*

Since h is a rational function on $\mathfrak{R}$, it holds that $h_\pm^{(0)}=h_\mp^{(i)}$ on $\left(a_{\vec c,i},b_{\vec c,i}\right)\cap U_e$. Then

$$\zeta_e(z):=\operatorname{sgn}\left(\operatorname{Im}(z)\right)i\int_e^z\left(h^{(0)}-h^{(i)}\right)(x)dx,\quad \operatorname{Im}(z)\neq 0, \tag{63}$$

extends to a conformal function in U_e vanishing at e. Define $\zeta_{\vec n,e}$ exactly as in (63) with h replaced by $h_{\vec n}$. Then it holds that

$$\zeta_{\vec n,e}(z)=\frac{\operatorname{sgn}\left(\operatorname{Im}(z)\right)i}{|\vec n|}\log\left(\Phi_{\vec n}^{(0)}(z)/\Phi_{\vec n}^{(i)}(z)\right),\quad \operatorname{Im}(z)\neq 0, \tag{64}$$

by (21). It follows from (19) and (14) that $\zeta_{\vec n,e}$ is real on $\left(a_{\vec c,i},b_{\vec c,i}\right)\cap U_e$. Moreover, since $U_e\setminus\left(a_{\vec c,i},b_{\vec c,i}\right)\subset D_i^+$, $\zeta_{\vec n,e}$ maps upper half-plane into the upper half-plane. In particular, $\zeta_{\vec n,e}(x)>0$ for $x\in\left(e,b_{\vec c,i}\right)\cap U_e$. Observe also that

$$\zeta_{\vec n,e}\to\zeta_e \tag{65}$$

holds uniformly on $\overline{U}_e$ by (19) since (19) is the statement about convergence of the imaginary parts of $\zeta_{\vec n,e}$ to the imaginary part of ζ_e.

9.2.2 *Matrix* $\mathbf{P}_e$

It follows from the way we extended ρ_i into $\Omega^{\pm}_{\vec n,i}$ that we can write

$$\rho_i(z) = \rho_{r,e}(z)\begin{cases} (e-z)^\alpha, & \mathrm{Re}(z) < e, \\ \beta(z-e)^\alpha, & \mathrm{Re}(z) > e, \end{cases} \tag{66}$$

where $\rho_{r,e}(x)$ is a holomorphic and non-vanishing function in U_e. Define r_e by

$$r_e(z) := \sqrt{\rho_{r,e}(z)}(z-e)^{\alpha/2},$$

where the square root is principal. Then r_e is a holomorphic and non-vanishing function in $U_e \setminus \{x : x < e\}$ that satisfies

$$\begin{cases} r_{e+}(x) r_{e-}(x) = \rho_i(x), & x \in \{x : x < e\} \cap U_e, \\ r_e^2(z) = \rho_i(z) e^{\pm \pi i \alpha}, & z \in \Gamma^{\pm}_{\vec n,ij} \cap U_e, \\ r_e^2(x) = \beta^{-1}\rho_i(x), & \left(\Gamma^{+}_{\vec n,ij+1} \cup \Gamma^{-}_{\vec n,ij+1} \cup \{x : x > e\}\right) \cap U_e. \end{cases} \tag{67}$$

It is a straightforward computation using (67) and (64) to verify that RHP-$\mathbf{P}_e$ is solved by

$$\mathbf{P}_e := \mathbf{E}_e \mathsf{T}_i \left(\mathbf{\Phi}_{\alpha,\beta}\left(|\vec n|\zeta_{\vec n,e}\right) r_e^{-\sigma_3} \left(\Phi^{(0)}_{\vec n} / \Phi^{(i)}_{\vec n}\right)^{-\sigma_3/2} \right) \mathbf{D},$$

where $\mathbf{\Phi}_{\alpha,\beta}$ is the solution of RHP-$\mathbf{\Phi}_{\alpha,\beta}$ and the holomorphic prefactor $\mathbf{E}_e$ chosen below to fulfill RHP-$\mathbf{P}_e$(d).

9.2.3 *Holomorphic Prefactor* $\mathbf{E}_e$

It follows from the properties of the branch of $(i\zeta)^{\log\beta\,\sigma_3/2\pi i}$ that

$$(i\zeta)_+^{\log\beta\,\sigma_3/2\pi i}\mathbf{B}_+ = (i\zeta)_-^{\log\beta\,\sigma_3/2\pi i}\mathbf{B}_- \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{on} \quad (-\infty, 0), \\ \begin{pmatrix} 0 & \beta \\ -1/\beta & 0 \end{pmatrix} & \text{on} \quad (0, \infty), \end{cases} \tag{68}$$

and it is holomorphic in $\mathbb{C} \setminus (-\infty, \infty)$. Therefore, it follows from RHP-$\mathbf{N}$(b) that

$$\mathbf{E}_e := \mathbf{M}\mathsf{T}_i \left((i|\vec n|\zeta_{\vec n,e})^{\log\beta\,\sigma_3/2\pi i} \mathbf{B}_\pm r_e^{-\sigma_3} \right)^{-1}, \quad \pm\mathrm{Im}(z) > 0,$$

is holomorphic in $U_e \setminus \{e\}$. Since $|r_e(z)| \sim |z-e|^{\alpha/2}$ and $\left|\zeta^{\log\beta/2\pi i}\right| \sim |\zeta|^{\arg(\beta)/2\pi}$, $\mathbf{E}_e$ is in fact holomorphic in U_e as claimed. Clearly, in this case it holds that $\varepsilon_{\vec n,e} = |\vec n|^{|\arg(\beta)|/\pi - 1}$.

9.3 *Hard Edge*

In this section we assume that $e \in E_{\vec c}$ and $e \notin \partial D_i^-$.

9.3.1 *Conformal Maps*

It follows from Proposition 3 that $b_{\vec c,i} = b_{\vec n,i} = b_i$ or $a_{\vec c,i} = a_{\vec n,i} = a_i$ for all $|\vec n|$ large in this case. Define

$$\zeta_e(z) := \left(\frac{1}{4} \int_e^z \left(h^{(0)} - h^{(i)}\right)(x)dx \right)^2, \quad z \in U_e. \tag{69}$$

Since $h^{(0)}_\pm = h^{(i)}_\mp$ on $(a_i, b_i) \cap U_e$, ζ_e is holomorphic in U_e. Moreover, since h has a pole at $\boldsymbol{e}$ (the corresponding branch point of $\mathfrak{R}$), ζ_e has a simple zero at e. Thus, we can choose U_e small enough so that ζ_e is conformal in $\overline{U}_e$.

Define $\zeta_{\vec n,e}$ as in (69) with h replaced by $h_{\vec n}$. The functions $\zeta_{\vec n,e}$ form a family of holomorphic functions in U_e, all having a simple zero at e. Moreover, (21) yields that

$$\zeta_{\vec n,e}(z) = \left(\frac{1}{4|\vec n|} \log\left(\Phi^{(0)}_{\vec n} / \Phi^{(i)}_{\vec n}\right) \right)^2, \quad z \in U_e, \tag{70}$$

which, together with (15) and (19), implies that $\zeta_{\vec{n},e}(x)$ is positive for $x \in \big(\mathbb{R}\setminus[a_i,b_i]\big)\cap U_e$ and is negative $x\in(a_i,b_i)\cap U_e$ (this also can be seen from (37) and (38)).

Considering $h_{\vec{n}}$ and h as defined on the same doubly circular neighborhood of e and recalling that their ratio converges to 1 on its boundary, we see that it converges to 1 uniformly throughout the neighborhood. The latter implies that (65) holds uniformly on $\overline{U}_e$. In particular, the functions $\zeta_{\vec{n},e}$ are conformal in $\overline{U}_e$ for all $\vec{n}$ large.

9.3.2 *Matrix* $\mathbf{P}_e$

In this case we can write

$$\rho_i(z)=\rho_{r,e}(z)\begin{cases}(e-z)^\alpha, & e=b_i,\\ (z-e)^\alpha, & e=a_i,\end{cases} \tag{71}$$

where $\rho_{r,e}$ is non-vanishing and holomorphic in U_e, $\alpha>-1$, and the α-roots are principal. Set

$$r_e(z):=\sqrt{\rho_{r,e}(z)}\begin{cases}(z-e)^{\alpha/2}, & e=b_i,\\ (e-z)^{\alpha/2}, & e=a_i,\end{cases} \tag{72}$$

where the branches are again principal. Then r_e is a holomorphic and non-vanishing function in $U_e\setminus[a_i,b_i]$ and satisfies

$$\begin{cases} r_{e+}(x)r_{e-}(x)=\rho_i(x), & x\in(a_i,b_i),\\ r_e^2(z)=\rho_i(z)e^{\pm\pi i\alpha}, & z\in\Gamma^{\pm}_{\vec{n},i}\cap U_e.\end{cases} \tag{73}$$

Then (70) and (73) imply that RHP-$\mathbf{P}_e$ is solved by

$$\mathbf{P}_e:=\mathbf{E}_e\mathbf{T}_i\left(\mathbf{\Psi}_e\left(|\vec{n}|^2\zeta_{\vec{n},e}\right)r_e^{-\sigma_3}\left(\Phi_{\vec{n}}^{(0)}/\Phi_{\vec{n}}^{(i)}\right)^{-\sigma_3/2}\right)\mathbf{D}, \tag{74}$$

where $\mathbf{\Psi}_e:=\mathbf{\Psi}_\alpha$ when $e=b_i$ and $\mathbf{\Psi}_e:=\sigma_3\mathbf{\Psi}_\alpha\sigma_3$ when $e=a_i$, and $\mathbf{\Psi}_\alpha$ solves RHP-$\mathbf{\Psi}_\alpha$, while $\mathbf{E}_e$ is a holomorphic prefactor chosen so that RHP-$\mathbf{P}_e$(d) is fulfilled.

9.3.3 *Holomorphic Prefactor* $\mathbf{E}_e$

As $\zeta_+^{1/4}=i\zeta_-^{1/4}$, it can be easily checked that

$$\frac{\zeta_+^{-\sigma_3/4}}{\sqrt{2}}\begin{pmatrix}1 & \pm i\\ \pm i & 1\end{pmatrix}=\frac{\zeta_-^{-\sigma_3/4}}{\sqrt{2}}\begin{pmatrix}1 & \pm i\\ \pm i & 1\end{pmatrix}\begin{pmatrix}0 & \pm 1\\ \mp 1 & 0\end{pmatrix}$$

on $(-\infty,0)$. Then RHP-$\mathbf{N}$(b) implies that

$$\mathbf{E}_e:=\mathbf{M}\mathbf{T}_i\left(\frac{\left(|\vec{n}|^2\zeta_{\vec{n},e}\right)^{-\sigma_3/4}}{\sqrt{2}}\begin{pmatrix}1 & \pm i\\ \pm i & 1\end{pmatrix}r_e^{-\sigma_3}\right)^{-1} \tag{75}$$

is holomorphic around in $U_e\setminus\{e\}$, where the sign $+$ is used around $e=b_i$ while the sign $-$ is used around $e=a_i$. Since $|r_e(z)|\sim|z-e|^{\alpha/2}$, $\mathbf{E}_e$ is in fact holomorphic in U_e as desired. Clearly, $\varepsilon_{\vec{n},e}=|\vec{n}|^{-1}$ in this case.

9.4 *Soft-Type Edge I*

Below, we assume that $e\in E_{\vec{c}}$ and $b_{\vec{n},i}\in\partial D^-_{\vec{n},i}$ or $a_{\vec{n},i}\in\partial D^-_{\vec{n},i}$.

9.4.1 *Conformal Maps*

By the condition of this section, it holds that $e\in\partial D_i^-$. Define

$$\zeta_e(z):=\left(-\frac{3}{4}\int_e^z\left(h^{(0)}-h^{(i)}\right)(x)dx\right)^{2/3},\quad z\in U_e. \tag{76}$$

Further, define $\zeta_{\vec{n},e}$ exactly as ζ_e only with h replaced by $h_{\vec{n}}$ and e replaced by $b_{\vec{n},i}$ if $e=b_{\vec{c},i}$ and by $a_{\vec{n},i}$ if $e=a_{\vec{c},i}$. It follows from (21) that

$$\zeta_{\vec{n},e}(z)=\left(-\frac{3}{4|\vec{n}|}\log\left(\Phi_{\vec{n}}^{(0)}(z)/\Phi_{\vec{n}}^{(i)}(z)\right)\right)^{2/3},\quad z\in U_e. \tag{77}$$

Analysis in (37) and (39) yields that these functions are conformal in $\overline{U}_e$ (make the radius smaller if necessary), are positive on $(\mathbb{R}\setminus[a_{\vec{n},i},b_{\vec{n},i}])\cap U_e$ and negative on $(a_{\vec{n},i},b_{\vec{n},i})\cap U_e$. Moreover, (65) holds as well.

9.4.2 *Matrix* $\mathbf{P}_e$

If $e=x_{ij}$ for some $j\in\{1,\dots,J_i-1\}$, set $\alpha:=\alpha_{ij}$ and $\beta:=\beta_{ij}$ when $e=b_{\vec{c},i}$ or $\beta:=1/\beta_{ij}$ when $e=a_{\vec{c},i}$, see (23); if $e\notin\{x_{ij}\}_{j=1}^{J_i-1}$ and $e\in(a_i,b_i)$, set $\alpha=0$ and $\beta=1$; if $e=a_i$, set $\alpha=\alpha_{i0}$ and $\beta=0$; if $e=b_i$, set $\alpha=\alpha_{iJ_i}$ and $\beta=0$. It follows from the way we extended ρ_i into $\Omega_{\vec{n},i}^{\pm}$ that

$$\rho_i(z)=\rho_{r,e}(z)\begin{cases}(e-z)^\alpha, & e=b_{\vec{c},i},\\ (z-e)^\alpha, & e=a_{\vec{c},i},\end{cases} \tag{78}$$

for $\mathsf{Re}(z)\in(a_{\vec{c},i},b_{\vec{c},i})$ and

$$\rho_i(z)=\beta\rho_{r,e}(z)\begin{cases}(z-e)^\alpha, & e=b_{\vec{c},i},\\ (e-z)^\alpha, & e=a_{\vec{c},i},\end{cases} \tag{79}$$

for $\mathsf{Re}(z)\notin[a_{\vec{c},i},b_{\vec{c},i}]$, where all the branches are principal. Define r_e by (72) with b_i and a_i replaced by $b_{\vec{c},i}$ and $a_{\vec{c},i}$. Then r_e is a holomorphic and non-vanishing function in $U_e\setminus[a_{\vec{c},i},b_{\vec{c},i}]$ that satisfies

$$\begin{cases} r_{e+}(x)r_{e-}(x)=\rho_i(x), & x\in(a_{\vec{c},i},b_{\vec{c},i})\cap U_e,\\ r_e^2(z)=\rho_i(z)e^{\pm\pi i\alpha}, & z\in\Gamma_{\vec{n},i}^{\pm}\cap U_e,\\ r_e^2(x)=\beta^{-1}\rho_i(x), & (\mathbb{R}\setminus(a_{\vec{c},i},b_{\vec{c},i}))\cap U_e.\end{cases} \tag{80}$$

Then one can check using (80) and (77) that RHP-$\mathbf{P}_e$ is solved by

$$\mathbf{P}_e:=\mathbf{E}_e\mathsf{T}_i\left(\mathbf{\Psi}_e\left(|\vec{n}|^{2/3}(\zeta_{\vec{n},e}-\zeta_{\vec{n},e}(e))\right)r_e^{-\sigma_3}\left(\Phi_{\vec{n}}^{(0)}/\Phi_{\vec{n}}^{(i)}\right)^{-\sigma_3/2}\right)\mathbf{D}, \tag{81}$$

where $\mathbf{\Psi}_e:=\mathbf{\Psi}_{\alpha,\beta}(\cdot;s_{\vec{n}})$ when $e=b_{\vec{c},i}$ and $\mathbf{\Psi}_e:=\sigma_3\mathbf{\Psi}_{\alpha,\beta}(\cdot;s_{\vec{n}})\sigma_3$ when $e=a_{\vec{c},i}$, $\mathbf{\Psi}_{\alpha,\beta}(\cdot;s)$ solves RHP-$\mathbf{\Psi}_{\alpha,\beta}$,

$$s_{\vec{n}}:=|\vec{n}|^{2/3}\zeta_{\vec{n},e}(e),$$

and $\mathbf{E}_e$ is a holomorphic prefactor chosen so RHP-$\mathbf{P}_e$(d) is satisfied.

9.4.3 *Holomorphic Prefactor* $\mathbf{E}_e$

If $s_{\vec{n}}=0$, then $\mathbf{E}_e$ is given by (75) with $|\vec{n}|^2$ replaced by $|\vec{n}|^{2/3}$. In this case we have by Theorem 7 that $\varepsilon_{\vec{n},e}=|\vec{n}|^{-1/3}$.

If $s_{\vec{n}}>0$, then (75) is no longer applicable as the matrix $\boldsymbol{M}$ has the jump only across $(a_{\vec{n},i},b_{\vec{n},i})$ while $r_e^{-\sigma_3}$ is discontinuous across $(a_{\vec{c},i},b_{\vec{c},i})\cap U_e$ where $b_{\vec{n},i}<b_{\vec{c},i}$ or $a_{\vec{n},i}>a_{\vec{c},i}$. Observe that

$$r_{e+}(x)=r_{e-}(x)e^{\alpha\pi i},\quad x\in\left((a_{\vec{c},i},b_{\vec{c},i})\setminus(a_{\vec{n},i},b_{\vec{n},i})\right)\cap U_e.$$

Therefore, define

$$G_\alpha(\zeta):=\exp\left\{-\pi i\alpha\sqrt{\zeta}\frac{1}{2\pi i}\int_0^1\frac{1}{\sqrt{x}}\frac{dx}{x-\zeta}\right\},\quad \zeta\in\mathbb{C}\setminus(-\infty,1].$$

It is quite easy to see that

$$\begin{cases} G_{\alpha+}G_{\alpha-}\equiv 1 & \text{on}\quad (-\infty,0),\\ G_{\alpha-}=G_{\alpha+}\pi i\alpha & \text{on}\quad (0,1).\end{cases}$$

Moreover, from the theory of singular integrals [13, Sec. 8.3] we know that G_α is bounded around the origin and behaves like $|\zeta-1|^{-\alpha/2}$ around 1. Then it can be checked using the above properties that the matrix function

$$\mathbf{E}_e := \mathbf{M}\mathsf{T}_i\left(\frac{\left(|\vec{n}|^{2/3}\zeta_{\vec{n},e}\right)^{-\sigma_3/4}}{\sqrt{2}}\begin{pmatrix}1 & \pm i\\ \pm i & 1\end{pmatrix}\left(G_\alpha\circ\left(\zeta_{\vec{n},e}/\zeta_{\vec{n},e}(e)\right)r_e\right)^{-\sigma_3}\right)^{-1}$$

is holomorphic in U_e. With such $\mathbf{E}_e$ it holds that

$$\mathbf{P}_e = \mathbf{M}\mathsf{T}_i\left(G_\alpha^{-\sigma_3}\circ\left(\zeta_{\vec{n},e}/\zeta_{\vec{n},e}(e)\right)\left(\mathbf{I}+\mathcal{O}\left(\varepsilon_{\vec{n},e}\right)\right)\right)\mathbf{D}$$

uniformly on $\partial U_e\setminus\left((a_i,b_i)\cup\Gamma^+_{\vec{n},i}\cup\Gamma^-_{\vec{n},i}\right)$, where

$$\varepsilon_{\vec{n},e} = \max\left\{|\zeta_{\vec{n},e}(e)|^{1/2}, |\vec{n}|^{-1/3}\right\}, \tag{82}$$

according to Theorem 7. To see that RHP-$\mathbf{P}_e$(d) is fulfilled it only remains to notice that $G_\alpha(\zeta) = 1+\mathcal{O}\left(\zeta^{-1/2}\right)$ as $\zeta\to\infty$ uniformly in $\mathbb{C}\setminus(-\infty,1]$.

If $s_{\vec{n}}<0$, we need to modify (75) again because $\mathbf{M}$ still has its jump over $\left(a_{\vec{n},i},b_{\vec{n},i}\right)$ while r_e over $\left(a_{\vec{c},i},b_{\vec{c},i}\right)$ where $b_{\vec{n},i}>b_{\vec{c},i}$ or $a_{\vec{n},i}<a_{\vec{c},i}$. Define

$$F_\beta(\zeta) := \beta^{1/2}\left(\frac{i+(\zeta-1)^{1/2}}{i-(\zeta-1)^{1/2}}\right)^{\log\beta/2\pi i}, \quad \zeta\in\mathbb{C}\setminus(-\infty,1]. \tag{83}$$

This function is holomorphic in the domain of its definition, tends to 1 as $\zeta\to\infty$, and satisfies

$$F_{\beta+}(x)F_{\beta-}(x) = \begin{cases}1, & x\in(-\infty,0),\\ \beta, & x\in(0,1).\end{cases}$$

Indeed, the function $(i+\sqrt{\zeta-1})/(i-\sqrt{\zeta-1})$ maps the complement of $(-\infty,1]$ to the lower half-plane, its traces on $(-\infty,1)$ are reciprocal to each other, are positive on $(0,1)$, and are negative on $(-\infty,0)$. The stated properties now easily follow if we take the principal branch of $\log\beta/2\pi i$ root of this function. Then

$$\mathbf{E}_e := \mathbf{M}\mathsf{T}_i\left(\frac{\left(|\vec{n}|^{2/3}\zeta_{\vec{n},e}\right)^{-\sigma_3/4}}{\sqrt{2}}\begin{pmatrix}1 & \pm i\\ \pm i & 1\end{pmatrix}\left(F_\beta\circ\left(\frac{\zeta_{\vec{n},e}(e)-\zeta_{\vec{n},e}}{\zeta_{\vec{n},e}(e)}\right)r_e\right)^{-\sigma_3}\right)^{-1}$$

is holomorphic in $U_e\setminus\{e\}$. Since $|r_e(z)|\sim|z-e|^{\alpha/2}$ as $z\to e$, one can deduce as before $\mathbf{E}_e$ is holomorphic in U_e. Moreover, exactly as in the case $s_{\vec{n}}>0$, we get that RHP-$\mathbf{P}_e$ holds with $\varepsilon_{\vec{n},e}$ given by (82) since $F_\beta(\zeta)=1+\mathcal{O}\left(\zeta^{-1/2}\right)$ as $\zeta\to\infty$.

9.5 *Soft-Type Edge II*

Let $e\in E_{\vec{c}}$, $e\in\partial D_i^-$, but $b_{\vec{n},i}\notin\partial D^-_{\vec{n},i}$ or $a_{\vec{n},i}\notin\partial D^-_{\vec{n},i}$. In this case it necessarily holds that $b_{\vec{n},i}=b_{\vec{c},i}=b_i$ or $a_{\vec{n},i}=a_{\vec{c},i}=a_i$.

9.5.1 *Conformal Maps*

By Proposition 3, h is bounded at $\boldsymbol{e}$ (the corresponding branch point of $\mathfrak{R}$) while $h_{\vec{n}}$ has a simple pole at $\boldsymbol{e}$ (this time $\boldsymbol{e}$ is a branch point of $\mathfrak{R}_{\vec{n}}$, but it has the same projection e) and a simple zero $\boldsymbol{z}_{\vec{n},i}$ or $\boldsymbol{z}_{\vec{n},i-1}$ that approaches $\boldsymbol{e}$. Hence, we can write

$$-\frac{3}{4}\int_e^z\left(h_{\vec{n}}^{(0)}-h_{\vec{n}}^{(i)}\right)(x)dx = \sqrt{z-e}\,(z-e-\epsilon_{\vec{n}})\,f_{\vec{n}}(z), \tag{84}$$

where $0\leqslant\epsilon_{\vec{n}}\to0$ as $|\vec{n}|\to\infty$ and $f_{\vec{n}}$ is non-vanishing in some neighborhood of e and is positive on the real line within this neighborhood (one can factor out $\sqrt{z-e}$ as the square of the left-hand side is holomorphic exactly as in (69) and (70)). Then there exist functions

$\zeta_{\vec{n},e}$, conformal in U_e, vanishing at e, real on $\mathbb{R}\cap U_e$, and positive for $x>e$ in U_e such that

$$-\frac{3}{4}\int_e^z\left(h_{\vec{n}}^{(0)}-h_{\vec{n}}^{(i)}\right)(x)dx=\zeta_{\vec{n},e}^{3/2}(z)-\zeta_{\vec{n},e}(e+\epsilon_{\vec{n}})\zeta_{\vec{n},e}^{1/2}(z). \tag{85}$$

Moreover, (65) holds, where ζ_e is defined by (76), and the left-hand side of (85) is equal to the right-hand side of (77). Indeed, consider the equation

$$u(z;\epsilon)\big(u(z;\epsilon)-p\big)^2=g(z;\epsilon),\quad g(z;\epsilon):=z(z-\epsilon)^2f(z;\epsilon), \tag{86}$$

where p is a parameter, $f(z;\epsilon)$ is positive on the real line in some neighborhood of zero and $g^{1/3}(z;0)$ is conformal in this neighborhood. The solution of (86) is given by

$$u(z;\epsilon)=2p+v^{1/3}(z;\epsilon)+p^2v^{-1/3}(z;\epsilon), \tag{87}$$

where $v(z;\epsilon)$ is the branch satisfying $v^{1/3}(0;\epsilon)=-p$ of

$$v(z;\epsilon)=g(z;\epsilon)-p^3+\sqrt{g(z;\epsilon)\big(g(z;\epsilon)-2p^3\big)}. \tag{88}$$

Choose p so that

$$2p^3=\max_{x\in[0,\epsilon]}g(x;\epsilon). \tag{89}$$

Conformality of $g^{1/3}(z;0)$ implies that there exists the unique $x_\epsilon>\epsilon$ such that

$$\begin{cases} g(x;\epsilon)\big(g(x;\epsilon)-2p^3\big)<0, & x\in(0,x_\epsilon)\setminus\{\epsilon\},\\ g(x;\epsilon)\big(g(x;\epsilon)-2p^3\big)>0, & x>x_\epsilon,\end{cases}$$

for all ϵ small enough. Then we can see from (88) that

$$|v_\pm(x;\epsilon)|^2=\big(g(x;\epsilon)-p^3\big)^2-g(x;\epsilon)\big(g(x;\epsilon)-2p^3\big)=p^6 \tag{90}$$

for $x\in[0,x_\epsilon]$. Moreover, it holds that

$$v_+(x;\epsilon)=p^2v_-^{-1}(x;\epsilon),\quad x\in[0,x_\epsilon]. \tag{91}$$

Finally, observe that the conformality of $g^{1/3}(z;0)$ yields that the change of the argument of $v_+(x;\epsilon)$ is 3π when x changes between 0 and x_ϵ. Hence, $v^{1/3}(z;\epsilon)$ is holomorphic off $[0,\epsilon]$ and its traces on $[0,\epsilon]$ map this interval onto the circle centered at the origin of radius p by (90). This together with (91) implies that $u(z;\epsilon)$ given by (87) is conformal in some neighborhood of the origin and $u(0;\epsilon)=0$. Thus, $\zeta_{\vec{n},e}$ in (85) is given by

$$\zeta_{\vec{n},e}(z)=u(z-e;\epsilon_{\vec{n}}),$$

where $u(z;\epsilon)$ is the solution given by (87) of (86) with $f(z;\epsilon):=f_{\vec{n}}^2(z-e)$ and the parameter p chosen as in (89).

9.5.2 *Matrix* $\mathbf{P}_e$

Clearly, formulae (71) and (72) remain valid in this case. Then (73) and (85) imply that the solution of RHP-$\mathbf{P}_e$ is given by

$$\mathbf{P}_e:=\mathbf{E}_e\mathrm{T}_i\left(\mathbf{\Psi}_e\left(|\vec{n}|^{2/3}\zeta_{\vec{n},e}\right)r_e^{-\sigma_3}\left(\Phi_{\vec{n}}^{(0)}/\Phi_{\vec{n}}^{(i)}\right)^{-\sigma_3/2}\right)\mathbf{D},$$

where $\mathbf{E}_e$ is given by (75) with $|\vec{n}|^2$ replaced by $|\vec{n}|^{2/3}$, $\mathbf{\Psi}_e=\widetilde{\mathbf{\Psi}}_{\alpha,0}(\cdot;s_{\vec{n}})$ when $e=b_i$ and $\mathbf{\Psi}_e=\sigma_3\widetilde{\mathbf{\Psi}}_{\alpha,0}(\cdot;s_{\vec{n}})\sigma_3$ when $e=a_i$,

$$s_{\vec{n}}:=-|\vec{n}|^{2/3}\zeta_{\vec{n},e}(e+\epsilon_{\vec{n}}),$$

and $\widetilde{\mathbf{\Psi}}_{\alpha,\beta}$ is the solution of RHP-$\widetilde{\mathbf{\Psi}}_{\alpha,\beta}$. In this case, it holds by Theorem 7 that

$$\varepsilon_{\vec{n},e}=\max\left\{\zeta_{\vec{n},e}^{1/2}(e+\epsilon_{\vec{n}}),|\vec{n}|^{-1/3}\right\}.$$

10 MODEL RIEMANN-HILBERT PROBLEM RHP-$\Psi_{\alpha,\beta}$

In this section we prove Theorem 7.

10.1 *Uniqueness and Existence*

Since all the jump matrices in RHP-$\Psi_{\alpha,\beta}$(b) have unit determinant, $\det(\Psi_{\alpha,\beta})$ is holomorphic in $\mathbb{C}\setminus\{0\}$. By RHP-$\Psi_{\alpha,\beta}$(d), it holds that $\det(\Psi_{\alpha,\beta})(\infty)=1$. It also follows from RHP-$\Psi_{\alpha,\beta}$(c) that $\det(\Psi_{\alpha,\beta})$ cannot have a polar singularity at the origin. Hence, $\det(\Psi_{\alpha,\beta})\equiv 1$. In particular, any solution of RHP-$\Psi_{\alpha,\beta}$ is invertible. If Ψ_1 and Ψ_2 are two such solutions, then it is easy to verify that $\Psi_1\Psi_2^{-1}$ is holomorphic in $\mathbb{C}$. Moreover, $\Psi_1\Psi_2^{-1}(\zeta)=\mathbf{I}+\mathcal{O}(1/\zeta)$ as $\zeta\to\infty$. Thus, $\Psi_1\Psi_2^{-1}=\mathbf{I}$, which proves uniqueness.

10.2 *Local Behavior*

To proceed with the existence, we need more detailed description of the behavior of $\Psi_{\alpha,\beta}$ at the origin. Denote by Ω_1, Ω_2, Ω_3, and Ω_4 consecutive sectors of $\mathbb{C}\setminus\big((-\infty,\infty)\cup I_-\cup I_+\big)$ staring with the one containing the first quadrant and continuing counter clockwise. Then we can write

$$\Psi_{\alpha,\beta}(\zeta)=\mathbf{E}(\zeta)\mathbf{S}_{\alpha,\beta}(\zeta)\mathbf{A}_j,\quad \zeta\in\Omega_j, \tag{92}$$

where $\mathbf{E}$ is a holomorphic matrix function,

$$\mathbf{A}_3=\mathbf{A}_4\begin{pmatrix}1&0\\e^{-\alpha\pi i}&1\end{pmatrix},\quad \mathbf{A}_4=\mathbf{A}_1\begin{pmatrix}1&-\beta\\0&1\end{pmatrix},\quad \mathbf{A}_1=\mathbf{A}_2\begin{pmatrix}1&0\\e^{\alpha\pi i}&1\end{pmatrix}, \tag{93}$$

and

$$\mathbf{A}_2=\begin{pmatrix}\frac{1}{2\cos(\alpha\pi/2)}\frac{1-\beta e^{\alpha\pi i}}{1-e^{\alpha\pi i}} & \frac{1}{2\cos(\alpha\pi/2)}\frac{\beta-e^{\alpha\pi i}}{1-e^{\alpha\pi i}}\\ -e^{\alpha\pi i/2} & e^{-\alpha\pi i/2}\end{pmatrix}\quad\text{while}\quad \mathbf{S}_{\alpha,\beta}(\zeta)=\zeta^{\alpha\sigma_3/2} \tag{94}$$

when α is not an integer,

$$\mathbf{A}_2=\begin{pmatrix}\frac12 e^{\alpha\pi i/2} & \frac12 e^{-\alpha\pi i/2}\\ -e^{\alpha\pi i/2} & e^{-\alpha\pi i/2}\end{pmatrix}\quad\text{while}\quad \mathbf{S}_{\alpha,\beta}(\zeta)=\begin{pmatrix}\zeta^{\alpha/2} & \frac{1-\beta}{2\pi i}\zeta^{\alpha/2}\log\zeta\\ 0 & \zeta^{-\alpha/2}\end{pmatrix} \tag{95}$$

when α is an even integer,

$$\mathbf{A}_2=\begin{pmatrix}0 & e^{-\alpha\pi i/2}\\ -e^{\alpha\pi i/2} & e^{-\alpha\pi i/2}\end{pmatrix}\quad\text{while}\quad \mathbf{S}_{\alpha,\beta}(\zeta)=\begin{pmatrix}\zeta^{\alpha/2} & \frac{1+\beta}{2\pi i}\zeta^{\alpha/2}\log\zeta\\ 0 & \zeta^{-\alpha/2}\end{pmatrix} \tag{96}$$

when α is an odd integer (observe that $\det(\mathbf{A}_j)=1$ for all $j\in\{1,2,3,4\}$). Indeed, equation (92) can be viewed as a definition of the matrix $\mathbf{E}$. Relations (93) are chosen so $\mathbf{E}$ is holomorphic across $I_\pm$ and $(0,\infty)$. Moreover, on $(-\infty,0)$ it holds that

$$\mathbf{E}_-^{-1}\mathbf{E}_+=\mathbf{S}_{\alpha,\beta-}\mathbf{A}_2\begin{pmatrix}1&0\\e^{\alpha\pi i}&1\end{pmatrix}\begin{pmatrix}1&-\beta\\0&1\end{pmatrix}\begin{pmatrix}1&0\\e^{-\alpha\pi i}&1\end{pmatrix}\begin{pmatrix}0&1\\-1&0\end{pmatrix}\mathbf{A}_2^{-1}\mathbf{S}_{\alpha,\beta+}^{-1}=\mathbf{I},$$

where the last equality is a tedious but straightforward computation. Hence, $\mathbf{E}$ is holomorphic in $\mathbb{C}\setminus\{0\}$. Using RHP-$\Psi_{\alpha,\beta}$(d) and (92), one can verify that $\mathbf{E}$ cannot have a polar singularity at 0 and therefore is entire as claimed.

10.3 *Vanishing Lemma*

The crucial step in showing solvability of RHP-$\Psi_{\alpha,\beta}$ is the following result. Assume $\mathbf{F}_{\alpha,\beta}$ satisfies RHP-$\Psi_{\alpha,\beta}$(a,b,c) and it holds that

$$\mathbf{F}_{\alpha,\beta}(\zeta)=\mathcal{O}\left(\zeta^{-1}\right)\frac{\zeta^{-\sigma_3/4}}{\sqrt2}\begin{pmatrix}1&i\\i&1\end{pmatrix}\exp\left\{-\left(\frac23\zeta^{3/2}+s\zeta^{1/2}\right)\sigma_3\right\} \tag{97}$$

as $\zeta \to \infty$ uniformly in $\mathbb{C} \setminus \big(I_+ \cup I_- \cup (-\infty,\infty)\big)$. Then $\mathbf{F}_{\alpha,\beta} \equiv 0$. To prove this claim, we follow the line of argument from [16] and [32]. Set, for brevity, $\theta(\zeta) := \left(\frac{2}{3}\zeta^{3/2} + s\zeta^{1/2}\right)$. Assuming $\mathrm{Re}(\beta) \geqslant 0$, define

$$\mathbf{G}_{\alpha,\beta}(\zeta) := \begin{cases} \mathbf{F}_{\alpha,\beta}(\zeta)e^{\theta(\zeta)\sigma_3}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & \zeta \in \Omega_1, \\ \mathbf{F}_{\alpha,\beta}(\zeta)e^{\theta(\zeta)\sigma_3}\begin{pmatrix} 1 & 0 \\ e^{\alpha\pi i}e^{2\theta(\zeta)} & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & \zeta \in \Omega_2, \\ \mathbf{F}_{\alpha,\beta}(\zeta)e^{\theta(\zeta)\sigma_3}\begin{pmatrix} 1 & 0 \\ -e^{-\alpha\pi i}e^{2\theta(\zeta)} & 1 \end{pmatrix}, & \zeta \in \Omega_3, \\ \mathbf{F}_{\alpha,\beta}(\zeta)e^{\theta(\zeta)\sigma_3}, & \zeta \in \Omega_4. \end{cases}$$

Then $\mathbf{G}_{\alpha,\beta}$ satisfies the following Riemann-Hilbert problem (RHP-$\mathbf{G}_{\alpha,\beta}$):

(a) $\mathbf{G}_{\alpha,\beta}$ is holomorphic in $\mathbb{C} \setminus (-\infty,\infty)$;
(b) $\mathbf{G}_{\alpha,\beta}$ has continuous traces on $(-\infty,0) \cup (0,\infty)$ that satisfy

$$\mathbf{G}_{\alpha,\beta+} = \mathbf{G}_{\alpha,\beta-}\begin{cases} \begin{pmatrix} 1 & -e^{\alpha\pi i}e^{2\theta_+} \\ e^{-\alpha\pi i}e^{2\theta_-} & 0 \end{pmatrix} & \text{on} \quad (-\infty,0), \\ \begin{pmatrix} \beta e^{-2\theta} & -1 \\ 1 & 0 \end{pmatrix} & \text{on} \quad (0,\infty); \end{cases}$$

(c) as $\zeta \to 0$ it holds that

$$\mathbf{G}_{\alpha,\beta}(\zeta) = \mathcal{O}\begin{pmatrix} |\zeta|^{\alpha/2} & |\zeta|^{\alpha/2} \\ |\zeta|^{\alpha/2} & |\zeta|^{\alpha/2} \end{pmatrix}$$

when $\alpha < 0$,

$$\mathbf{G}_{\alpha,\beta}(\zeta) = \mathcal{O}\begin{pmatrix} 1 & 1+(1-\beta)\log|\zeta| \\ 1 & 1+(1-\beta)\log|\zeta| \end{pmatrix} \quad \text{and} \quad \mathbf{G}_{\alpha,\beta}(\zeta) = \mathcal{O}\begin{pmatrix} 1+(1-\beta)\log|\zeta| & 1 \\ 1+(1-\beta)\log|\zeta| & 1 \end{pmatrix}$$

when $\alpha = 0$, for $\mathrm{Im}(\zeta) < 0$ and $\mathrm{Im}(\zeta) > 0$, respectively, and

$$\mathbf{G}_{\alpha,\beta}(\zeta) = \mathcal{O}\begin{pmatrix} |\zeta|^{\alpha/2} & |\zeta|^{-\alpha/2} \\ |\zeta|^{\alpha/2} & |\zeta|^{-\alpha/2} \end{pmatrix} \quad \text{and} \quad \mathbf{G}_{\alpha,\beta}(\zeta) = \mathcal{O}\begin{pmatrix} |\zeta|^{-\alpha/2} & |\zeta|^{\alpha/2} \\ |\zeta|^{-\alpha/2} & |\zeta|^{\alpha/2} \end{pmatrix}$$

when $\alpha > 0$, for $\mathrm{Im}(\zeta) < 0$ and $\mathrm{Im}(\zeta) > 0$, respectively;
(d) $\mathbf{G}_{\alpha,\beta} = \mathcal{O}(\zeta^{-3/4})$ as $\zeta \to \infty$.

Properties RHP-$\mathbf{G}_{\alpha,\beta}$(a,b) can be easily verified using RHP-$\mathbf{\Psi}_{\alpha,\beta}$(a,b), the definition of $\mathbf{G}_{\alpha,\beta}$, and the fact that $\theta_+ + \theta_- \equiv 0$ on $(-\infty,0)$. To show RHP-$\mathbf{G}_{\alpha,\beta}$(c), observe that the representations (92)–(96) holds for $\mathbf{F}_{\alpha,\beta}$ as well. They imply that

$$\mathbf{G}_{\alpha,\beta} = \mathbf{E}\mathbf{S}_{\alpha,\beta}\mathbf{A}_1\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}e^{-\theta\sigma_3} \quad \text{and} \quad \mathbf{G}_{\alpha,\beta} = \mathbf{E}\mathbf{S}_{\alpha,\beta}\mathbf{A}_4 e^{\theta\sigma_3}$$

in $\Omega_1 \cup \Omega_2$ and $\Omega_3 \cup \Omega_4$, respectively. Since $[\mathbf{A}_1]_{21} = [\mathbf{A}_4]_{21} = 0$, RHP-$\mathbf{G}_{\alpha,\beta}$(c) follows. Finally, RHP-$\mathbf{G}_{\alpha,\beta}$(d) is the consequence of the fact that $\mathrm{Re}(\theta) < 0$ in $\Omega_2 \cup \Omega_3$.

For the next step of the proof consider the matrix function

$$\mathbf{G}_{\alpha,\beta}(\zeta)\big(\mathbf{G}_{\alpha,\beta}(\bar{\zeta})\big)^*, \quad \zeta \notin (-\infty,\infty),$$

where $\big(\mathbf{G}_{\alpha,\beta}\big)^*$ is the hermitian conjugate of $\mathbf{G}_{\alpha,\beta}$. This matrix function is holomorphic off the real line, has integrable traces on the real line by RHP-$\mathbf{G}_{\alpha,\beta}$(c) (recall that $\alpha > -1$), and vanishes at infinity as $\zeta^{-3/2}$ by RHP-$\mathbf{G}_{\alpha,\beta}$(d). Thus, we deduce from Cauchy's theorem that the integral of its traces over the real line is zero, i.e.,

$$0 = \int_{-\infty}^{\infty}\mathbf{G}_{\alpha,\beta+}(x)\big(\mathbf{G}_{\alpha,\beta-}(x)\big)^* dx = \int_{-\infty}^{\infty}\mathbf{G}_{\alpha,\beta-}(x)\big(\mathbf{G}_{\alpha,\beta+}(x)\big)^* dx. \tag{98}$$

Adding the last two integrals together and using RHP-$\mathbf{G}_{\alpha,\beta}$(b), we get

$$0 = \int_{-\infty}^{0} \mathbf{G}_{\alpha,\beta -}(x) \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} (\mathbf{G}_{\alpha,\beta -}(x))^* dx + \int_0^{\infty} \mathbf{G}_{\alpha,\beta -}(x) \begin{pmatrix} (\beta + \bar{\beta}) e^{-2\theta(x)} & 0 \\ 0 & 0 \end{pmatrix} (\mathbf{G}_{\alpha,\beta -}(x))^* dx.$$

The above equality yields that the first column of $\mathbf{G}_{\alpha,\beta -}$ vanishes identically on $(-\infty, 0)$ (on the whole real line if β is not purely imaginary). In any case, since $\mathbf{G}_{\alpha,\beta -}$ consists of traces of holomorphic functions, its first column vanishes identically in the lower half-plane. Thus, by RHP-$\mathbf{G}_{\alpha,\beta}$(b), the second column of $\mathbf{G}_{\alpha,\beta}$ vanishes identically in the upper half-plane. To finish proving that $\mathbf{G}_{\alpha,\beta} \equiv 0$ in the case $\mathrm{Re}(\beta) \geqslant 0$ (and therefore $\mathbf{F}_{\alpha,\beta} \equiv 0$), set

$$g_i(\zeta) := \begin{cases} [\mathbf{G}_{\alpha,\beta}]_{i1}(\zeta), & \mathrm{Im}(\zeta) > 0, \\ [\mathbf{G}_{\alpha,\beta}]_{i2}(\zeta), & \mathrm{Im}(\zeta) < 0. \end{cases}$$

Both functions g_i are holomorphic in $\mathbb{C} \setminus (-\infty, 0]$, satisfy $g_i(\zeta) = \mathcal{O}(\zeta^{-3/4})$ as $\zeta \to \infty$ and $g_i(\zeta) = \mathcal{O}(|\zeta|^{-|\alpha|/2})$ as $\zeta \to 0$, while their traces are related by the formula

$$g_{i+}(x) = g_{i-}(x) e^{-\alpha\pi i} e^{2\theta_-(x)}, \quad x \in (-\infty, 0).$$

The latter is possible only if $g_i \equiv 0$ as shown in [16, Def. (2.26) and below].

When $\mathrm{Re}(\beta) < 0$ and $\mathrm{Im}(\beta) \neq 0$, let us redefine $\mathbf{G}_{\alpha,\beta}$ in Ω_1 and Ω_2 by setting

$$\mathbf{G}_{\alpha,\beta}(\zeta) := \begin{cases} \mathbf{F}_{\alpha,\beta}(\zeta) e^{\theta(\zeta)\sigma_3} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \zeta \in \Omega_1, \\ \mathbf{F}_{\alpha,\beta}(\zeta) e^{\theta(\zeta)\sigma_3} \begin{pmatrix} 1 & 0 \\ e^{\alpha\pi i} e^{2\theta(\zeta)} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \zeta \in \Omega_2. \end{cases}$$

This newly defined function $\mathbf{G}_{\alpha,\beta}$ still satisfies RHP-$\mathbf{G}_{\alpha,\beta}$ except for RHP-$\mathbf{G}_{\alpha,\beta}$(b), which now becomes

$$\mathbf{G}_{\alpha,\beta +} = \mathbf{G}_{\alpha,\beta -} \begin{cases} \begin{pmatrix} 1 & e^{\alpha\pi i} e^{2\theta_+} \\ e^{-\alpha\pi i} e^{2\theta_-} & 0 \end{pmatrix} & \text{on} \quad (-\infty, 0), \\ \begin{pmatrix} \beta e^{-2\theta} & 1 \\ 1 & 0 \end{pmatrix} & \text{on} \quad (0, \infty). \end{cases}$$

Observe that (98) remains valid. Thus, by taking the difference of the integrals in (98), we arrive at

$$0 = \int_0^{\infty} \mathbf{G}_{\alpha,\beta -}(x) \begin{pmatrix} (\beta - \bar{\beta}) e^{-2\theta(x)} & 0 \\ 0 & 0 \end{pmatrix} (\mathbf{G}_{\alpha,\beta -}(x))^* dx.$$

This again allows us to conclude that the first column of $\mathbf{G}_{\alpha,\beta}$ vanishes identically in the lower half-plane and the second column vanishes in the upper half-plane. The remaining part of the proof is now exactly the same as in the case $\mathrm{Re}(\beta) \geqslant 0$.

10.4 *Existence*

For $\mathbf{A}_j$ and $\mathbf{S}_{\alpha,\beta}$ as in (93)–(96), define

$$\mathbf{M}_{\alpha,\beta}(\zeta) := \begin{cases} \mathbf{\Psi}_{\alpha,\beta}(\zeta) \mathbf{A}_j^{-1} \mathbf{S}_{\alpha,\beta}^{-1}(\zeta), & \zeta \in \Omega_j \cap \{|\zeta| < 1\}, \\ \mathbf{\Psi}_{\alpha,\beta}(\zeta) e^{\theta(\zeta)\sigma_3} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \zeta^{\sigma_3/4}, & \zeta \in \Omega_j \cap \{|\zeta| \geqslant 1\}. \end{cases}$$

Further, let the contour $\Sigma_{\mathbf{M}}$ be as on Figure 4 with its subarcs oriented so that $\mathbb{C} \setminus \Sigma_{\mathbf{M}} = \Omega_+ \cup \Omega_-$, where $\Sigma_{\mathbf{M}}$ is positively oriented boundary of Ω_+ and is negatively oriented boundary of Ω_-. If $\mathbf{\Psi}_{\alpha,\beta}$ uniquely solves RHP-$\mathbf{\Psi}_{\alpha,\beta}$, then $\mathbf{M}_{\alpha,\beta}$ uniquely solves the following Riemann-Hilbert problem (RHP-$\mathbf{M}_{\alpha,\beta}$):

(a) $\mathbf{M}_{\alpha,\beta}$ is holomorphic in $\mathbb{C} \setminus \Sigma_{\mathbf{M}}$ and $\mathbf{M}_{\alpha,\beta}(\zeta) = \mathbf{I} + \mathcal{O}(1/\zeta)$ as $\zeta \to \infty$;

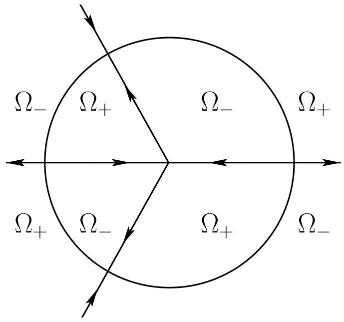

FIGURE 4. Contours $\Sigma_{\mathbf{M}}$.

(b) $\mathbf{M}_{\alpha,\beta}$ has continuous traces on $\Sigma_{\mathbf{M}}$ that satisfy $\mathbf{M}_{\alpha,\beta+} = \mathbf{M}_{\alpha,\beta-}\mathbf{J}$, where

$$\mathbf{J}(\zeta) = \left(\mathbf{S}_{\alpha,\beta}(\zeta)\mathbf{A}_j e^{\theta(\zeta)\sigma_3}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\\ -i & 1\end{pmatrix}\zeta^{\sigma_3/4}\right)^{\pm 1},$$

on $\Omega_j \cap \{|\zeta| = 1\}$ with the exponent 1 for $j = 1, 3$ and the exponent -1 for $j = 2, 4$, and on the rest of the contour $\Sigma_{\mathbf{M}}$ the jump is equal to

$$\mathbf{J}(\zeta) = \mathbf{I} + e^{-2\theta(\zeta)}\begin{cases} 0, & \zeta \in \Sigma_{\mathbf{M}} \cap \{|\zeta| < 1\} \text{ or } \zeta \in (-\infty, -1),\\ \frac{\beta}{2}\begin{pmatrix}-i & \zeta^{-1/2}\\ \zeta^{1/2} & i\end{pmatrix}, & \zeta \in (1, \infty),\\ \frac{e^{\pm\alpha\pi i}}{2}\begin{pmatrix}i\zeta^{-1/2} & 1\\ 1 & -i\zeta^{1/2}\end{pmatrix}, & \zeta \in I_{\pm} \cap \{|\zeta| > 1\}.\end{cases}$$

According to [18, Appendix], see also [16, Prop. 2.4], the unique solution of RHP-$\mathbf{M}_{\alpha,\beta}$ is given by the formula

$$\mathbf{M}_{\alpha,\beta}(\zeta) = \mathbf{I} + \mathcal{C}\big(\mathbf{M}(\mathbf{W}_+ + \mathbf{W}_-)\big)(\zeta), \quad \zeta \in \mathbb{C} \setminus \Sigma_{\mathbf{M}},$$

where $\mathbf{J} = (\mathbf{I} - \mathbf{W}_-)^{-1}(\mathbf{I} - \mathbf{W}_+)$ is a factorization of the jump $\mathbf{J}$ for some $\mathbf{W}_\pm \in L^2(\Sigma_{\mathbf{M}}) \cap L^\infty(\Sigma_{\mathbf{M}})$, $\mathcal{C}$ is a Cauchy operator

$$\mathcal{C}\mathbf{H}(\zeta) = \frac{1}{2\pi i}\int_{\Sigma_{\mathbf{M}}}\frac{\mathbf{H}(s)}{s - \zeta}ds, \quad \mathbf{H} \in L^2(\Sigma_{\mathbf{M}}), \quad \zeta \in \mathbb{C} \setminus \Sigma_{\mathbf{M}},$$

and $\mathbf{M} \in \mathbf{I} + L^2(\Sigma_{\mathbf{M}})$ is the solution of the singular integral equation

$$(\mathcal{I} - \mathcal{C}_{\mathbf{W}})\mathbf{M} = \mathbf{I} \tag{99}$$

for the singular operator $\mathcal{C}_{\mathbf{W}} : L^2(\Sigma_{\mathbf{M}}) \to L^2(\Sigma_{\mathbf{M}})$ given by

$$\mathcal{C}_{\mathbf{W}}\mathbf{H} := \mathcal{C}_+(\mathbf{H}\mathbf{W}_-) + \mathcal{C}_-(\mathbf{H}\mathbf{W}_+), \quad \mathbf{H} \in L^2(\Sigma_{\mathbf{M}}),$$

provided this solution exists and is unique. Indeed, given such $\mathbf{M}$, it holds that

$$\mathbf{I} + \mathcal{C}_\pm\big(\mathbf{M}(\mathbf{W}_+ + \mathbf{W}_-)\big) = \mathbf{I} + \mathcal{C}_{\mathbf{W}}\mathbf{M} + \big(\mathcal{C}_\pm - \mathcal{C}_\mp\big)(\mathbf{M}\mathbf{W}_\pm) = \mathbf{M} \pm \mathbf{M}\mathbf{W}_\pm$$

by (99) and Sokhotski-Pemelj formulae [13, Section 4.2]. Then

$$\big(\mathbf{I} + \mathcal{C}_-\big(\mathbf{M}(\mathbf{W}_+ + \mathbf{W}_-)\big)\big)^{-1}\big(\mathbf{I} + \mathcal{C}_+\big(\mathbf{M}(\mathbf{W}_+ + \mathbf{W}_-)\big)\big) = (\mathbf{I} - \mathbf{W}_-)^{-1}(\mathbf{I} - \mathbf{W}_+) = \mathbf{J}$$

as desired. Thus, only the unique solvability of (99) needs to be shown. The sufficient condition for the latter is bijectivity of the operator $\mathcal{I} - \mathcal{C}_{\mathbf{W}}$, which can be established by showing that $\mathcal{I} - \mathcal{C}_{\mathbf{W}}$ is Fredholm with index zero and trivial kernel.

To this end, let us specify $\mathbf{W}_\pm$. Away from the points of self-intersection of $\Sigma_{\mathbf{M}}$, set

$$\mathbf{W}_+ := \mathbf{J} - \mathbf{I} \quad \text{and} \quad \mathbf{W}_- = 0.$$

Around the points of self-intersection, we chose W_+ to be continuous along the boundary of Ω_+ and W_- to be continuous along the boundary of Ω_-. The latter is possible because around each point of self-intersection of Σ_M, the jumps satisfy the cyclic relation

$$J_1 J_2^{-1} J_3 J_4^{-1} = I,$$

where we label the four arcs meeting at the point of self-intersection counter-clockwise starting with an arc oriented away from the point and denote by J_i the jump across the i-th arc. Clearly, $W_\pm \in L^2(\Sigma_M) \cap L^\infty(\Sigma_M)$. To show that $\mathcal{I} - \mathcal{C}_W$ is Fredholm, one needs to construct its pseudoinverse. The latter is given by $\mathcal{I} - \mathcal{C}_{\widetilde{W}}$, where

$$\widetilde{W}_+ := \big(I + W_+\big)^{-1} - I \quad \text{and} \quad \widetilde{W}_- := I - \big(I - W_-\big)^{-1},$$

as explained in [16, Eq. (2.39)–(2.42)]. The index of $\mathcal{I} - \mathcal{C}_W$ is equal to the winding number of $\det(J)$, which is zero since $\det(J) \equiv 1$. Finally, the kernel of $\mathcal{I} - \mathcal{C}_W$ is trivial if and only if the homogeneous Riemann-Hilbert problem corresponding to RHP-$M_{\alpha,\beta}$ has only trivial solutions. Correspondence between RHP-$\Psi_{\alpha,\beta}$ and RHP-$M_{\alpha,\beta}$ implies that the kernel is trivial if and only if the solution of RHP-$\Psi_{\alpha,\beta}$ with RHP-$\Psi_{\alpha,\beta}$(d) replaced by (97) has only trivial solutions. This is precisely the content of the preceding subsection. This finishes the proof of the first claim of Theorem 7.

10.5 *Asymptotics of* RHP-$\Psi_{\alpha,\beta}$ *for* $s > 0$

It is known that $\mathcal{O}\left(\eta^{-1}\right)$ is uniform for s on compact subsets of the real line [16]. Thus, we only need to prove (33) for s large.

10.5.1 *Renormalized* RHP-$\Psi_{\alpha,\beta}$

Set $\widehat{\Gamma}_\pm := \{\eta : \arg(\eta + 1) = \pm 2\pi/3\}$ and let $\widehat{\Omega}_j$, $j \in \{1,2,3,4\}$, be the domains comprising $\mathbb{C} \setminus \big((-\infty,\infty) \cup \widehat{\Gamma}_+ \cup \widehat{\Gamma}_-\big)$, numbered counter-clockwise and so that $\widehat{\Omega}_1$ contains the first quadrant. Define

$$g(\eta) = \frac{2}{3}(\eta+1)^{3/2}, \quad \eta \in \mathbb{C} \setminus (-\infty,-1],$$

to be the principal branch and set for convenience $\tau := s^{3/2}$. Let

$$\widehat{\Psi}_{\alpha,\beta}(\eta;\tau) = s^{\sigma_3/4} \Psi_{\alpha,\beta}(s\eta; s) \begin{cases} I & \text{in} \quad \Omega_1 \cup \Omega_4 \cup \widehat{\Omega}_2 \cup \widehat{\Omega}_3, \\ \begin{pmatrix} 1 & 0 \\ \pm e^{\pm\alpha\pi i} & 1 \end{pmatrix} & \text{in} \quad \Omega_2 \setminus \widehat{\Omega}_2,\ \Omega_3 \setminus \widehat{\Omega}_3, \end{cases} \tag{100}$$

where the sign $+$ is used in $\Omega_2 \setminus \widehat{\Omega}_2$ and the sign $-$ in $\Omega_3 \setminus \widehat{\Omega}_3$. Then $\widehat{\Psi}_{\alpha,\beta}$ solves the following Riemann-Hilbert problem (RHP-$\widehat{\Psi}_{\alpha,\beta}$):

(a) $\widehat{\Psi}_{\alpha,\beta}$ is holomorphic in $\mathbb{C} \setminus \big(\widehat{\Gamma}_+ \cup \widehat{\Gamma}_- \cup (-\infty,\infty)\big)$;

(b) $\widehat{\Psi}_{\alpha,\beta}$ has continuous traces on $\widehat{\Gamma}_+ \cup \widehat{\Gamma}_- \cup (-\infty,-1) \cup (-1,0) \cup (0,\infty)$ that satisfy

$$\widehat{\Psi}_{\alpha,\beta+} = \widehat{\Psi}_{\alpha,\beta-} \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{on} \quad (-\infty,-1), \\ \begin{pmatrix} e^{\alpha\pi i} & 1 \\ 0 & e^{-\alpha\pi i} \end{pmatrix} & \text{on} \quad (-1,0), \\ \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} & \text{on} \quad (0,\infty), \\ \begin{pmatrix} 1 & 0 \\ e^{\pm\alpha\pi i} & 1 \end{pmatrix} & \text{on} \quad \widehat{\Gamma}_\pm; \end{cases}$$

(c) as $\eta \to 0$ it holds that

$$\widehat{\Psi}_{\alpha,\beta}(\eta;\tau) = \widehat{E}(\eta) S_{\alpha,\beta}(\eta) A_j, \quad \eta \in \widehat{\Omega}_j, \quad j \in \{1,4\},$$

where $\widehat{E}$ is holomorphic, and $S_{\alpha,\beta}$, A_1 and A_4 are the same as in RHP-$\Psi_{\alpha,\beta}$(c);

(d) $\widehat{\Psi}_{\alpha,\beta}$ has the following behavior near ∞:

$$\widehat{\Psi}_{\alpha,\beta}(\eta;\tau) = \left(\mathbf{I} + \mathcal{O}\left(\eta^{-1}\right)\right) \frac{\eta^{-\sigma_3/4}}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} e^{-\tau g(\eta)\sigma_3}$$

uniformly in $\mathbb{C} \setminus \left(\widehat{\Gamma}_+ \cup \widehat{\Gamma}_- \cup (-\infty,\infty)\right)$.

10.5.2 *Global Parametrix*

Let

$$\begin{aligned} \widehat{\Psi}^{(\infty)}(\eta;\tau) &:= \begin{pmatrix} 1 & 0 \\ \alpha i & 1 \end{pmatrix} \frac{(\eta+1)^{-\sigma_3/4}}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \left(\frac{(\eta+1)^{1/2}+1}{(\eta+1)^{1/2}-1}\right)^{-\alpha\sigma_3/2} e^{-\tau g(\eta)\sigma_3} \\ &=: F^{(\infty)}(\tau) e^{-\tau g(\eta)\sigma_3}. \end{aligned}$$

Then, as is explained in [17, Section 2.4.1], this matrix-valued function solves the following Riemann-Hilbert problem:

(a) $\widehat{\Psi}^{(\infty)}$ is holomorphic in $\mathbb{C} \setminus (-\infty, 0]$;

(b) $\widehat{\Psi}^{(\infty)}$ has continuous traces on $(-\infty,-1) \cup (-1,0)$ that satisfy

$$\widehat{\Psi}^{(\infty)}_+ = \widehat{\Psi}^{(\infty)}_- \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{on} \quad (-\infty,-1), \\ e^{\alpha\pi i\sigma_3} & \text{on} \quad (-1,0), \end{cases}$$

(c) as $\eta \to 0$ it holds that $\widehat{\Psi}^{(\infty)}(\eta;\tau) = \widehat{E}^{(\infty)}(\eta)\eta^{\alpha\sigma_3/2}$, where $\widehat{E}^{(\infty)}$ is holomorphic and non-vanishing around zero;

(d) $\widehat{\Psi}^{(\infty)}$ satisfies RHP-$\widehat{\Psi}_{\alpha,\beta}$(d) uniformly in $\mathbb{C} \setminus (-\infty; 0]$ and the term $\mathcal{O}(\eta^{-1})$ does not depend on τ.

Notice that $F^{(\infty)}$ has the same jumps as $\widehat{\Psi}^{(\infty)}$.

10.5.3 *Local Parametrix Around -1*

The solution $\Psi_{Ai} := \Psi_{0,1}(\cdot;0)$ is known explicitly and is constructed with the help of the Airy function and its derivative [9]. Set

$$\widehat{\Psi}^{(-1)}(\eta;\tau) := \widehat{E}^{(-1)}(\eta)\Psi_{Ai}\left(s(\eta+1)\right)e^{\pm\alpha\pi i\sigma_3/2}, \quad \pm\operatorname{Im}(\eta) > 0,$$

where $\widehat{E}^{(-1)}$ is holomorphic around -1 and is given by

$$\widehat{E}^{(-1)}(\eta) := F^{(\infty)}(\eta)\left(\frac{\left(s(\eta+1)\right)^{-\sigma_3/4}}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} e^{\pm\alpha\pi i\sigma_3/2}\right)^{-1}, \quad \pm\operatorname{Im}(\eta) > 0.$$

Let U_{-1} be the disk of radius $1/4$ centered at -1 with boundary oriented counter-clockwise. Then it is shown in [17, Section 2.4.2] that $\widehat{\Psi}^{(-1)}$ satisfies

(a) $\widehat{\Psi}^{(-1)}$ is holomorphic in $U_{-1} \setminus \left(\widehat{\Gamma}_+ \cup \widehat{\Gamma}_- \cup (-\infty,\infty)\right)$;

(b) $\widehat{\Psi}^{(-1)}$ has continuous traces on $U_{-1} \cap \left(\widehat{\Gamma}_+ \cup \widehat{\Gamma}_- \cup (-\infty,\infty)\right)$ that satisfy RHP-$\widehat{\Psi}_{\alpha,\beta}$(b);

(c) it holds that

$$\widehat{\Psi}^{(-1)}(\eta;\tau) = F^{(\infty)}(\eta)\left(\mathbf{I} + \mathcal{O}\left(\tau^{-1}\right)\right) e^{-\tau g(\eta)\sigma_3}$$

as $\tau \to \infty$, uniformly for $\eta \in \partial U_{-1} \setminus \left(\widehat{\Gamma}_+ \cup \widehat{\Gamma}_- \cup (-\infty,\infty)\right)$.

10.5.4 *Local Parametrix Around 0*

Define

$$\widehat{\Psi}^{(0)}(\eta;\tau) := \widehat{E}^{(0)}(\eta)\mathbf{S}_{\alpha,\beta}(\tau) \begin{cases} \mathbf{A}_1, & \operatorname{Im}(\eta) > 0, \\ \mathbf{A}_4, & \operatorname{Im}(\eta) < 0, \end{cases}$$

where $\mathbf{S}_{\alpha,\beta}$ and $\mathbf{A}_j$ are the same as in RHP-$\Psi_{\alpha,\beta}$(c) and

$$\widehat{E}^{(0)}(\eta) := \widehat{\Psi}^{(\infty)}(\eta;\tau)\eta^{-\alpha\sigma_3/2}\begin{pmatrix} [\mathbf{A}_1]_{11}^{-1} & 0 \\ 0 & [\mathbf{A}_1]_{22}^{-1} \end{pmatrix},$$

which is a holomorphic function around the origin by the properties of $\widehat{\Psi}^{(\infty)}$. Let U_0 be the disk of radius $1/4$ centered at 0 with boundary oriented counter-clockwise. Then $\widehat{\Psi}^{(0)}$ possesses the following properties:

(a) $\widehat{\Psi}^{(0)}$ is holomorphic in $U_0 \setminus (-1/4, 1/4)$;
(b) $\widehat{\Psi}^{(0)}$ has continuous traces on $(-1/4,0)\cup(0,1/4)$ that satisfy RHP-$\widehat{\Psi}_{\alpha,\beta}$(b);
(c) $\widehat{\Psi}^{(0)}$ satisfies RHP-$\widehat{\Psi}_{\alpha,\beta}$(c) with $\widehat{E}$ replaced by $\widehat{E}^{(0)}$;
(d) it holds that

$$\widehat{\Psi}^{(0)}(\eta;\tau) = F^{(\infty)}(\eta)\left(\mathbf{I} + \mathcal{O}\left(e^{-c\tau}\right)\right)e^{-\tau g(\eta)\sigma_3}$$

as $\tau\to\infty$ for some $c>0$, uniformly for $\eta\in\partial U_0\setminus\{-1/4,1/4\}$.

Indeed, properties (a,b,c) easily follow from RHP-$\widehat{\Psi}_{\alpha,\beta}$(b,c) and the holomorphy of $\widehat{E}^{(0)}$. To get (d), write $[\mathbf{S}_{\alpha,\beta}]_{12}(\eta) = \eta^{\alpha/2}\kappa(\eta)$, where

$$\kappa(\eta) = 0, \quad \kappa(\eta) = \frac{1-\beta}{2\pi i}\log\eta, \quad \text{or} \quad \kappa(\eta) = \frac{1+\beta}{2\pi i}\log\eta$$

depending on whether α is not an integer, an even integer, or an odd integer. Recall also that $\mathbf{A}_1$ and $\mathbf{A}_4$ are upper triangular matrices and $[\mathbf{A}_1]_{ii} = [\mathbf{A}_4]_{ii}$ for $i\in\{1,2\}$. Then

$$\begin{aligned}\widehat{\Psi}^{(0)}(\eta;\tau) &= F^{(\infty)}(\eta)e^{-\tau g(\eta)\sigma_3}\begin{pmatrix} [\mathbf{A}_j]_{11}^{-1} & 0 \\ 0 & [\mathbf{A}_j]_{22}^{-1}\end{pmatrix}\begin{pmatrix} 1 & \kappa(\eta) \\ 0 & 1\end{pmatrix}\mathbf{A}_j \\ &= F^{(\infty)}(\eta)\begin{pmatrix} 1 & e^{-2\tau g(\eta)}\left([\mathbf{A}_j]_{22}\kappa(\eta) + [\mathbf{A}_j]_{12}\right)/[\mathbf{A}_j]_{11} \\ 0 & 1\end{pmatrix}e^{-\tau g(\eta)\sigma_3},\end{aligned}$$

from which property (d) can be easily deduced as $\tau>0$ and $\mathrm{Re}(g(\eta))>0$ for $\eta\in\partial U_0$.

10.5.5 *Asymptotics of* RHP-$\Psi_{\alpha,\beta}$

Denote by

$$\Sigma(\mathbf{R}_{\alpha,\beta}) := \partial U_{-1}\cup\partial U_0\cup\left(\left(\widehat{I}_-\cup\widehat{I}_+\cup(-1,\infty)\right)\cap\left(\mathbb{C}\setminus\left(\overline{U}_{-1}\cup\overline{U}_0\right)\right)\right),$$

and let $\Sigma^\circ(\mathbf{R}_{\alpha,\beta})$ be $\Sigma(\mathbf{R}_{\alpha,\beta})$ with the points of self-intersection removed. Put

$$\mathbf{R}_{\alpha,\beta}(\eta;\tau) := \widehat{\Psi}_{\alpha,\beta}(\eta;\tau)\begin{cases}\widehat{\Psi}^{(-1)}(\eta;\tau)^{-1}, & \eta\in U_{-1},\\ \widehat{\Psi}^{(0)}(\eta;\tau)^{-1}, & \eta\in U_0,\\ \widehat{\Psi}^{(\infty)}(\eta;\tau)^{-1}, & \eta\in\mathbb{C}\setminus\left(\overline{U}_0\cup\overline{U}_{-1}\right).\end{cases}$$

Then $\mathbf{R}_{\alpha,\beta}$ has the following properties:

(a) $\mathbf{R}_{\alpha,\beta}$ is holomorphic in $\mathbb{C}\setminus\Sigma(\mathbf{R}_{\alpha,\beta})$;
(b) $\mathbf{R}_{\alpha,\beta}$ has continuous traces on $\Sigma^\circ(\mathbf{R}_{\alpha,\beta})$ that satisfy $\mathbf{R}^{(0)}_{\alpha,\beta+} := \mathbf{R}^{(0)}_{\alpha,\beta-}\left(\mathbf{I}+\mathcal{O}\left(\tau^{-1}\right)\right)$ as $\tau\to\infty$;
(c) it holds that $\mathbf{R}_{\alpha,\beta}(\eta;\tau) = \mathbf{I}+\mathcal{O}\left(\eta^{-1}\right)$ as $\eta\to\infty$ uniformly in $\mathbb{C}\setminus\Sigma(\mathbf{R}_{\alpha,\beta})$.

Property (a) follows from the facts that $\widehat{\Psi}^{(e)}$ has the same jumps as $\widehat{\Psi}_{\alpha,\beta}$ in U_e, $e\in\{-1,0\}$, $\widehat{\Psi}^{(\infty)}$ has the same jump across $(-\infty,-1)$ as $\widehat{\Psi}_{\alpha,\beta}$, and $\widehat{\Psi}^{(0)}$ has the same local behavior around 0 as $\widehat{\Psi}_{\alpha,\beta}$. Property (c) follows easily from the fact that both $\widehat{\Psi}^{(\infty)}$ and $\widehat{\Psi}_{\alpha,\beta}$ satisfy RHP-$\widehat{\Psi}_{\alpha,\beta}$(d). Property (b) on ∂U_e, $e\in\{-1,0\}$, is the consequence of the fact

$$\mathbf{R}^{-1}_{\alpha,\beta-}\mathbf{R}_{\alpha,\beta+} = \widehat{\Psi}^{(\infty)}\widehat{\Psi}^{(e)-1} = \mathbf{I} + F^{(\infty)}\mathcal{O}\left(\tau^{-1}\right)F^{(\infty)-1}.$$

Finally, on the rest of $\Sigma(\mathbf{R}_{\alpha,\beta})$ it holds that

$$\mathbf{R}_{\alpha,\beta+} = \mathbf{R}_{\alpha,\beta-} \begin{cases} \mathbf{I} + \mathbf{F}_-^{(\infty)} \begin{pmatrix} 0 & e^{-2\tau g} \\ 0 & 0 \end{pmatrix} \mathbf{F}_+^{(\infty)-1} & \text{on} \quad (-3/4, -1/4), \\ \mathbf{I} + \mathbf{F}^{(\infty)} \begin{pmatrix} 0 & \beta e^{-2\tau g} \\ 0 & 0 \end{pmatrix} \mathbf{F}^{(\infty)-1} & \text{on} \quad (1/4, \infty), \\ \mathbf{I} + \mathbf{F}^{(\infty)} \begin{pmatrix} 0 & 0 \\ e^{\pm\alpha\pi i} e^{2\tau g} & 0 \end{pmatrix} \mathbf{F}^{(\infty)-1} & \text{on} \quad \widehat{I}_\pm \setminus \overline{U}_{-1}. \end{cases}$$

As $g(\eta) > 0$ for $\eta \in (-1, \infty)$ and $g(\eta) < 0$ for $\eta \in \widehat{I}_\pm$, the last part of the property (b) follows. Given (a,b,c) it is by now standard to conclude that

$$\mathbf{R}_{\alpha,\beta}(\eta; \tau) = \mathbf{I} + \mathcal{O}\left(\frac{1}{\tau(1+|\eta|)}\right)$$

as $\tau \to \infty$ uniformly for $\eta \in \mathbb{C} \setminus \Sigma(\mathbf{R}_{\alpha,\beta})$. Thus,

$$\begin{aligned} \widehat{\mathbf{\Psi}}_{\alpha,\beta}(\eta;\tau) &= \frac{\eta^{-\sigma_3/4}}{\sqrt{2}} \left(\mathbf{I} + \mathcal{O}\left(\frac{1}{\tau\sqrt{1+|\eta|}}\right)\right) \left(\mathbf{I} + \mathcal{O}\left(\eta^{-1/2}\right)\right) \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} e^{-\tau g(\eta)\sigma_3} \\ &= \frac{\eta^{-\sigma_3/4}}{\sqrt{2}} \left(\mathbf{I} + \mathcal{O}\left(\eta^{-1/2}\right)\right) \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} e^{-\tau g(\eta)\sigma_3} \qquad (101) \end{aligned}$$

as $\eta \to \infty$ uniformly for $\eta \in \mathbb{C} \setminus \Sigma(\mathbf{R}_{\alpha,\beta})$ and τ large. Estimate (33) now follows from (100).

10.6 *Asymptotics of* RHP-$\mathbf{\Psi}_{\alpha,\beta}$ *for* $s < 0$

In this section we assume that $\beta \neq 0$ and define

$$\log \beta = \log|\beta| + i\arg(\beta), \quad \arg(\beta) \in (-\pi, \pi).$$

Again, we only need to prove (33) when $s \to -\infty$.

10.6.1 *Renormalized* RHP-$\mathbf{\Psi}_{\alpha,\beta}$

Set $\widehat{J}_\pm$ to be two Jordan arcs connecting 0 and 1, oriented from 0 to 1, and lying in the first (+) and the fourth (−) quadrants. Denote further by $\Omega_\pm$ the domains delimited by $\widehat{J}_\pm$ and $[0,1]$. Define

$$g(\eta) = \frac{2}{3}(\eta - 1)^{3/2}, \quad \eta \in \mathbb{C} \setminus (-\infty, 1],$$

to be the principal branch and set for convenience $\tau := (-s)^{3/2}$. Let

$$\widehat{\mathbf{\Psi}}_{\alpha,\beta}(\eta;\tau) = (-s)^{\sigma_3/4} \mathbf{\Psi}_{\alpha,\beta}(-s\eta; s) \begin{cases} \begin{pmatrix} 1 & 0 \\ \mp 1/\beta & 1 \end{pmatrix} & \text{in} \quad \Omega_\pm, \\ \mathbf{I} & \text{otherwise.} \end{cases} \qquad (102)$$

Put for brevity $\Sigma(\widehat{\mathbf{\Psi}}_{\alpha,\beta}) := I_+ \cup I_- \cup (-\infty, \infty) \cup \widehat{J}_+ \cup \widehat{J}_-$. Then $\widehat{\mathbf{\Psi}}_{\alpha,\beta}$ solves the following Riemann-Hilbert problem (RHP-$\widehat{\mathbf{\Psi}}_{\alpha,\beta}$):

(a) $\widehat{\mathbf{\Psi}}_{\alpha,\beta}$ is holomorphic in $\mathbb{C} \setminus \Sigma(\widehat{\mathbf{\Psi}}_{\alpha,\beta})$;

(b) $\widehat{\mathbf{\Psi}}_{\alpha,\beta}$ has continuous traces on $\Sigma(\widehat{\mathbf{\Psi}}_{\alpha,\beta}) \setminus \{0, 1\}$ that satisfy

$$\widehat{\mathbf{\Psi}}_{\alpha,\beta+} = \widehat{\mathbf{\Psi}}_{\alpha,\beta-} \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{on} \quad (-\infty, 0), \\ \begin{pmatrix} 0 & \beta \\ -1/\beta & 0 \end{pmatrix} & \text{on} \quad (0, 1), \\ \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} & \text{on} \quad (1, \infty), \end{cases}$$

and

$$\widehat{\Psi}_{\alpha,\beta+} = \widehat{\Psi}_{\alpha,\beta-} \begin{cases} \begin{pmatrix} 1 & 0 \\ 1/\beta & 1 \end{pmatrix} & \text{on} \quad \widehat{J}_\pm, \\ \begin{pmatrix} 1 & 0 \\ e^{\pm\alpha\pi i} & 1 \end{pmatrix} & \text{on} \quad I_\pm; \end{cases}$$

(c) as $\eta \to 0$ it holds that

$$\widehat{\Psi}_{\alpha,\beta}(\eta;\tau) = \mathcal{O}\begin{pmatrix} |\zeta|^{\alpha/2} & |\zeta|^{\alpha/2} + |\zeta|^{-\alpha/2} \\ |\zeta|^{\alpha/2} & |\zeta|^{\alpha/2} + |\zeta|^{-\alpha/2} \end{pmatrix} \quad \text{and} \quad \widehat{\Psi}_{\alpha,\beta}(\eta;\tau) = \mathcal{O}\begin{pmatrix} 1 & \log|\zeta| \\ 1 & \log|\zeta| \end{pmatrix}$$

when $\alpha \neq 0$ and $\alpha = 0$, respectively;

(d) $\widehat{\Psi}_{\alpha,\beta}$ has the following behavior near ∞:

$$\widehat{\Psi}_{\alpha,\beta}(\eta;\tau) = \left(\mathbf{I} + \mathcal{O}\left(\eta^{-1}\right)\right) \frac{\eta^{-\sigma_3/4}}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} e^{-\tau g(\eta)\sigma_3}$$

uniformly in $\mathbb{C} \setminus \left(I_+ \cup I_- \cup (-\infty,\infty)\right)$.

10.6.2 *Global Parametrix*

Set

$$\widehat{\Psi}^{(\infty)}(\eta;\tau) := \mathbf{F}^{(\infty)}(\eta) e^{-\tau g(\eta)\sigma_3},$$

where

$$\mathbf{F}^{(\infty)}(\eta) := \begin{pmatrix} 1 & 0 \\ -\frac{1}{\pi i}\log\beta & 1 \end{pmatrix} \frac{(\eta-1)^{-\sigma_3/4}}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} F_\beta^{-\sigma_3}(\eta)$$

and the function F_β is give by (83). Now, it is a straightforward verification to see that

(a) $\widehat{\Psi}^{(\infty)}$ is holomorphic in $\mathbb{C} \setminus (-\infty, 1]$;

(b) $\widehat{\Psi}^{(\infty)}$ has continuous traces on $(-\infty, 1)$ that satisfy

$$\widehat{\Psi}^{(\infty)}_+ = \widehat{\Psi}^{(\infty)}_- \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{on} \quad (-\infty, 0), \\ \begin{pmatrix} 0 & \beta \\ -1/\beta & 0 \end{pmatrix} & \text{on} \quad (0,1); \end{cases}$$

(c) $\widehat{\Psi}^{(\infty)}$ satisfies RHP-$\widehat{\Psi}_{\alpha,\beta}$(d) uniformly in $\mathbb{C} \setminus (-\infty; 1]$ and the term $\mathcal{O}\left(\eta^{-1}\right)$ does not depend on τ.

Again, notice that $\widehat{\Psi}^{(\infty)}$ and $\mathbf{F}^{(\infty)}$ satisfy the same jump relations.

10.6.3 *Local Parametrix Around* 1

Denote by U_1 the disk centered at 1 of radius $1/4$ with boundary oriented counter-clockwise. Choose arcs $\widehat{J}_\pm$ so that

$$\left\{\eta - 1 : \ \eta \in \widehat{J}_\pm \cap U_1\right\} \subset I_\pm.$$

Let, as before, $\Psi_{Ai} = \Psi_{0,1}(\cdot;0)$. Set

$$\widehat{\Psi}^{(1)}(\eta;\tau) := \widehat{\mathbf{E}}^{(1)}(\eta)\Psi_{Ai}\left(-s(\eta-1)\right)\beta^{-\sigma_3/2},$$

where $\widehat{\mathbf{E}}^{(1)}$ is holomorphic around 1 and is given by

$$\widehat{\mathbf{E}}^{(1)}(\eta) := \mathbf{F}^{(\infty)}(\eta)\left(\frac{\left(-s(\eta-1)\right)^{-\sigma_3/4}}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \beta^{-\sigma_3/2}\right)^{-1}.$$

Then it can be checked that $\widehat{\Psi}^{(1)}$ satisfies

(a) $\widehat{\Psi}^{(1)}$ is holomorphic in $U_1 \setminus \Sigma(\widehat{\Psi}_{\alpha,\beta})$;

(b) $\widehat{\Psi}^{(1)}$ has continuous traces on $U_1 \cap \Sigma(\widehat{\Psi}_{\alpha,\beta})$ that satisfy RHP-$\widehat{\Psi}_{\alpha,\beta}$(b);

(c) it holds that

$$\widehat{\Psi}^{(1)}(\eta;\tau) = \mathbf{F}^{(\infty)}(\eta)\left(\mathbf{I} + \mathcal{O}\left(\tau^{-1}\right)\right)e^{-\tau g(\eta)\sigma_3}$$

as $\tau \to \infty$, uniformly for $\eta \in \partial U_1 \setminus \Sigma(\widehat{\Psi}_{\alpha,\beta})$.

10.6.4 *Local Parametrix Around 0*

Denote by U_0 the disk centered at 0 of radius $1/4$ whose boundary oriented counter-clockwise. Let

$$m(\eta) := 3 \mp 2ig(\eta), \quad \pm\mathrm{Im}(\eta) > 0.$$

Then m is conformal in U_0, $m(0) = 0$, and $m(x) > 0$ for $x \in (0, 1/4)$. Choose the arcs $\widehat{J}_\pm$ so that $m(\widehat{J}_\pm) \subset J_\pm$. Define

$$\widehat{\Psi}^{(0)}(\eta;\tau) := \widehat{\mathbf{E}}^{(0)}(\eta)\mathcal{D}\left(\mathbf{\Phi}_{\alpha,\beta}(\tau m(\eta))\right),$$

where $\mathbf{\Phi}_{\alpha,\beta}$ is the solution of RHP-$\mathbf{\Phi}_{\alpha,\beta}$, $\mathcal{D}\left(\mathbf{\Phi}_{\alpha,\beta}(\tau m)\right)$ is a holomorphic deformation of $\mathbf{\Phi}_{\alpha,\beta}(\tau m)$ that moves the jumps from $(\tau m)^{-1}(I_\pm)$ to $I_\pm$, and $\widehat{\mathbf{E}}^{(0)}$ is holomorphic around 0 and is given by

$$\widehat{\mathbf{E}}^{(0)}(\eta) := \mathbf{F}^{(\infty)}(\eta)\left(e^{-3\tau i\sigma_3/2}(i\tau m(\eta))^{\log\beta\,\sigma_3/2\pi i}\mathbf{B}_\pm\right)^{-1} \tag{103}$$

(the constant matrices $\mathbf{B}_\pm$ were also defined in RHP-$\mathbf{\Phi}_{\alpha,\beta}$). To see that $E^{(0)}$ is indeed holomorphic recall that

$$\mathbf{B}_+ = \mathbf{B}_-\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad (ix)_-^{\log\beta/2\pi i} = \beta\,(ix)_+^{\log\beta/2\pi i}$$

for $x > 0$, which implies that the function in parenthesis in (103) has the same jump as $\mathbf{F}^{(\infty)}$ on $(-1/4, 1/4)$. Observe further that

$$\mathbf{B}_\pm e^{\mp i\tau m(\eta)\sigma_3/2} = e^{3\tau i\sigma_3/2}\mathbf{B}_\pm e^{-\tau g(\eta)\sigma_3}, \quad \pm\mathrm{Im}(\eta) > 0.$$

Therefore, it follows from RHP-$\mathbf{\Phi}_{\alpha,\beta}$(d) that

$$\widehat{\Psi}^{(0)}(\eta;\tau) = \mathbf{F}^{(\infty)}(\eta)\left(e^{-3\tau i\sigma_3/2}(i\tau m(\eta))^{\log\beta\,\sigma_3/2\pi i}\mathbf{B}_\pm\right)^{-1}\left(\mathbf{I} + \mathcal{O}\left(\tau^{-1}\right)\right)\times$$
$$\times\left(e^{-3\tau i\sigma_3/2}(i\tau m(\eta))^{\log\beta\,\sigma_3/2\pi i}\mathbf{B}_\pm\right)e^{-\tau g(\eta)\sigma_3}.$$

Finally, notice that

$$\left|\tau^{\log\beta/2\pi i}\right| = \tau^{\arg(\beta)/2\pi}, \quad \arg(\beta) \in (-\pi, \pi).$$

Thus, $\widehat{\Psi}^{(0)}$ has the following properties:

(a) $\widehat{\Psi}^{(0)}$ is holomorphic in $U_0 \setminus \Sigma(\widehat{\Psi}_{\alpha,\beta})$;
(b) $\widehat{\Psi}^{(0)}$ satisfies RHP-$\widehat{\Psi}_{\alpha,\beta}$(b) on $\Sigma(\widehat{\Psi}_{\alpha,\beta}) \cap U_0$;
(c) $\widehat{\Psi}^{(0)}$ satisfies RHP-$\widehat{\Psi}_{\alpha,\beta}$(c) within U_0 (by RHP-$\mathbf{\Phi}_{\alpha,\beta}$(c));
(d) it holds that

$$\widehat{\Psi}^{(0)}(\eta;\tau) = \mathbf{F}^{(\infty)}(\eta)\left(\mathbf{I} + \mathcal{O}\left(\tau^{\arg(\beta)/\pi - 1}\right)\right)e^{-\tau g(\eta)\sigma_3}$$

as $\tau \to \infty$ uniformly on $\partial U_0 \setminus \Sigma(\widehat{\Psi}_{\alpha,\beta})$.

10.6.5 *Asymptotics of* RHP-$\Psi_{\alpha,\beta}$

Define

$$\mathbf{R}_{\alpha,\beta}(\eta;\tau) := \widehat{\Psi}_{\alpha,\beta}(\eta;\tau)\begin{cases} \widehat{\Psi}^{(0)}(\eta;\tau)^{-1}, & \eta \in U_0, \\ \widehat{\Psi}^{(1)}(\eta;\tau)^{-1}, & \eta \in U_1, \\ \widehat{\Psi}^{(\infty)}(\eta;\tau)^{-1}, & \eta \in \mathbb{C} \setminus (\overline{U}_0 \cup \overline{U}_1). \end{cases}$$

Notice that the jumps of $\mathbf{R}_{\alpha,\beta}$ across $\widehat{J}_\pm \setminus (\overline{U}_0 \cup \overline{U}_1)$ are equal to

$$\mathbf{I} + \mathbf{F}^{(\infty)-1}\begin{pmatrix} 0 & 0 \\ e^{2\tau g} & 0 \end{pmatrix}\mathbf{F}^{(\infty)}.$$

Since $\mathrm{Re}(g) < 0$ there, we get exactly as in the case $s > 0$ that

$$\mathbf{R}_{\alpha,\beta}(\eta;\tau) = \mathbf{I} + \mathcal{O}\left(\frac{1}{\tau^{1-\arg(\beta)/\pi}(1+|\eta|)}\right)$$

as $\tau \to \infty$ uniformly for $\eta \in \mathbb{C} \setminus \left(\partial U_0 \cup \partial U_1 \cup \left(\Sigma(\widehat{\Psi}_{\alpha,\beta}) \setminus (\overline{U}_0 \cup \overline{U}_1)\right)\right)$. Hence, (101) still holds and therefore (33) follows from (102).

10.7 *Asymptotics of* RHP-$\widetilde{\Psi}_{\alpha,\beta}$

Below, we assume that $\beta = 0$. As before, we only need to prove (34) when $s \to -\infty$.

10.7.1 *Renormalized* RHP-$\widetilde{\Psi}_{\alpha,\beta}$

Define

$$g(\eta) = \frac{2}{3}\eta^{1/2}(\eta - 1), \quad \eta \in \mathbb{C} \setminus (-\infty, 1],$$

to be the principal branch and set for convenience $\tau := (-s)^{3/2}$. Let

$$\widehat{\Psi}_\alpha(\eta;\tau) = (-s)^{\sigma_3/4}\widetilde{\Psi}_{\alpha,0}(-s\eta; s). \tag{104}$$

Then $\widehat{\Psi}_\alpha$ solves the following Riemann-Hilbert problem (RHP-$\widehat{\Psi}_{\alpha,\beta}$):

(a) $\widehat{\Psi}_\alpha$ is holomorphic in $\mathbb{C} \setminus (I_+ \cup I_- \cup (-\infty, 0])$;

(b) $\widehat{\Psi}_\alpha$ has continuous traces on $I_+ \cup I_- \cup (-\infty, 0)$ that satisfy

$$\widehat{\Psi}_{\alpha+} = \widehat{\Psi}_{\alpha-}\begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{on} \quad (-\infty, 0), \\ \begin{pmatrix} 1 & 0 \\ e^{\pm\alpha\pi i} & 1 \end{pmatrix} & \text{on} \quad I_\pm; \end{cases}$$

(c) as $\eta \to 0$ it holds that

$$\widehat{\Psi}_\alpha(\eta;\tau) = \mathcal{O}\begin{pmatrix} |\zeta|^{\alpha/2} & |\zeta|^{\alpha/2} + |\zeta|^{-\alpha/2} \\ |\zeta|^{\alpha/2} & |\zeta|^{\alpha/2} + |\zeta|^{-\alpha/2} \end{pmatrix} \quad \text{and} \quad \widehat{\Psi}_\alpha(\eta;\tau) = \mathcal{O}\begin{pmatrix} 1 & \log|\zeta| \\ 1 & \log|\zeta| \end{pmatrix}$$

when $\alpha \neq 0$ and $\alpha = 0$, respectively;

(d) $\widehat{\Psi}_\alpha$ has the following behavior near ∞:

$$\widehat{\Psi}_\alpha(\eta;\tau) = \left(\mathbf{I} + \mathcal{O}\left(\eta^{-1}\right)\right)\frac{\eta^{-\sigma_3/4}}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}e^{-\tau g(\eta)\sigma_3}$$

uniformly in $\mathbb{C} \setminus (I_+ \cup I_- \cup (-\infty, \infty))$.

10.7.2 *Global Parametrix*

Set

$$\widehat{\Psi}^{(\infty)}(\eta;\tau) := \frac{\eta^{-\sigma_3/4}}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}e^{-\tau g(\eta)\sigma_3} =: \mathbf{F}^{(\infty)}(\eta)e^{-\tau g(\eta)\sigma_3}.$$

It is a straightforward verification to see that

(a) $\widehat{\Psi}^{(\infty)}$ is holomorphic in $\mathbb{C} \setminus (-\infty, 0]$;

(b) $\widehat{\Psi}^{(\infty)}$ has continuous traces on $(-\infty, 0)$ that satisfy $\widehat{\Psi}^{(\infty)}_+ = \widehat{\Psi}^{(\infty)}_-\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$;

(c) $\widehat{\Psi}^{(\infty)}$ satisfies RHP-$\widehat{\Psi}_\alpha$(d) with $\mathcal{O}(\eta^{-1}) \equiv 0$.

10.7.3 *Local Parametrix Around 0*

Denote by U_0 the disk centered at 0 of small enough radius so that $g^2(\eta)$ is conformal in U_0. Notice that $g^2(x) > 0$ for $\{x > 0\} \cap U_0$. Define

$$\widehat{\Psi}^{(0)}(\eta;\tau) := \widehat{E}^{(0)}(\eta)\mathcal{D}\left(\Psi_\alpha\left((\tau g(\eta)/2)^2\right)\right),$$

where Ψ_α is the solution of RHP-Ψ_α, $\mathcal{D}\left(\Psi_\alpha\left((\tau g/2)^2\right)\right)$ is a holomorphic deformation of $\Psi_\alpha\left((\tau g/2)^2\right)$ that moves the jumps from $(\tau^2 g^2/4)^{-1}(I_\pm)$ to $I_\pm$, and $\widehat{E}^{(0)}$ is holomorphic around 0 and is given by

$$\widehat{E}^{(0)}(\eta) := F^{(\infty)}(\eta)\mathcal{D}\left(F^{(\infty)-1}\left((\tau g/2)^2\right)\right).$$

Clearly, $\widehat{\Psi}^{(0)}$ has the following properties:

(a) $\widehat{\Psi}^{(0)}$ is holomorphic in $U_0 \setminus \left(I_+ \cup I_- \cup (-\infty,\infty)\right)$;
(b) $\widehat{\Psi}^{(0)}$ satisfies RHP-$\widehat{\Psi}_\alpha$(b) on $\left(I_+ \cup I_- \cup (-\infty,\infty)\right) \cap U_0$;
(c) $\widehat{\Psi}^{(0)}$ satisfies RHP-$\widehat{\Psi}_\alpha$(c) within U_0 (by RHP-Ψ_α(c));
(d) it holds that

$$\widehat{\Psi}^{(0)}(\eta;\tau) = F^{(\infty)}(\eta)\left(I + \mathcal{O}\left(\tau^{-1}\right)\right)e^{-\tau g(\eta)\sigma_3}$$

as $\tau \to \infty$ uniformly on $\partial U_0 \setminus \left(I_+ \cup I_- \cup (-\infty,\infty)\right)$.

10.7.4 *Asymptotics of* RHP-$\widetilde{\Psi}_{\alpha,\beta}$

Define

$$R_\alpha(\eta;\tau) := \widehat{\Psi}_\alpha(\eta;\tau)\begin{cases}\widehat{\Psi}^{(0)}(\eta;\tau)^{-1}, & \eta \in U_0,\\ \widehat{\Psi}^{(\infty)}(\eta;\tau)^{-1}, & \eta \in \mathbb{C}\setminus\overline{U}_0.\end{cases}$$

Exactly as before we have that

$$R_\alpha(\eta;\tau) = I + \mathcal{O}\left(\frac{1}{\tau(1+|\eta|)}\right)$$

as $\tau \to \infty$ uniformly for $\eta \in \mathbb{C}\setminus\left(\partial U_0 \cup \left(\left(I_+ \cup I_- \cup (-\infty,\infty)\right)\setminus\overline{U}_0\right)\right)$. Hence, (34) follows from (104).

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